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Backward selfsimilar solutions of supercritical parabolic equations

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ABSTRACT

We consider the exponential reaction–diffusion equation in space-dimension $n \in (2, 10)$. We show that for any integer $k \geq 2$ there is a backward selfsimilar solution which crosses the singular steady state k-times. The same holds for the power nonlinearity if the exponent is supercritical in the Sobolev sense and subcritical in the Joseph–Lundgren sense.

1. Introduction

By a backward selfsimilar solution of the equation

$$u_t = u_{rr} + \frac{n-1}{r} u_r + |u|^{p-1}u, \quad r > 0,\ p > 1, \tag{1}$$

we mean a solution of the form

$$u(r,t) = (T-t)^{-\frac{1}{p-1}} \psi(y), \quad y = \frac{r}{\sqrt{T-t}},\ T \in \mathbb{R},\ t < T,$$

where ψ is a solution of the ODE

$$\psi'' + \left(\frac{n-1}{y} - \frac{y}{2}\right)\psi' + |\psi|^{p-1}\psi - \frac{1}{p-1}\psi = 0, \quad y > 0. \tag{2}$$

Backward selfsimilar solutions play an important role in the analysis of the asymptotic behaviour of solutions of (1) which blow up in finite time, see [1], for instance.

Bounded solutions of (2) satisfy the initial conditions

$$\psi(0) = \alpha, \qquad \psi'(0) = 0. \tag{3}$$

In the case $n = 1, 2$ or $n > 2$ and $p \leq p_S := (n+2)/(n-2)$, the only bounded solutions of (2) are the constants $\psi \equiv 0$, $\psi \equiv \pm\kappa$, $\kappa := (p-1)^{-1/(p-1)}$, see [2]. On the other hand, for $p_S < p < p^*$,

$$p^* := \begin{cases} \infty & \text{if } n \leq 10, \\ 1 + \dfrac{4}{n-4-2\sqrt{n-1}} & \text{if } n > 10, \end{cases} \tag{4}$$

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there exists an increasing sequence $\{\alpha_k\}_{k=1}^{\infty}$, $\alpha_k \to \infty$, such that the solution $\psi = \psi_k$ of (2) and (3) with $\alpha = \alpha_k$ satisfies:

$$\psi(y) > 0 \quad \text{for } y > 0, \qquad y^{2/(p-1)}\psi(y) \to c \quad \text{as } y \to \infty \tag{5}$$

for some $c = c_k > 0$, see [3–5]. For $n > 10$ and $p^* \le p < p_L := 1 + 6/(n-10)$ there exist solutions of (2) and (3), satisfying (5), see [6]. If $p_S < p < p_L$ then all nonconstant positive bounded solutions of (2) intersect the explicit singular solution

$$\psi_\infty(y) := Ly^{-\frac{2}{p-1}}, \qquad L := \left(\frac{2}{p-1}\left(n-2-\frac{2}{p-1}\right)\right)^{\frac{1}{p-1}}, \tag{6}$$

at least twice, see [3–6]. If $n > 2$ and $p_S < p < p^*$ then for every even positive integer k and for every large odd integer k there is a bounded solution of (2) which intersects the explicit singular solution k-times and satisfies (5), see [4].

In this paper we show the following:

Theorem 1.1. *Assume that $n > 2$ and $p_S < p < p^*$. Then for every integer $k \ge 2$ there is a bounded solution of (2) which has k intersections with the singular solution ψ_∞ and satisfies (5) with some $c = c_k > 0$.*

We also establish a result on the existence of solutions with odd number of intersections with ψ_∞ for some $p^* \le p < p_L$ and $n > 10$, see Corollary 2.8.

In [7], Mizoguchi showed the nonexistence of positive bounded solutions of (2) which intersect ψ_∞ at least twice for $p > 1 + 7/(n-11)$, $n > 11$. A numerical study of Plecháč and Šverák ([8]) suggests that this is true if $p > p_L$, $n > 10$.

