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Derivation of generalized quantum jump operators and comparison of the microscopic single photon detector models

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Abstract. The recent experiment of Parigi et al. [Science **317**, 1890 (2007)] shows, in agreement with theory, that subtraction of one photon can increase the expectation value of the number of photons in the thermal state. This observation agrees with the standard photon counting model in which the quantum jump superoperator (QJS) gives a count rate proportional to the number of photons. An alternate model for indirect photon counting has been introduced by Dodonov et al. [Phys. Rev. A **72**, 023816 (2005)]. In their model the count rate is proportional to the probability that there are photons in the cavity, and the cavity field is bidirectionally coupled with a two state quantum system which is unidirectionally coupled to a counting device. We give a consistent first principle derivation of the QJSs for the indirect photon counting scheme and establish the complete relations between the physical measurement setup and the QJSs. It is shown that the time-dependent probability for photoelectron emission event must include normalization of the conditional probability. This normalization was neglected in the previous derivation of the QJSs. We include the normalization and obtain the correct photoelectron emission rates and the correct QJSs and show in which coupling parameter regimes these QJSs are applicable. Our analytical results are compared with the exact numerical solution of the Lindblad equation of the system. The derived QJSs enable analysis of experimental photon count rates in a case where a one-to-one correspondence does not exist between the decay of photons and the detection events.

PACS. 42.50.Ar Photon statistics and coherence theory – 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps – 03.65.Ta Foundations of quantum mechanics; measurement theory

1 Introduction

In a recent experiment Parigi et al. [1] probed the quantum commutation rules by adding and subtracting photons to/from a light field. Their results showed that a subtraction of one photon from a thermal field state can increase the expectation value of the number of photons. This result agrees with the predictions of the quantum trajectory model of photon counting developed by Srinivas and Davies (the SD model) [2] and further elaborated by Ueda et al. [3]. In the SD model the time evolution of the field is determined by a linear (in photon number) one-count and a no-count superoperators. Recently another superoperator model, where the one-count operator is a nonlinear saturating function of the number of photons, called the E model has been presented by de Oliveira et al. [4] and Dodonov et al. [5]. While the presently available experimental results [1,6,7] agree with the predictions of the SD model it would be important to know the exact regimes of the validity of both the SD and the E models – this is the goal of our work.

We start from the Lindblad master equation and show, using the conditional probability formulas of the detection events, how the SD and E models are obtained as asymptotic limits of the detection probabilities. In particular our approach gives the correct dependence of the quantum jump superoperators on the coupling constants of the theoretical measurement apparatus. This allows us to determine the validity regimes of the SD and E models. We also show that the SD and E model cannot be considered simply as a low intensity and a high intensity model, respectively, even for the particular single atom detector considered in this manuscript.

2 Background

2.1 Origin of the Srinivas and Davies model and the E model of quantum jump superoperators

Srinivas and Davies [2] originally considered *a phenomenological* cavity photon counting model. The detector was described by a single model parameter representing the

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strength of the field-detector coupling and a one-to-one correspondence was assumed between the photon absorption and the count events. The count rate in the SD model is proportional to the number of photons and hence unbounded. Later Ueda et al. [3] calculated, by using the SD model, the time-evolution of the photon statistics for a number of common single-mode cavity fields and Imoto et al. [8] presented a microscopic derivation for the SD model using the perturbation theory and a schematic detector consisting of an atomic beam interacting with the cavity field.

Recently de Oliveira et al. [4] and Dodonov et al. [9] postulated a new cavity photon counting model by replacing the bosonic annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators used in the SD model with nonlinear bounded operators $\hat{E} = (\hat{a}^\dagger\hat{a}+1)^{-1/2}\hat{a}$ and $\hat{E}^\dagger = \hat{a}^\dagger(\hat{a}^\dagger\hat{a}+1)^{-1/2}$, respectively, resulting in the so called E model. Later they discussed the microscopic background of their model [5]. Their motivation for the E model was to obtain a count rate which is bounded for all fields, and also to avoid some of the predictions of the SD model they thought to be inconsistent [10]. In the E model the photon counting rate is proportional to the probability that there are photons (one or more) in the field state. Therefore, the count rate for high intensity fields saturates to a maximum value which is a model parameter.

2.2 Difference between the SD and the E model

2.2.1 Definitions

The SD and the E model originally shared four main assumptions [2,4]:

1. The absorption of photons by the detector takes place as instantaneous events represented by the one-count superoperator $\hat{J}\hat{\rho}_f(t) = \gamma\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger$ (γ is defined in item 4). The one-count operator $\hat{J}$ is a quantum jump superoperator i.e. the density operator jumps from $\hat{\rho}_f(t)$ to $\hat{J}\hat{\rho}_f(t)$ in an infinitesimal time interval $[t, t+dt]$. In the SD model $\hat{A} \equiv \hat{a}$. The E model is obtained by substituting $\hat{A} \equiv \hat{E} = (\hat{a}^\dagger\hat{a}+1)^{-1/2}\hat{a}$. The normalized operators $\hat{E}$ and $\hat{E}^\dagger$ obey the relations $\hat{E}|0\rangle = 0$, $\hat{E}|n>0\rangle = |n-1\rangle$ and $\hat{E}^\dagger|n\rangle = |n+1\rangle$. The probability of the one-count during $[t, t+dt]$ is in both models given by $\gamma\mathrm{Trace}\{\hat{A}\hat{\rho}_f(t)\hat{A}^\dagger\}dt$.
2. Between two consecutive count events during $[t, t+\tau]$ the density operator evolves according to the no-count superoperator $\hat{S}(\tau)\hat{\rho}_f(t) = \hat{U}(\tau)\hat{\rho}_f(t)\hat{U}^\dagger(\tau)$, where $\hat{U}(\tau) = \exp\Big(-\mathrm{i}\hat{H}_0\tau/\hbar - \gamma\hat{A}^\dagger\hat{A}\tau/2\Big)$ and $\hat{H}_0 = \hbar\omega\hat{a}^\dagger\hat{a}$ is the field Hamiltonian. Here τ is not necessarily a differential time interval.
3. After measuring an event (one-count or no-count) corresponding to the operator $\hat{\mathcal{O}}$ ($\hat{\mathcal{O}} = \hat{J}$ or $\hat{S}$), the density operator is $\hat{\rho}_f = \hat{\mathcal{O}}\hat{\rho}_f(t)/\mathrm{Trace}\{\hat{\mathcal{O}}\hat{\rho}_f(t)\}$.
4. The coupling between the detector and the field is parameterized using a model dependent coupling coefficient γ. The γ_{sd} and γ_e are not necessary equal but if we equate the absorption rates for a one photon Fock state (i.e. $|1\rangle$) we obtain $\gamma_{sd} = \gamma_e$. The one-count rates of the SD and E models are given by $\gamma_{sd}\bar{n}(t)$ and $\gamma_e(1-p_0(t))$ ($p_0(t)$ is the probability of the vacuum state in $\hat{\rho}_f$), respectively, and this is the fundamental difference of the models.

