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Optimal confinement potential in quantum Hall droplets

E. Tölö*, A. Harju

Helsinki Institute of Physics and Department of Applied Physics, Helsinki University of Technology, P.O. Box 4100, FI-02015 HUT, Finland

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ABSTRACT

We find that the confinement potential of a few electron quantum dot can be tuned to significantly increase the overlap with certain quantum Hall trial wave functions. Besides manipulating inter-electron interaction, this approach may prove useful in quantum point contact experiments, which involve narrow constrictions.

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1. Introduction

Fractional quantum Hall effect is observed in high-mobility two-dimensional electron systems at low temperature and strong perpendicular magnetic field as the interactions between the electrons induce a gap in the bulk, and the transverse conductance is quantized universally as $G = \nu e^2/h$ with certain rational numbers ν . Each filling fraction ν possesses quasiparticle excitations with characteristic fractional charge and statistics.

Recently, there has been a major technological interest in suspected non-abelian phases, in particular at filling fraction $\nu = \frac{5}{2}$, the quasiparticle properties of which could proposedly be applied in topologically protected quantum gates, robust against decoherence and local perturbations [1–3]. The ongoing experimental studies of the statistics of the $\frac{5}{2}$ state make use of quantum point contacts [4–7], in which the electrons are confined away from the tunneling region. However, experiments suggest that as the quantum point contact shrinks in size, the $\frac{5}{2}$ state eventually becomes unstable. Engineering the form of the confinement potential as to stabilize the non-abelian state in the quantum point contact region could therefore prove advantageous.

In this paper, we find the confinement potentials that maximize the overlap of the wave function of a few electron quantum Hall droplet with the Laughlin state at $\nu = \frac{1}{3}$ [8]

$$\Psi_L(z_1, z_2, \dots, z_N) = \prod_{i < j} z_{ij}^3 e^{-\sum_i z_i \bar{z}_i / 2}, \quad (1)$$

where $z = x + iy$ and $z_{ij} = z_i - z_j$, and the Moore–Read state on the lowest Landau level (LLL) [9]

$$\Psi_{M-R}^{LLL}(z_1, z_2, \dots, z_N) = \text{Pf}(z_{ij}^{-1}) \prod_{i < j} z_{ij}^2 e^{-\sum_i z_i \bar{z}_i / 2} \quad (2)$$

and on the second lowest Landau level (SLL). The latter is obtained by applying the Landau level raising operator $\partial_z - \bar{z}/2$ to each electron coordinate in Ψ_{M-R}^{LLL} and assuming the lowest Landau level of both spin types completely filled and inert.

The rest of the manuscript is organized as follows. The model and the optimization method is explained in Section 2. Section 3 contains the results and Section 4 the summary.

2. Model and optimization

The system is modeled by an effective-mass Hamiltonian

$$H = \sum_{i=1}^N \left[\frac{(\mathbf{p}_i + \frac{e}{c} \mathbf{A}_i)^2}{2m^*} + V(r_i) \right] + \sum_{i < j} \frac{e^2}{\epsilon r_{ij}}, \quad (3)$$

where N is the number of electrons and \mathbf{A} the vector potential of the homogeneous magnetic field \mathbf{B} perpendicular to the sample plane. The material dependent parameters are $m^* = 0.067m_e$, the effective mass of an electron, and $\epsilon = 12.7$, the dielectric constant of GaAs semiconductor medium. The confinement potential is expanded as

$$V(r) = \frac{m^* \omega_0^2 r^2}{2} + \sum_{k=-n}^{k_{\max}} \frac{a_k n! m^* \omega^2 l^2}{(k+n)!} (r/l)^{2k} e^{-(r/l)^2} (L_n^k((r/l)^2))^2, \quad (4)$$

where the last term is motivated by the form of the single-particle basis. k_{\max} is set to the maximum single-particle angular momentum (finite for a fixed total angular momentum), and n is the Landau level index. L_n^m are the generalized Laguerre polynomials, $L_0^0(x) = 1$ and $L_1^1(x) = -x + m + 1$. The optimal parameters a_k are searched by a random walk whereby new value is accepted whenever the resulting overlap is higher.

In the calculations, we set the confinement strength $\hbar\omega_0$ to 2 meV as its scaling should merely shift the ranges of magnetic fields for different phases. Lengths are written in units of

* Corresponding author. Tel.: +358 9 451 5157; fax: +358 9 451 3241.
E-mail address: eero.tolo@tkk.fi (E. Tölö).

oscillator length $l = \sqrt{\hbar/(m^* \omega)}$, where $\omega = \sqrt{\omega_0^2 + (\omega_c/2)^2}$ and $\omega_c = eB/(m^*c)$ is the cyclotron frequency. The lowest energy state of Eq. (3) with given angular momentum is solved by performing Lanczos diagonalization for the many-body Hamiltonian matrix constructed in the basis of a spin-polarized Fock–Darwin band corresponding to fixed n . This constitutes a Landau level projection, an approximation that is valid at the high magnetic field regime and for small enough deviation from the ideal parabolic potential. The single-particle wave functions at Fock–Darwin band n and angular momentum quantum number m are written in oscillator units as

$$\langle z|n, m\rangle = \sqrt{\frac{n!}{\pi(n+m)!}} z^m L_n^m(z\bar{z}) e^{-z\bar{z}/2}, \quad m \geq -n. \quad (5)$$