By a backward selfsimilar solution of the equation

$$u_t = u_{rr} + \frac{n-1}{r}u_r + e^u, \quad r > 0, \tag{7}$$

we mean a solution of the form

$$u(r,t) = -\log(T-t) + \psi(y), \qquad y = \frac{r}{\sqrt{T-t}},\ T \in \mathbb{R},\ t < T,$$

where ψ is a solution of the ODE

$$\psi'' + \left(\frac{n-1}{y} - \frac{y}{2}\right)\psi' + e^{\psi} - 1 = 0, \quad y > 0. \tag{8}$$

We are interested in solutions of (8) which satisfy

$$\psi(0) = \alpha \ge 0, \qquad \psi'(0) = 0, \tag{9}$$

and

$$\lim_{y\to\infty}\left(1 + \frac{y}{2}\psi'(y)\right) = 0. \tag{10}$$

Condition (10) arises naturally (see [1, p. 70]) and it means in particular that if u is a backward selfsimilar solution of (7) with ψ satisfying (10) then $\lim_{t\to T-} u(r,t)$ exists and is finite for $r > 0$.

In the case $n = 1, 2$, there is no solution of (8), (9), (10), see [1], [9]. On the other hand, for $2 < n < 10$, there exists an increasing sequence $\{\alpha_k\}_{k=1}^{\infty}$, $\alpha_k \to \infty$, such that the solution ψ_k of (8), (9) satisfies (10), see [10]. Lacey and Tzanetis proved in [11] that there is a solution $\psi = \psi_\alpha$ of (8), (9), (10) and a negative constant C such that

$$\lim_{y\to\infty}(\psi(y) + 2\log y - \log 2(n-2)) = C. \tag{11}$$

We prove the following:

Theorem 1.2. *Assume that $2 < n < 10$. Then for every integer $k \ge 2$ there exists $\alpha = \alpha_k$ such that the solution of (8), (9) has k intersections with the singular solution $\psi_\infty(y) := -2\log y + \log 2(n-2)$ and satisfies (11) for some constant $C = C_k$.*

2. Intersections with the singular steady state

Let ψ be a solution of problem (2), (3) or (8), (9). If ψ satisfies (2), we define $\phi = \psi - \kappa$ and if ψ satisfies (8), we merely let $\phi = \psi$. Therefore we are considering the solutions of the equation

$$\phi'' + \left(\frac{n-1}{y} - \frac{y}{2}\right)\phi' + G(\phi) = 0, \quad y > 0, \tag{12}$$

with initial conditions

$$\phi(0) = \alpha - K \ge 0, \qquad \phi'(0) = 0, \tag{13}$$

where either $G(\phi) = -\frac{1}{p-1}(\phi + K) + (\phi + K)^p$ and $K = \kappa$, or $G(\phi) = e^{\phi} - 1$ and $K = 0$. We will let $\phi^*(y) = Ly^{-2/(p-1)} - \kappa$ if the nonlinearity G is algebraic and $\phi^*(y) = -2\log y + \log 2(n-2)$ if G is exponential.

If G is algebraic then it is only defined for $\phi \ge -\kappa$. If it then happens that $\phi(y_0) = -\kappa$ for some $y_0 > 0$, we make a formal extension $\phi(y) = -\infty$ for $y > y_0$. This is just to be able to handle the exponential and power cases both at the same time. If there is a need for the explicit writing of the initial condition we will let $\phi_\alpha = \phi$ with $\phi(0) = \alpha - K$.

We will frequently use the following comparison lemma which is well known, see [12], for instance.

Lemma 2.1. *Suppose that $-\infty < y_0 < y_\infty \le \infty$, $a, b \in C([y_0, y_\infty))$ and that $f, g \in C^2([y_0, y_\infty))$ satisfy*

$$\begin{cases} f'' + af' + bf \ge 0, & g'' + ag' + bg \le 0, \quad \text{in } (y_0, y_\infty), \\ g > 0, \ \text{in } (y_0, y_\infty), & f(y_0) = g(y_0), f'(y_0) \ge g'(y_0) > 0. \end{cases}$$

Then $f \ge g$ and $f'g \ge fg'$ in (y_0, y_∞).

The next proposition limits the number of zeros of ϕ near 0.