2.2.2 Comparison of the models and experimental results

The SD model predicts that immediately after a one-count event (i.e. in the photon subtracted state) the expectation value of the number of the photons is $\bar{n}_0 - 1$, $\bar{n}_0$, and $2\bar{n}_0$ for the Fock state, coherent state, and thermal state, respectively (see Ref. [3]). Here $\bar{n}_0$ is the expectation value of the number of photons in the initial state. In contrast E model predicts $\bar{n}_0-1$, $\bar{n}_0/(1-\exp(-\bar{n}_0))-1$, and $\bar{n}_0$ for the Fock state, coherent state, and thermal state, respectively (see Ref. [9]).

Although the predictions of the SD model for the thermal field may seem counter intuitive they are not unphysical. The increase of the expectation value of the number of the photons in thermal field was theoretically explained in reference [3] and recently verified experimentally [1]. Parigi et al. [1] added/subtracted a photon to/from the light pulse, and measured the photon statistics of the final field state with a homodyne detector (see Ref. [1] for details). They were able to show that after subtracting one photon from a thermal field the expectation value of the number of photons doubled as predicted by Ueda et al. [3] using the SD model. Furthermore, Parigi et al. [1] verified experimentally the non-commutativity of the bosonic annihilation and creation operators (i.e. $[\hat{a}, \hat{a}^\dagger] \neq 0$) and showed that the simple view of the classical particle addition and subtraction is incorrect in this case. The results of Parigi et al. [1] show that the photon added and the photon subtracted states are given by $\hat{a}^\dagger\hat{\rho}\hat{a}$ and $\hat{a}\hat{\rho}\hat{a}^\dagger$ (unnormalized), respectively, as predicted by the SD model.

Note that the photon creation ($\hat{a}^\dagger$) and annihilation ($\hat{a}$) operators are the same for the cavity field and the light pulse measured by Parigi et al. [1]. Therefore, the results of Parigi et al. can be applied to test photon statistics modeled in this work even though the experimental detector setup is completely different [11]. The measurements done in reference [1] were aimed to demonstrate the measurement back action to light field through the state of the light field after subtraction/addition of a photon. The SD and E photon counting models give the state of the light field after photon subtraction. Therefore, the models should predict the measured photon subtracted states in the setup of Parigi et al. [1].

3 Theory of the field-atom-detector dynamics

In the following we consider a special measurement apparatus consisting of a single atom and a photoelectron detector. Properties of the apparatus are similar to the apparatus introduced by Imoto et al. [8] and generalized by Dodonov et al. [5]. The total system consists of three

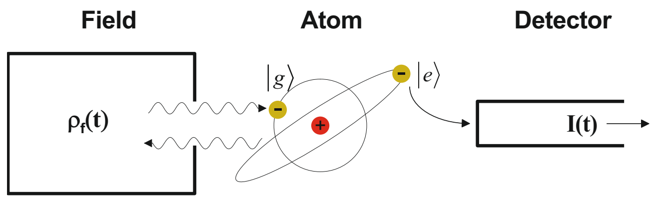

Fig. 1. (Color online) The schematic detector setup for indirect photon counting: (1) the photons are absorbed by the atom which is excited to an autoionizing state. (2) The autoionizing state [12] decays either by electron emission recorded by the detector as $\mathbf{I(t)}$ or emission of the photon back to the cavity. It is assumed that the atom is quickly neutralized if the electron emission (autoionizing) occurs.

parts: the field (f), the atom (A), and the detector (d). The field is coupled to the atom (coupling constant g) and the atom to the reservoir (coupling constant λ). The photoelectron detector is assumed ideal: each photoelectron is recorded. It is also assumed that no thermal radiation enters the cavity. The detector and the atom are called together as a measurement apparatus. From dissipation point of view the detector plays a role of an infinite reservoir. See Figure 1 for a schematic detector setup.

The measurement apparatus defined above is mainly a theoretical configuration for measuring the cavity photon statistics. It is, however, quite general in the sense that the exact physical implementation is not important. The dynamics of the system is governed by the physical device parameters: the field-atom coupling g and the atom-detector coupling λ. The main property of the measurement apparatus is that it allows the *interaction* between the photon field and the atom *during* the measurement. This, as we will show, is one crucial requirement for the E model to apply. A detector system with approximately the same physical properties should be experimentally feasible, and the apparatus of Figure 1 thus serves as a workhorse to highlight the difference between the photoelectron statistics and the cavity photon statistics.

3.1 Field-atom-reservoir system

The dynamics of the system formed by the cavity field and the atom interacting with the reservoir is described by the well known master equation [5]

$$\frac{d\hat{\rho}_s(t)}{dt} = -\frac{\mathrm{i}}{\hbar}\left(\hat{H}\hat{\rho}_s(t) - \hat{\rho}_s(t)\hat{H}^{\dagger}\right) + 2\lambda\sigma_-\hat{\rho}_s(t)\sigma_+, \quad (1)$$

where the density operator $\hat{\rho}_s(t)$ describes both the field (f) and the atom (A). In equation (1) the system Hamiltonian $\hat{H}$ is the standard Jaynes-Cummings Hamiltonian of an atom with eigenstates $\pm\hbar\omega_0/2$ coupled to photon mode with $E = \hbar\omega$ with an additional dissipative term

$$\hat{H} = \frac{1}{2}\hbar\omega_0\sigma_0 + \hbar\omega\hat{a}^{\dagger}\hat{a} + \hbar g(\hat{a}\sigma_+ + \hat{a}^{\dagger}\sigma_-) - \mathrm{i}\hbar\lambda\sigma_+\sigma_-, \quad (2)$$

where $\sigma_+ = |e\rangle\langle g|$, $\sigma_- = |g\rangle\langle e|$ and $\sigma_0 = |e\rangle\langle e| - |g\rangle\langle g|$. Equation (1) is a special case of the Lindblad equation,

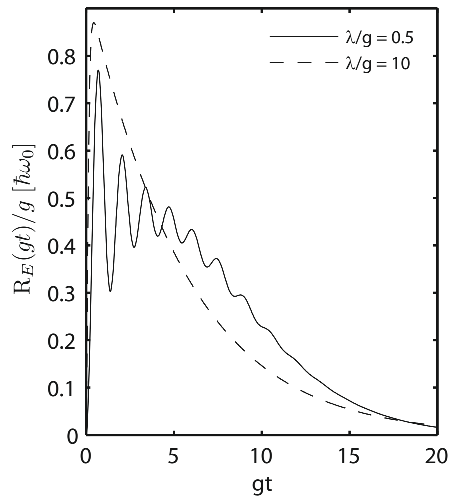

Fig. 2. The exact rate of energy dissipation calculated numerically from Lindblad equation (1) with two different ratios of coupling constants, $\lambda/g = 0.5$ (solid line) and $\lambda/g = 10$ (dashed line). The field is initially in the Fock state $|5\rangle$. For $\lambda/g = 0.5$ the rate oscillates due to the Rabi oscillations. For $\lambda/g = 10$ the rate decays exponentially since in the regime $\lambda \gg g$ there are no Rabi oscillations and the number of photons decay exponentially. In this regime the behavior of the field-atom system coincides with the SD model (see below).

which is known as the most general form of a memoryless coupling between a system and an (infinite) reservoir [13]. Master equation (1) implies a linear coupling between the system and the reservoir but does not otherwise limit the strength of the interaction as long as the state of the reservoir remains unchanged. In this formalism the one-count event recorded by the detector is the emission of the photoelectron from the excited state of the atom to the reservoir.

3.2 Rabi oscillations and energy dissipation

From equation (1) it follows that the rate of the photoelectron emission is given by

$$\mathrm{Trace}_{f,A}\{2\lambda\sigma_-\hat{\rho}_s(t)\sigma_+\} = 2\lambda p_e(t), \quad (3)$$

where $p_e(t)$ is the probability that the atom is in the excited state at time t. On the other hand, the rate of energy dissipation from the cavity-atom system is given by the rate of change of the total energy of the field-atom system and fulfills the differential equation

$$R_E(t) = -\hbar\omega_0\left(\frac{dp_e(t)}{dt} + \frac{d\bar{n}(t)}{dt}\right). \quad (4)$$

Equations (3) and (4) are related by $R_E(t) = 2\lambda p_e(t)\hbar\omega_0$. Figure 2 shows the energy dissipation rate solved numerically from the Lindblad master equation (1) at resonance $\omega_0 = \omega$ using the finite difference method. Note that the result in equation (4) differs from the SD model

($R_E^{sd}(t) = \hbar\omega_0\gamma_{sd}\bar{n}(t)$) and also from the E model ($R_E^e(t) = \hbar\omega_0\gamma_e(1 - p_0(t))$). For $\lambda/g = 10$ the rate decays exponentially while for $\lambda/g = 0.5$ the rate oscillates during the decay (see Fig. 2). The difference is due to the Rabi oscillations which allow the atom to emit the absorbed photon back to the cavity mode in the case of small λ/g. In the SD model the rate decays exponentially and in the E model it decreases monotonically since no photons are emitted back to the cavity in the original SD and E models.

3.3 Quantum trajectories of the field vs. quantum trajectories of the field-atom system

The average evolution (i.e. the evolution without observation) of the system is obtained by solving the Lindblad master equation. To follow the conditional evolution of the system, quantum trajectory approach is used. In the quantum trajectory approach [14] the one-count operator $\hat{J}_s$ (one photoelectron is emitted) and the no-count operator $\hat{S}_s$ (no photoelectron emission) are needed. If we account for all the possible trajectories and calculate the average we obtain the same result as by directly solving the Lindblad master equation.

Assuming that the system at time t is described by $\hat{\rho}_s(t)$ it evolves in an infinitesimal time δt to

$$\hat{\rho}_s(t+\delta t) = \hat{J}_s\hat{\rho}_s(t)\delta t + \hat{S}_s(\delta t)\hat{\rho}_s(t), \tag{5}$$

where the two possible trajectories are given by the field-atom system's one-count and no-count operators

$$\hat{J}_s\hat{\rho}_s(t) = 2\lambda\hat{\sigma}_-\hat{\rho}_s(t)\hat{\sigma}_+ \tag{6}$$

$$\hat{S}_s(\delta t)\hat{\rho}_s(t) = \hat{U}(\delta t)\hat{\rho}_s(t)\hat{U}^\dagger(\delta t) \tag{7}$$

instead of the definitions 1 and 2 in Section 2.2. Furthermore, the time evolution operator $\hat{U}(t)$ in the no-count operator is defined using the dissipative system Hamiltonian (2) $\hat{U}(t)=\exp(-\mathrm{i}\hat{H}t/\hbar)$. If we detect the one-count event (i.e. the emission of the photoelectron) the system is projected into a state $\hat{\rho}_s(t+\delta t) = \hat{J}_s\hat{\rho}_s(t)/\mathrm{Tr}\{\hat{J}_s\hat{\rho}_s(t)\}$. Similarly, after an observed no-count event the system collapses into the state $\hat{\rho}_s(t+\delta t) = \hat{S}_s(\delta t)\hat{\rho}_s(t)/\mathrm{Tr}\{\hat{S}_s(\delta t)\hat{\rho}_s(t)\}$.