Proposition 2.2. *If ϕ satisfies (12) then it cannot have more than one zero in $(0, \sqrt{2n})$.*

Proof. Assume that $\phi(y_1) = \phi(y_2) = 0$ for some $0 < y_1 < y_2 < \sqrt{2n}$ with $\phi(y) < 0$ for $y \in (y_1, y_2)$. Let $v(y) = y^2 - 2n$ so that it satisfies

$$v'' + \left(\frac{n-1}{y} - \frac{y}{2}\right) v' + v = 0, \quad y > 0. \tag{14}$$

Clearly ϕ verifies

$$\phi'' + \left(\frac{n-1}{y} - \frac{y}{2}\right) \phi' + \phi = (1 - G'(\eta))\phi,$$

for some $\eta = \eta(y) \in [0, \phi(y)]$. Since $G'(\phi) < 1$ for every $\phi < 0$, we have that

$$\phi'' + \left(\frac{n-1}{y} - \frac{y}{2}\right) \phi' + \phi < 0, \tag{15}$$

for $y \in (y_1, y_2)$. Let $v_\varepsilon = \varepsilon v$ and take $\varepsilon > 0$ small enough such that it holds that $v_\varepsilon(y_1 + \varepsilon_1) = \phi(y_1 + \varepsilon_1)$ with $v_\varepsilon'(y_1 + \varepsilon_1) > \phi'(y_1 + \varepsilon_1)$ and $v_\varepsilon(y_2 - \varepsilon_2) = \phi(y_2 - \varepsilon_2)$ with $v_\varepsilon'(y_2 - \varepsilon_2) < \phi'(y_2 - \varepsilon_2)$ for some $\varepsilon_1, \varepsilon_2 > 0$ and $y_1 + \varepsilon_1 < y_2 - \varepsilon_2$. Then we can use Lemma 2.1 with $y_0 = y_1 + \varepsilon_1$ and $y_\infty = y_2$ to conclude that $\phi(y) < v_\varepsilon(y)$ for every $y \in (y_1 + \varepsilon_1, y_2)$ which is a contradiction since $v_\varepsilon(y_2) < 0$. □

Proposition 2.3. *If ϕ has a zero at $y_1 > \sqrt{2n}$ then there exist $C > 0$ and $y_2 \ge y_1$ such that $\phi(y) \le C(2n - y^2)$ for $y > y_2$.*

Proof. If $\phi'(y_1) > 0$ then there exists $y_2 > y_1$ such that $\phi(y_2) = 0$ and $\phi'(y_2) < 0$. If $\phi'(y_1) < 0$ then take $y_2 = y_1$.

Let $M = -\infty$ if the exponential equation is under consideration and $M = -\kappa$ if we are dealing with the power equation. Let $y_\infty = \sup\{\widetilde{y} > y_2 : M \le \phi(y) < 0 \text{ in } (y_2, \widetilde{y})\}$. Let $v_\varepsilon = \varepsilon(2n - y^2)$ and so v_ε satisfies (14). We also have that ϕ verifies (15) in (y_2, y_∞). Taking then $\varepsilon > 0$ small enough such that $v_\varepsilon(y_2 + \varepsilon_2) = \phi(y_2 + \varepsilon_2)$ and $v_\varepsilon'(y_2 + \varepsilon_2) > \phi'(y_2 + \varepsilon_2)$ for some $\varepsilon_2 > 0$, we can use the comparison lemma above to obtain that $\phi(y) < v_\varepsilon(y)$ for every $y \in (y_2 + \varepsilon_2, y_\infty)$.

In the exponential case, if $y_\infty < \infty$, then it must hold that $\phi(y_\infty) = 0$ which is a contradiction since by comparison we have $\phi(y_\infty) \le v_\varepsilon(y_\infty) < 0$. Therefore the claim holds.