In the general case of the field-atom-detector system the photoabsorption process and the electron emission process do not have a one-to-one correspondence since the absorbed photon may be emitted back to the cavity. Therefore, the dissipative term $2\lambda\hat{\sigma}_-\hat{\rho}_s\hat{\sigma}_+$ in equation (1) enables us only to define a photoelectron emission operator, which is the one-count operator $\hat{J}_s$ (Eq. (6)) of the quantum trajectory of the field-atom system. The corresponding no-count operator is $\hat{S}_s$ (Eq. (7)) defined above. The fundamental problem of indirect photon counting is whether or not we can find the exact quantum trajectory dynamics of the field density operator $\hat{\rho}_f$ i.e. if the Lindblad equation (1) enables us to write operators $\hat{J}_f$ and $\hat{S}_f$ such that $\hat{\rho}_f(t+\delta t) = \hat{J}_f\hat{\rho}_f(t)\delta t + \hat{S}_f(\delta t)\hat{\rho}_f(t)$. The obvious answer to this question is no since the quantum trajectories of the field and the atom are entangled. One may then ask if we still can find some average behavior of the field density operator $\hat{\rho}_f$ i.e. define average operators $\hat{\bar{J}}_f$ and $\hat{\bar{S}}_f$ such that $\hat{\bar{\rho}}_f(t+\delta t) = \hat{\bar{J}}_f\hat{\bar{\rho}}_f(t)\delta t + \hat{\bar{S}}_f(\delta t)\hat{\bar{\rho}}_f(t)$.

3.4 Averaging of the quantum jump superoperator

In order to remove the entanglement of the field's and the atom's quantum trajectories Dodonov et al. [5] defined a time-dependent one-count operator and averaged it over time. They considered a quantum trajectory that starts at $t_0 = 0$ when a photoelectron has been emitted (or the initial field is prepared), and ends at t when the next photoelectron emission takes place. Therefore, the atom is in the ground state at $t_0 = 0$ and during the time interval $(0,t)$ the atom goes through an indefinite number of Rabi oscillations (repeated absorption and re-emission of one photon) until it emits a photoelectron and returns to the ground state. They defined a time dependent one-count superoperator $\hat{J}_f(t)$ by using the expression of the probability of the next photoelectron emission to occur at $[t, t+dt)$ as

$$P(t) = \mathrm{Trace}_{f,A}\{\hat{J}_s\hat{S}_s(t)\hat{\rho}_s(0)\}dt \tag{8a}$$

$$= \mathrm{Trace}_f\{\hat{J}_f(t)\hat{\rho}_f(0)\}dt. \tag{8b}$$

By the definition in equation (8b) $\hat{J}_f(t)$ is a combination of the no-count and one-count operators of the field-atom system reduced to describe the field between two jumps. By substituting $\hat{J}_s$ and $\hat{S}_s(t)$ and taking the trace over atomic coordinates they obtained [5]

$$\hat{J}_f(t)\hat{\rho}_f(0) = 2\lambda e^{-\lambda t}e^{-\mathrm{i}\omega\hat{n}t}\hat{X}_{\hat{n}+1}(t)\hat{a}\hat{\rho}_f(0)\hat{a}^\dagger\hat{X}^\dagger_{\hat{n}+1}(t)e^{+\mathrm{i}\omega\hat{n}t}. \tag{9}$$

For the present purpose we assume the exact resonance $\omega_0 = \omega$. Then we have $\hat{X}_{\hat{n}}(t) = \sin\left(gt\sqrt{\hat{n}-(\lambda/(2g))^2}\right)/\sqrt{\hat{n}-(\lambda/(2g))^2}$ with $\hat{n} = \hat{a}^\dagger\hat{a}$.

The trace of equation (9) gives the probability per unit time of observing the first photoelectron at t. Therefore, Dodonov et al. [5] suggested that if the exact instant of the detection event is unknown within $[0,T)$, the detection events happen randomly with uniform probability density given by the trace of equation

$$\hat{\bar{J}}_{f,T}\hat{\rho}_f = \frac{1}{T}\int_0^T \hat{J}_f(t)\hat{\rho}_f(0)dt. \tag{10}$$

Assuming that T is very large but still finite they obtained a one-count superoperator similar to that given by the E model [5]

$$\hat{\bar{J}}_{f,T}\hat{\rho}_f = \frac{1}{T}\sum_{n=1}^{\infty} p_{n,n}|n-1\rangle\langle n-1| = \frac{1}{T}\mathrm{diag}(\hat{E}\hat{\rho}_f\hat{E}^\dagger), \tag{11}$$

where the operator $\hat{E} = (\hat{n}+1)^{-1/2}\hat{a}$. Note that this result depends on the inverse of the averaging interval but it is independent of the coupling parameters g and λ. Therefore, operator $\hat{\bar{J}}_{f,T}$ in equation (10) can only reproduce the exact evolution of the system calculated from the Lindblad equation when fitting T to the coupling constants g and λ. Furthermore, if we calculate the average (10) using a large time interval $T \gg 1$, the average (10) approaches zero. This is clearly non-physical: if there are photons in the cavity the average one-count operator cannot be zero. The same averaging method was also applied to generalized Jaynes-Cummings Hamiltonian by Lu et al. [15]. They followed the derivation in reference [5] and obtained the SD model from the standard Jaynes-Cummings Hamiltonian and the E model from an intensity dependent Jaynes-Cummings Hamiltonian [15].

The QJS in equation (9) acts on the density operator $\hat{\rho}_f(0)$ during $[0, t+dt)$ since it is a combination of the no-count during $[0,t)$ and the one-count during $[t, t+dt)$. It is more useful, however, to define an abrupt jump operator acting on $\hat{\rho}_f(t)$ during $[t, t+dt)$. Then $\hat{J}_f(t)$ defined using equation (8b) and given in equation (9) cannot be used for deriving the averaged QJS defined in equation (10), even though equation (8a) is a correct expression for the probability of observing the first one-count event at $[t, t+dt)$.

Equation (8a) gives the joint probability of a process in which there is (A) no photoelectron emission during $[0,t)$ and (B) the photoelectron emission occurs at $[t, t+dt)$. Therefore, to define the quantum jump superoperator which describes only the emission at $[t, t+dt)$, we must consider the joint process as a *conditional* photoelectron emission event with the *condition* that there is no photoelectron emission during $[0,t)$. According to the principles of the probability theory (see for example [16]) the conditional probability that event B happens with the condition that event A has happened is $P(B|A) = P(A \cap B)/P(A)$, where $P(A \cap B)$ is the probability that both A and B happen. Therefore, the conditional probability of the quantum jump is

$$P_{\hat{J}|\hat{S}}(t) = \frac{\mathrm{Trace}_{f,A}\{\hat{J}_s \hat{S}_s(t)\hat{\rho}_s(0)\}dt}{\mathrm{Trace}_{f,A}\{\hat{S}_s(t)\hat{\rho}_s(0)\}}. \qquad (12)$$

In order to derive the QJS for the field we must use equation (12) instead of $\hat{J}_f(t)$ in equation (8b) since our objective is to take an average over the random duration of the no-count event before the photoelectron emission. The photoelectron one-count event occurs during $[t, t+dt)$ only if it has not occurred during $[0,t)$. Therefore, the probability must be normalized with the probability of the no-count event during $[0,t)$.

We can consider the conditional probability also in the following way: during $[t, t+dt)$ the system evolves according to $\hat{\rho}_s(t+dt) = \hat{J}_s\hat{\rho}_s(t)dt + \hat{S}_s(dt)\hat{\rho}_s(t)$. Our interest is limited to the quantum trajectories in which there are no counts between $[0,t)$ since no photoelectron emission event have been observed during $[0,t)$. The state at $t+dt$ is then given by

$$\begin{aligned}\hat{\rho}_s(t+dt) &= \hat{J}_s\hat{\rho}_s(t)dt + \hat{S}_s(dt)\hat{\rho}_s(t)\\ &= \frac{\hat{J}_s\hat{S}_s(t)\hat{\rho}_s(0)dt}{\mathrm{Tr}\{\hat{S}_s(t)\hat{\rho}_s(0)\}} + \frac{\hat{S}_s(t+dt)\hat{\rho}_s(0)}{\mathrm{Tr}\{\hat{S}_s(t)\hat{\rho}_s(0)\}}.\end{aligned} \qquad (13)$$

The reduced QJS can be obtained by calculating the trace of the first term on the right side of equation (13) $\mathrm{Tr}\{\hat{J}_s\hat{S}_s(t)\hat{\rho}_s(0)\}dt\ /\mathrm{Tr}\{\hat{S}_s(t)\hat{\rho}_s(0)\}$. This is the probability of observing a photoelectron emission during $[t, t+dt)$ with the condition that it has not been observed during $[0,t)$. Note the difference of using equations (8b) and (12): the QJS obtained using $\hat{J}_f(t)$ in equation (8b) acts during $[0, t+dt)$ while the QJS obtained from equation (12) acts only during $[t, t+dt)$.

For the numerator of equation (12) we obtain

$$\mathrm{Trace}_{f,A}\{\hat{J}_s\hat{S}_s(t)\hat{\rho}_s(0)\} = 8\lambda e^{-\lambda t} \times \sum_{n=0}^{\infty}(n+1)p_{n+1}\frac{\left|\sin\left(\frac{gt}{2}\sqrt{4(n+1)-(\lambda/g)^2}\right)\right|^2}{|4(n+1)-(\lambda/g)^2|} \qquad (14)$$

and for the denominator

$$\begin{aligned}\mathrm{Trace}_{f,A}\{\hat{S}_s(t)\hat{\rho}_s(0)\} &= e^{-\lambda t}\\ &\times\sum_{n=0}^{\infty}\Bigg[p_n\Bigg|\cos\left(\frac{gt}{2}\sqrt{4n-(\lambda/g)^2}\right)\\ &+\frac{\lambda}{g}\frac{\sin\left(\frac{gt}{2}\sqrt{4n-(\lambda/g)^2}\right)}{\sqrt{4n-(\lambda/g)^2}}\Bigg|^2\\ &+4(n+1)p_{n+1}\frac{\left|\sin\left(\frac{gt}{2}\sqrt{4(n+1)-(\lambda/g)^2}\right)\right|^2}{|4(n+1)-(\lambda/g)^2|}\Bigg].\end{aligned} \qquad (15)$$

The conditional photoelectron emission rate (i.e. the conditional probability density) is obtained from the probability (12) by dividing with dt:

$$r(t) = \frac{P_{\hat{J}|\hat{S}}(t)}{dt} = \frac{\mathrm{Trace}_{f,A}\{\hat{J}_s\hat{S}_s(t)\hat{\rho}_s(0)\}}{\mathrm{Trace}_{f,A}\{\hat{S}_s(t)\hat{\rho}_s(0)\}}. \qquad (16)$$

In the following sections we consider this rate in detail.

As will be shown below there is an important difference between the conditional probability in equation (12) and the joint probability in equation (8b). Consistent formulation of the time averaged quantum jump superoperators can only be derived from equation (12).