In the power case it holds that $y_\infty < \infty$ and $\phi(y_\infty) = -\kappa$, since $\varepsilon(2n - y^2) < -\kappa$ for y large enough. Therefore $\phi(y) < \varepsilon(2n - y^2)$ for $y \in (y_2 + \varepsilon_2, y_\infty]$ and $\phi(y) = -\infty$ for $y > y_\infty$ which gives the claim. □

Define y^* by the equation $\phi^*(y^*) = 0$ which implies $(y^*)^2 = 2(n-2) - 4K^{p-1} < 2n$. Then the number of crossings of ϕ and ϕ^* in the interval (y^*, ∞) is limited as follows.

Proposition 2.4. *Assume that $\phi(y_1) = \phi^*(y_1)$ for some $y_1 > y^*$. Then there is a constant $C > 0$ such that $\phi(y) \le C(2n - y^2)$ for y large enough. Moreover, the following hold:*

(i) *If $\phi'(y_1) > (\phi^*)'(y_1)$ then there exist exactly two points $y_2, y_3 > y_1$ such that $\phi(y_2) = \phi(y_3) = 0$ and exactly one point $y_4 > y_1$ such that $\phi(y_4) = \phi^*(y_4)$.*
(ii) *If $\phi'(y_1) < (\phi^*)'(y_1)$, then ϕ does not cross ϕ^* for $y > y_1$.*

Proof. Assume that $\phi'(y_1) > (\phi^*)'(y_1)$. Then $y_2 = \sup\{\widetilde{y} > y_1 : \phi^*(y) < \phi(y) < 0 \text{ in } (y_1, \widetilde{y})\} \le \infty$ is well-defined because $y_1 > y^*$. Define $g = \phi^*\phi' - (\phi^*)'\phi$ and let $\rho = \rho(y) = y^{n-1}e^{-y^2/4}$. Then

$$\begin{aligned} (\rho g)' &= \rho' g + \rho g' = \left(\frac{n-1}{y} - \frac{y}{2}\right)\rho g + \rho(\phi^*\phi'' - (\phi^*)''\phi) \\ &= -\rho\phi^* G(\phi) + \rho\phi G(\phi^*) = \rho\phi\phi^* \left(\frac{G(\phi^*)}{\phi^*} - \frac{G(\phi)}{\phi}\right), \end{aligned}$$

and since the function $G(x)/x$ is increasing for $x < 0$ such that $G(x)$ is defined, we obtain that $(\rho g)' < 0$ in (y_1, y_2). Therefore we have that $(\rho g)(y) < (\rho g)(y_1)$ for every $y \in (y_1, y_2)$ and so

$$\left(\frac{\phi}{\phi^*}\right)' = \frac{g}{(\phi^*)^2} < \frac{(\rho g)(y_1)}{\rho(\phi^*)^2},$$

in (y_1, y_2). This implies that

$$\frac{\phi(y)}{\phi^*(y)} < \frac{\phi(y_1)}{\phi^*(y_1)} + \int_{y_1}^{y} \frac{(\rho g)(y_1)}{\rho(s)\phi^*(s)^2} \mathrm{d}s,$$

for every $y \in (y_1, y_2)$ and since $(\rho g)(y_1) = \rho(y_1)\phi^*(y_1)(\phi'(y_1) - (\phi^*)'(y_1)) < 0$, we have

$$\phi(y) > \phi^*(y) \left(1 + (\rho g)(y_1) \int_{y_1}^{y} s^{1-n} \mathrm{e}^{s^2/4} \phi^*(s)^{-2} \mathrm{d}s\right) > \phi^*(y), \tag{16}$$

for every $y \in (y_1, y_2)$. Clearly $y_2 < \infty$ since the integral part of (16) tends to ∞ as $y \to \infty$ and so it also has to hold that $\phi(y_2) = 0$ with $\phi'(y_2) > 0$. On the other hand, since ϕ has to be negative for large y (see (5) and (10)), we know that ϕ crosses 0 again at some $y_3 > y_2$.