3.5 Weak atom-detector coupling regime ($\lambda \ll g$)

Let us first consider the limit where the atom-reservoir coupling λ is much weaker than the field-atom coupling g i.e. $\lambda/g \ll 1$. The conditional photoelectron emission rate is then given by

$$r(t) = 2\lambda\sum_{n=0}^{\infty}p_{n+1}\sin^2\left(g\sqrt{n+1}t\right). \qquad (17)$$

The time dependence of the rate is due to the Rabi oscillations. Since the amplitude of the oscillation is constant we can average over infinitely many oscillation periods. The time average of the rate is obtained from equation (17) in the limit $T \to \infty$ by using $\frac{1}{T}\int_{t=0}^{T} \sin^2\left(g\sqrt{n+1}t\right) dt = 1/2$. Thus, the average of the conditional rate is

$$\bar{r} = \lambda \sum_{n=0}^{\infty} p_{n+1} = \lambda(1 - p_0). \tag{18}$$

This result is easily understandable as follows: if we know, that initially there is at least one photon in the cavity and that no photoelectron emission has occurred, the atom stays half of the time in the excited state due to the Rabi oscillations. Therefore, the average probability that the atom is in the excited state is $\bar{p}_e = 1/2$. On the other hand, it follows from the Lindblad equation that the exact rate of electron emission is given by $2\lambda p_e(t)$ (see Eq. (3)) giving the average rate λ if there are photons in the cavity. Thus, the average rate is $\bar{r} = \lambda(1-p_0)$ as is *intuitively obvious* even without the quantum trajectory considerations above.

From equation (18) we can conclude that the averaged one-count operator $\hat{\hat{J}}_f$, which gives the average rate $\bar{r} = \text{Trace}\{\hat{\hat{J}}_f \hat{\rho}\}$ in equation (18), for the field is

$$\hat{\hat{J}}_f \hat{\rho} = \lambda \hat{E} \hat{\rho} \hat{E}^\dagger. \tag{19}$$

Therefore, in the one-count operator of the E model one *must have* the model parameter $\gamma_e = \lambda$ to make it consistent with the Lindblad equation (1). See Figure 3a for comparison of the rates given by the exact model and the E model. In this regime of coupling parameters λ and g the E model gives the average behavior of the system.

3.6 Strong atom-detector coupling regime ($\lambda \gg$ g)

If the atom-reservoir coupling λ is much stronger than the field-atom coupling g i.e. $\lambda/g \gg 1$, the conditional photoelectron emission rate equation (16) is proportional to the expectation value of the number of photons:

$$r = \frac{2g^2}{\lambda}\bar{n}. \tag{20}$$

This rate is independent of time so no averaging is needed. In this limit the excited atom always dissipates its energy (infinitely) fast into the reservoir and there are no Rabi oscillations. Since the absorption probability of a photon is directly proportional to the number of photons and each photon absorption process leads to a photoelectron emission, the photoelectron emission rate is directly proportional to the number of photons. This corresponds to the direct photon counting limit for which the SD model is valid as seen below.

The one-count operator for the field is

$$\hat{\hat{J}}_f \hat{\rho} = \frac{2g^2}{\lambda} \hat{a} \hat{\rho} \hat{a}^\dagger. \tag{21}$$

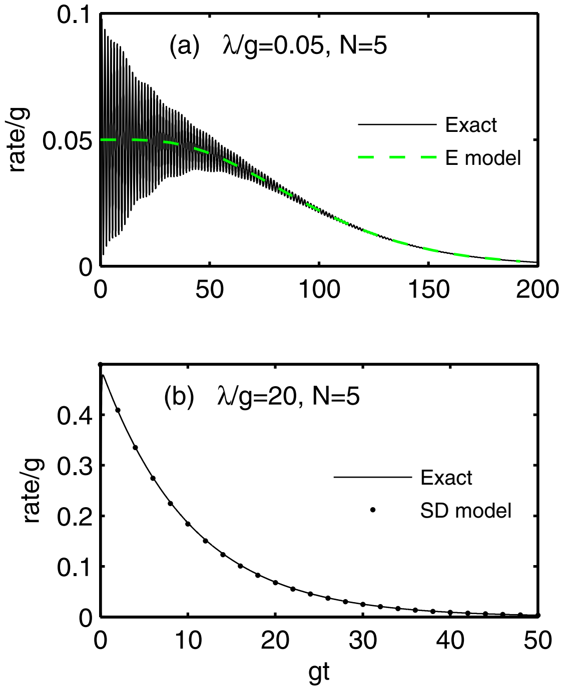

Fig. 3. (Color online) The photoelectron emission rate with two different ratios of the coupling constants: (a) $\lambda/g = 0.05$ and (b) $\lambda/g = 20$. The field being initially in the Fock state $|5\rangle$. The numerically calculated exact rate from Lindblad equation (Eq. (1), solid line) is compared to (a) the rate given by the E model (Eq. (19), dashed line), and to (b) the rate given by the SD model (Eq. (21), dots).

We thus reproduce the SD model with model parameter $\gamma_{sd} = 2g^2/\lambda$. See Figure 3b for comparison of the rates given by the field-atom-reservoir model and the SD model. In this regime of coupling parameters λ and g the rate given by the SD model coincides with the rate obtained from Lindblad equation (Eq. (1)).

3.7 Field intensity and the validity of the photon counting models

Above we used the ratio of the coupling constants λ and g to separate the asymptotic limits $\lambda/g \ll 1$ (saturated detector, Sect. 3.5) and $\lambda/g \gg 1$ (direct photocounting, Sect. 3.6). These conditions are actually somewhat simplified. The exact conditions following from equations (14) and (15) are $\lambda/g \ll 2\sqrt{N_{min}}$ and $\lambda/g \gg 2\sqrt{N_{max}}$, respectively. Here N_{min} (N_{max}) corresponds to Fock state $|N_{min}\rangle$ ($|N_{max}\rangle$) with minimum (maximum) number of photons in the state mixture $\hat{\rho}_f$. In the intermediate region where λ/g satisfies neither condition, neither the QJS of the SD model nor the QJS of the E model gives the correct result. It may be possible to find a simple QJS valid for all regimes of λ/g but it is not in the scope of this work. See Figures 4–6 for the comparison of the exact solution and the SD and the E models at different intensities and coupling constant regimes.

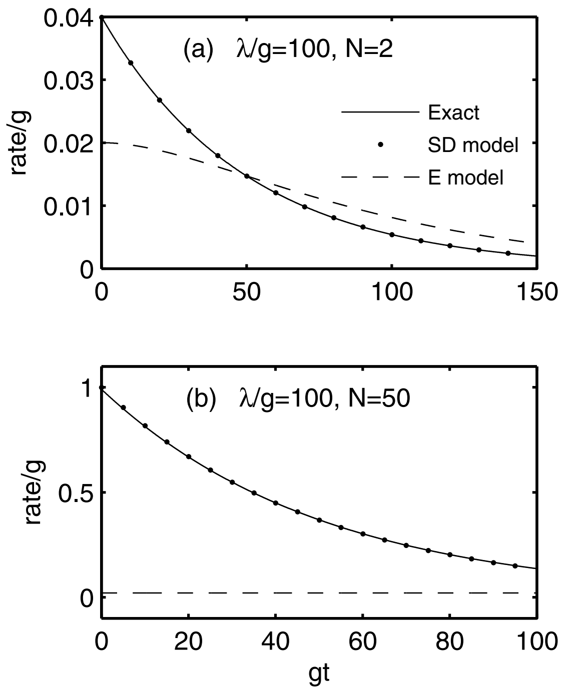

Fig. 4. Field intensity and validity of the photon counting models for $\lambda/g = 100$ i.e. for the strong atom-reservoir coupling. Solid line: the exact photoelectron emission rate solved numerically from Lindblad equation (Eq. (1)), dots: the rate given by the SD model ($\frac{2g^2}{\lambda}\bar{n}(t)$), and dashed line: the rate given by the E model ($\frac{2g^2}{\lambda}(1-p_0(t))$). The field is initially in the Fock state $|2\rangle$ (a) and $|50\rangle$ (b). In both cases the results given by the SD model agree with the exact numerical solution.

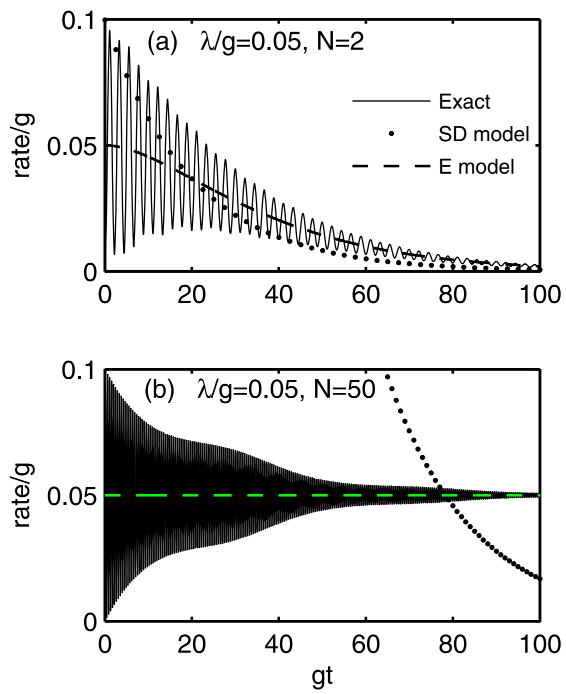

Fig. 5. (Color online) Field intensity and validity of the photon counting models for $\lambda/g = 0.05$ i.e. for the weak atom-reservoir coupling. Solid line: the exact photoelectron emission rate solved numerically from Lindblad equation (Eq. (1)), dots: the rate of the SD model ($\lambda\bar{n}(t)$), and dashed line: the rate of the E model ($\lambda(1-p_0(t))$). The field is initially at the Fock state $|2\rangle$ (a) and $|50\rangle$ (b). The E model gives the correct average behavior of the numerical solution at both the high and the low intensities.

3.8 Comparison of the microscopic theories of photon counting

In reference [5] the time dependent one-count operator was derived using the unnormalized probability (8b) as a starting point. Assuming a low intensity limit and a short interaction time $t = \delta t$ it was obtained from equations (8b)–(9) that the time-dependent one-count operator is

$$\hat{J}_f(\delta t)\hat{\rho}_f = 2\lambda(g\delta t)^2 e^{-\mathrm{i}\omega\hat{n}\delta t}\hat{a}\hat{\rho}_f\hat{a}^\dagger e^{+\mathrm{i}\omega\hat{n}\delta t}, \qquad (22)$$

which corresponds to the SD model with a model parameter $\gamma'_{sd} = 2\lambda(g\delta t)^2$. We obtained the one-count operator (21) corresponding to the SD model with a model parameter $\gamma_{sd} = 2g^2/\lambda$ by assuming $\lambda/g \gg 1$ (see Fig. 4). Note that γ'_{sd} includes fitting parameter δt while our result γ_{sd} depends only on the coupling constants g and λ of the Lindblad equation (1).

The one-count operator (11) was obtained in reference [5] assuming high field intensity. Operator in equation (11) corresponds to the E model with a model parameter $\gamma'_e = 1/T$, where $T \gg 1$ is the integration time. Note that this result does not depend on the system parameters g and λ. By assuming $\lambda/g \ll 1$ we obtained the one count operator (19) corresponding to the E model with a model parameter $\gamma_e = \lambda$ (see Fig. 5) without fitting.

The differences between our results and the results given in reference [5] (like Eq. (22) above) are due to the normalization we used. Note also, that by the arguments given in Section 3.7, the E model and the SD model can not be simply considered as a high intensity and a low intensity photon counting models.

4 Discussion

4.1 Comparison of the analytical and numerical results

Figures 4 and 5 show the exact rate $2\lambda p_e(t)$ solved numerically from equation (1), the rate $\gamma_{sd}\bar{n}(t)$ given by the SD model, and the rate $\gamma_e(1-p_0(t))$ given by the E model. In Figure 4 the field is initially in the Fock state $|2\rangle$ (Fig. 4a), and $|50\rangle$ (Fig. 4b), the atom is initially in the ground state, and the ratio of the coupling parameters is $\lambda/g = 100$. Figure 5 shows the same information as Figure 4 but for $\lambda/g = 1/20$.