By Proposition 2.2, we obtain that $y_3 > \sqrt{2n}$ and so by Proposition 2.3 we have that $\phi(y) < C(2n-y^2)$ for y large enough. Therefore there exists y_4 such that $\phi(y_4) = \phi^*(y_4)$. Using the same function g as above and precisely the same estimates but with $(\rho g)(y_4) > 0$ and $(\rho g)' > 0$, we arrive at the inequality

$$\phi(y) < \phi^*(y) \left(1 + (\rho g)(y_4) \int_{y_4}^{y} s^{1-n} \mathrm{e}^{s^2/4} \phi^*(s)^{-2} \mathrm{d}s\right) < \phi^*(y), \tag{17}$$

for every $y \in (y_4, y_\infty)$ where $y_\infty = \sup\{\widetilde{y} > y_4 : M < \phi(y) < \phi^*(y) \text{ in } (y_4, \widetilde{y})\}$, where again $M = -\infty$ for the exponential and $M = -\kappa$ for the power. Therefore we conclude that ϕ does not cross ϕ^* again after y_4 and $\phi < C(2n - y^2)$ for y large enough.

Assuming that $\phi'(y_1) < (\phi^*)'(y_1)$ we just replace y_4 by y_1 in (17) and that proves the claim. □

Denote by $z_\#(f)$ the number of zeros of the function f in the interval $(0, \infty)$.

Proposition 2.5. *Assume that $z_\#(\phi_{\alpha_{2k}} - \phi^*) = 2k$ and that $z_\#(\phi_\alpha - \phi^*) > 2k$ for $\alpha - \alpha_{2k} > 0$ small enough. Then there exists $\alpha_{2k+1} > \alpha_{2k}$ such that $z_\#(\phi_\alpha - \phi^*) = 2k + 2$ for $\alpha \in (\alpha_{2k}, \alpha_{2k+1})$ and $z_\#(\phi_{\alpha_{2k+1}} - \phi^*) \in \{2k, 2k + 1\}$.*

Proof. Define $I_k(a) = \{\alpha > a : z_\#(\phi_\alpha - \phi^*) \neq k\}$. Let $\{y_i(\alpha)\}_i$ be the zeros of $\phi_\alpha - \phi^*$ for any α and assume $y_j(\alpha) < y_{j+1}(\alpha)$ for any $j \le z_\#(\phi_\alpha - \phi^*) - 1$.

Since $y_{2k+1}(\alpha)$ exists for $\alpha - \alpha_{2k} > 0$ small enough, we obtain by continuity that $y_{2k+1}(\alpha) \to \infty$ as $\alpha \searrow \alpha_{2k}$. Therefore for α close to α_{2k}, we have that $y_{2k+1}(\alpha) > y^*$ and $(\phi_\alpha)'(y_{2k+1}(\alpha)) > (\phi^*)'(y_{2k+1}(\alpha))$ (due to continuity with respect to α). So by Proposition 2.4, we have another zero $y_{2k+2}(\alpha)$ of $\phi_\alpha - \phi^*$ and points $\widetilde{y}_2(\alpha), \widetilde{y}_3(\alpha) \in (y_{2k+1}(\alpha), y_{2k+2}(\alpha))$ such that $\phi_\alpha(\widetilde{y}_2(\alpha)) = \phi_\alpha(\widetilde{y}_3(\alpha)) = 0$. Hence there exists $\alpha_{2k+1} = \inf I_{2k+2}(\alpha_{2k})$ such that $z_\#(\phi_\alpha - \phi^*) = 2k+2$ for $\alpha \in (\alpha_{2k}, \alpha_{2k+1})$.

Assume that $z_\#(\phi_{\alpha_{2k+1}} - \phi^*) = 2k + 2$. Then by continuity, $z_\#(\phi_\alpha - \phi^*) > 2k + 2$ for $\alpha - \alpha_{2k+1} > 0$ small enough and by the same argument that we used above, it must hold $z_\#(\phi_\alpha - \phi^*) \ge 2k + 4$ for $\alpha - \alpha_{2k+1} > 0$ small enough.