In the regime where $\lambda \gg g$ (Fig. 4) the photoelectron emission rate is proportional to the number of photons. The number of photons decays exponentially and is *exactly* described by the SD model as expected based on the analysis in Section 3.

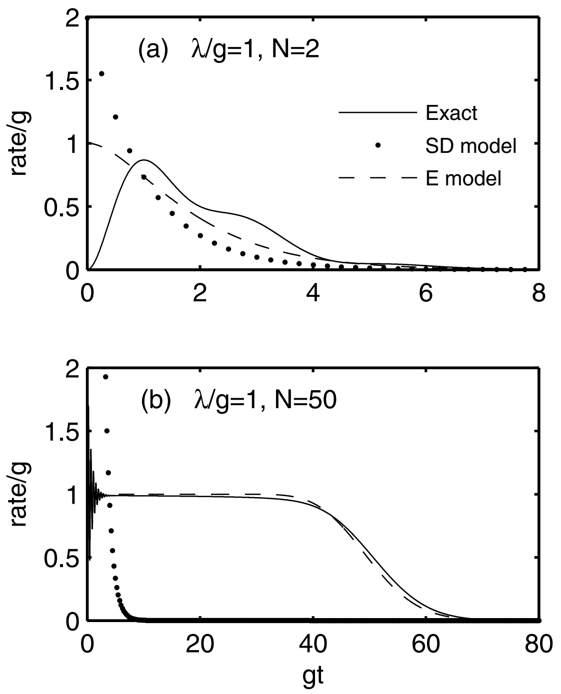

Fig. 6. Intermediate case $\lambda/g = 1$. Solid line: the exact photoelectron emission rate ($2\lambda p_e(t)$) solved numerically from Lindblad master equation (Eq. (1)), dots: the rate of the SD model ($\lambda\bar{n}(t)$), and dashed line: the rate of the E model ($\lambda(1-p_0(t))$). The field is initially in the Fock state $|2\rangle$ (a) and $|50\rangle$ (b).

In the regime $\lambda \ll g$ (Fig. 5) the coupling between the field and the atom is much stronger than the coupling between the atom and the reservoir. When the atom is excited it more likely emits a photon back to the cavity than emits a photoelectron to the reservoir. At limit $\lambda \to 0$ one obtains the standard non-dissipative Jaynes-Cummings model. Therefore, when $\lambda \ll g$ the atom goes through several Rabi oscillations before it emits a photoelectron. In this regime the averaged quantum jump superoperator (i.e. the E model) gives reasonable *average* results as expected based on the analysis in Section 3.

In Figure 6 the field is initially in the Fock state $|2\rangle$ (Fig. 6a), and $|50\rangle$ (Fig. 6b), the atom is initially in the ground state, and the ratio of the coupling parameters is $\lambda/g = 1$. With the high photon number (case (b)) the E model agrees reasonably well with the average of the exact numerical result, again in agreement with the results of Section 3. However, note that in this intermediate regime neither the SD nor the E model has general validity since conditions used in deriving the SD and E models are not fulfilled. No simple form of the QJS in this intermediate regime was found.

4.2 Experimental setup and choice of photon counting model

In a typical photon counting experiment a photomultiplier tube (PMT) can be used as a photon counter. The PMT can be easily cooled to liquid helium temperatures and it will generally not emit absorbed photons back to the field. It can also operate without saturation up to high intensities. If necessary, beam splitters can be used to divide the photon flux to several PMTs to enable their use in single photon counting mode (see for example [17]). The SD model was originally developed for direct photon counting experiments where a photon absorption event has a one-to-one correspondence to the one-count event. This is experimentally very well realizable by using PMTs. If an analogy is to be found between the single photon counting by PMTs and the field-atom-reservoir apparatus (indirect counting) of Figure 1 one has to assume that the excitation of the atom leads immediately to photoelectron emission and simultaneous return of the atom back to the ground state (cf. Fig. 4). Thus, we can write the dissipative Lindblad master equation for the field density operator without accounting for the atom part of the system. The saturation effects and the Rabi oscillations will then never appear and the SD model of the direct detection scheme is applicable.

At indirect photon counting limit corresponding to a weak atom-reservoir coupling the excitation of the atom does not necessarily lead to a photoelectron emission process (cf. Fig. 5). Therefore, the detector states must be included in the Lindblad master equation and only the *averaged* QJS operating on the field state (i.e. the E model) can be obtained.

5 Conclusions

The two photon counting models, the SD model and the E model, have been derived using the conditional photoelectron emission rate (Eq. (16)) and calculating the averaged quantum jump superoperator in a consistent way. The models have been shown to describe the behavior of a single atom detector in the asymptotic limits of weak field-atom/strong atom-reservoir and strong field-atom/weak atom-reservoir coupling, respectively. The first limit ($\lambda/g \gg 1$), the SD model, is more general and the only choice if one wants to model accurately the stochastics of the photons escaping from the cavity.

Our detailed derivation gives the relations $\gamma_{sd} = 2g^2/\lambda$ and $\gamma_e = \lambda$ between the coupling constants g and λ of the Lindblad equation (Eq. (1)) and the model parameters γ_{sd} and γ_e. With these model parameters the analytical results agree with the exact numerical solution of the Lindblad equation within their application regimes. It has been shown that the E model describes the *average* behavior of the field adequately if $\lambda/g \ll 1$ while the SD model coincides exactly with the solution of the Lindblad equation if $\lambda/g \gg 1$. We emphasize that even for $\lambda/g \ll 1$ the E model gives only an *average* of the field statistics, it never reproduces the exact field statistics. Furthermore, the E model describes the special case of a saturating single atom detector which obviously has only limited practical use in describing the statistics of electromagnetic fields. We finally reiterate that for $\lambda/g \gg 1$ the SD model is an exact quantum mechanical result within the Lindblad equation description of dissipation. The predictions of the SD model are neither weird nor inconsistent but an excellent demonstration of the quantum theory of measurement as verified experimentally by Parigi et al. [1].

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