Since $y_{2k+2}(\alpha)$ is continuous in $(\alpha_{2k}, \alpha_{2k+1}]$, there exists a constant $D(\varepsilon) > 0$, such that $y_{2k+2}(\alpha) < D(\varepsilon)$ for every $\alpha \in [\alpha_{2k} + \varepsilon, \alpha_{2k+1}]$. Also by continuity, $\phi'(\widetilde{y}_2(\alpha)) > 0$ for every $\alpha \in (\alpha_{2k}, \alpha_{2k+1}]$, since otherwise $\phi_{\widehat{\alpha}}(\widetilde{y}_2(\widehat{\alpha})) = \phi'_{\widehat{\alpha}}(\widetilde{y}_2(\widehat{\alpha})) = 0$ for some $\widehat{\alpha}$, which is clearly a contradiction. Therefore there exists a point $\widetilde{y}_1(\alpha)$ such that $\phi(\widetilde{y}_1(\alpha)) = 0$ and $\widetilde{y}_1(\alpha) < \sqrt{2n} < \widetilde{y}_2(\alpha) < \widetilde{y}_3(\alpha)$ for every $\alpha \in (\alpha_{2k}, \alpha_{2k+1})$ by Propositions 2.2 and 2.4 above.

We have, due to $\phi_\alpha(0)$, thus obtained that $\sqrt{2n} \le \widetilde{y}_2(\alpha) < \widetilde{y}_3(\alpha) < y_{2k+2}(\alpha) < D(\varepsilon)$ for every $\alpha \in [\alpha_{2k} + \varepsilon, \alpha_{2k+1}]$. However, the fact that $y_{2k+2}(\alpha_{2_{k+1}}) > \sqrt{2n} > y^*$ implies that $\phi_\alpha - \phi^*$ has at least 3 zeros after the point $y = y^*$ for $\alpha - \alpha_{2k+1} > 0$ small enough. This is a contradiction by Proposition 2.4.

Assume then that $z_\#(\phi_{\alpha_{2k+1}} - \phi^*) > 2k + 2$. Then by continuity, $z_\#(\phi_\alpha - \phi^*) > 2k$ also for $\alpha_{2k+1} - \alpha > 0$ small enough which contradicts the definition of α_{2k+1}.

Assume that $z_\#(\phi_{\alpha_{2k+1}} - \phi^*) < 2k$. Then by continuity, $y_{2k}(\alpha), y_{2k+1}(\alpha), y_{2k+2}(\alpha) > y^*$ for $\alpha_{2k+1} - \alpha > 0$ small enough. This contradicts Proposition 2.4. Now the claim is proved. □

Proposition 2.6. *Assume that $z_\#(\phi_{\alpha_{2k+1}} - \phi^*) = 2k + 1$ and that $z_\#(\phi_\alpha - \phi^*) > 2k + 1$ for $\alpha - \alpha_{2k+1} > 0$ small enough. Then there exists $\alpha_{2k+2} > \alpha_{2k+1}$ such that $z_\#(\phi_\alpha - \phi^*) = 2k + 2$ for $\alpha \in (\alpha_{2k+1}, \alpha_{2k+2})$.*

Proof. If $z_\#(\phi_\alpha - \phi^*) > 2k + 2$ for $\alpha - \alpha_{2k+1} > 0$ small, then there exist two zeros of $\phi_\alpha - \phi^*$ that satisfy $y^* < y_{2k+2}(\alpha) < y_{2k+3}(\alpha)$ and $\phi'(y_{2k+2}(\alpha)) < (\phi^*)'(y_{2k+2}(\alpha))$ and $\phi'(y_{2k+3}(\alpha)) > \phi'(y_{2k+3}(\alpha))$ which is a contradiction with Proposition 2.4. □

Theorem 2.7. *Assume that there exists a solution ϕ_{α_m} of (12), (13) with $z_\#(\phi_{\alpha_m} - \phi^*) = m \geq 5$. Then for any integer $k \in [2, m-2]$ there exists $\alpha_k > 0$ such that $z_\#(\phi_{\alpha_k} - \phi^*) = k$. Moreover, there is a constant $c = c_k > 0$ such that $\psi = \phi_{\alpha_k} + \kappa$ satisfies (5) if G is algebraic or a constant $C = C_k$ such that $\psi = \phi_{\alpha_k}$ satisfies (11) if G is exponential.*

Proof. By Propositions 2.5 and 2.6, the function $z_\#(\phi_\alpha - \phi^*)$ can only increase by at most 2 as α increases. By Proposition 2.4 and continuity, the function $z_\#(\phi_\alpha - \phi^*)$ can only decrease by at most 2 as α increases because there can be at most two crossings of ϕ_α and ϕ^* in (y^*, ∞).

For $\alpha > 0$ small enough, we know that $z_\#(\phi_\alpha - \phi^*) = 2$, cf. [6,11]. Suppose that there exists an integer $k \in [2, m-2]$ such that there is no solution of (12), (13) which intersects with the singular solution k-times. Then there exist values $\{\alpha_{k-1}^{(i)}\}_i$ such that $\phi_\alpha - \phi^*$ has $k-1$ zeros for $\alpha_{k-1}^{(i)} - \alpha > 0$ small, and $k+1$ zeros for $\alpha - \alpha_{k-1}^{(i)} > 0$ small. Since there is a solution ϕ_{α_m} with m intersections with ϕ^*, there exist $\alpha_{k-1} \in \{\alpha_{k-1}^{(i)}\}_i$ and $\alpha_{k+1} > \alpha_{k-1}$ such that $z_\#(\phi_{\alpha_{k-1}} - \phi^*) = k-1$, while $z_\#(\phi_\alpha - \phi^*) = k+1$ for $\alpha \in (\alpha_{k-1}, \alpha_{k+1})$ and $z_\#(\phi_\alpha - \phi^*) > k+1$ for $\alpha - \alpha_{k+1} > 0$ small.

If $k-1$ is odd we have a contradiction by Proposition 2.6. If $k-1$ is even we obtain a contradiction by Proposition 2.5. This proves that for every integer $k \in [2, m-2]$ there is a solution ϕ_{α_k} of (12) such that ϕ_{α_k} crosses the singular solution k-times.

It remains to prove that there exist solutions with k intersections satisfying (5) or (11).

For the solutions ϕ_α that have an odd number of intersections with the singular solution ϕ^* this follows from [4] or [11]. For the power case the claim was proved for even k in [4,6].

For the exponential nonlinearity it was proved in [10] that with $a_{2k} = \inf J_{2k+1} = \inf\{\alpha : \phi_\alpha$ crosses the singular solution at least $2k+1$ times$\}$ it holds that $\phi_{a_{2k}}$ satisfies (10). By the above definition we have that $z_\#(\phi_{a_{2k}} - \phi^*) \in \{2k-1, 2k\}$ and $z_\#(\phi_\alpha - \phi^*) \in \{2k+1, 2k+2\}$ for $\alpha - a_{2k} > 0$ small enough. If $z_\#(\phi_{a_{2k}} - \phi^*) = 2k-1$, then by Proposition 2.6 we have that $z_\#(\phi_\alpha - \phi^*) = 2k$ for $\alpha - a_{2k} > 0$ small enough which is a contradiction. Therefore it has to hold that $z_\#(\phi_{a_{2k}} - \phi^*) = 2k$. This finishes the proof. □

Theorems 1.1 and 1.2 follow now from Theorem 2.7 and [4,10]. We also have the following:

Corollary 2.8. *Let $m \geq 6$ be an even integer and let $p = p_m \in [p^*, p_L)$ and $n = n_m > 10$ be such that there is a bounded solution of (2) which has m intersections with the singular solution ψ_∞ and satisfies (5) with some $c = c_m > 0$. Then for every odd $k \in \{3, \ldots, m-3\}$ there is a bounded solution of (2) which has k intersections with the singular solution ψ_∞ and satisfies (5) with some $c = c_k > 0$.*

Proof. It was shown in [6] that for every even integer $m \geq 2$ there are $p = p_m \in [p^*, p_L)$ and $n = n_m > 10$ such that for every even $k \in \{2, 3, \ldots, m\}$ there is a bounded solution of (2) which has k intersections with the singular solution ψ_∞ and satisfies (5) with some $c = c_k > 0$. If $k \in \{3, \ldots, m-3\}$ is odd then the existence follows from Theorem 2.7. □

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