The many-body Hamiltonian is written in second quantized formalism as

$$H = \sum_{ij} (\delta_{ij} \epsilon_i^n + v_{ij}^n) c_i^{n\dagger} c_j^n + \sum_{ijkl} v_{ijkl}^n c_i^{n\dagger} c_j^{n\dagger} c_l^n c_k^n, \quad (6)$$

where $\epsilon_m^n = (2n+1)\hbar\omega + l\hbar(\omega - \omega_c/2)$ are the single-particle energies in absence of the optimization potential and $v_{mm'}^n = \langle n, m' | \sum_{k=-n}^{k_{\max}} (a_k n! m^* \omega^2 l^2 / (k+n)!) (r/l)^{2k} e^{-(r/l)^2} (L_n^k((r/l)^2))^2 |n, m\rangle$ are calculated analytically for the two lowest Landau levels (see Appendix). The non-trivial quantities are the interaction matrix elements v_{ijkl}^n , which are calculated utilizing Tsiper's analytic formula [10].

In the light of the numerical study [11] and the recent tunneling experiment [6], it seems plausible that the Landau level mixing, which breaks the particle–hole symmetry, favors the particle–hole conjugated Moore–Read state termed the anti-Pfaffian [12]. The potential could then be optimized in similar fashion for anti-Pfaffian or any other candidate state.

3. Results

The obtained optimal overlaps are presented in Table 1. For the Laughlin state, the initial overlaps are in general the highest, yet the lack of overlap decreases on average by 68%. In the lowest Landau level, the initial overlaps of the Moore–Read are quite modest but average 63% reduction in the lack of overlap makes a notable increase. On the second lowest Landau level the reduction averages at about 84% and the overlaps are even higher. Similarly increased overlaps have been reported earlier for modified Coulomb interactions whereby thickness or screening is added [13–15].

However, the relevance of few electron results for moderately large electron numbers remains uncertain. Firstly, the smaller

the droplet, the larger the finite size effects. Even if the overlap is quite low, the state may still belong to the same topological phase as the trial wave function. Secondly, despite the high overlap the new state could be energetically unstable relative to states with different angular momentum than that of the trial wave function. Extrapolation of the actual ground state to larger particle numbers then yields an instability to another filling fraction or a compressible state. Alternatively, one might therefore try to maximize the excitation gap of the ground state phase keeping only a moderate overlap with the trial state.

The modified potentials and their effect to the electron densities are presented in Fig. 1. The potentials are shifted by a constant as to ensure that $V(r)=0$ at $r=0$. As seen in Fig. 1(a),(e), and (i), the difference to the clean parabolic potential reduces as the particle number is increased. This is likely to be related to smoothening of the electronic density as the particle number is increased. In the second Landau level there are more fluctuations in the optimized potentials, since the basis functions there contain the first non-trivial generalized Laguerre polynomial factor leading to an additional maximum in the single particle densities. Looking at the electronic densities in Fig. 1, the improvement in densities is clearly visible in all cases except for Fig. 1(f). Note, however, that due to the disc geometry, the optimal densities tend to be more optimal near the edge of the droplet where most of the density is accumulated.

For the potentials optimized for the Laughlin state, in Fig. 1(a), a common feature is the sharpness of the potential compared to the parabolic potential near the edge of the droplet (see Fig. 1(b)–(d)) around $r=4l$. As noted in earlier works [16,17], this compression leads to a renormalization of the tunneling exponent [18,19] characterizing the non-Ohmic charge transport to the fractional quantum Hall edge. In this case, the increase in the initially high overlaps seems marginal but the experimental consequences may not be.

For the Moore–Read state in the lowest Landau level, the resulting densities (Fig. 1(f)–(h)) match the trial density almost as well as for the Laughlin state despite the notably lower initial overlaps. The form of the potential near the edge is again slightly sharper than a parabolic (see Fig. 1(e)). A remarkable improvement occurs in the lowest Landau level for eight electrons (Fig. 1(h)) where the overlap climbs up to 0.974. This raises question, whether filling fraction $\nu = \frac{1}{2}$ state with Pfaffian wave function (Eq. (2)) could be realized in a suitably tuned quantum Hall droplet. Our preliminary analysis, however, indicates that the obtained state is unstable against higher angular momentum states and thus not immediately realized as the ground state. However, if slight variations of the potential landscape are still allowed without destroying the state, not to mention a larger droplet with more electrons, this instability might be surmountable. A lowest Landau level non-abelian state would be especially welcome since the Landau level mixing could then be well neglected.

In the second lowest Landau level, the Moore–Read state attains even larger overlaps and the densities seem to match reasonably well (Fig. 1(j)–(l)). However, the form of the optimal potentials is a bit more complicated with oscillatory behavior (see Fig. 1(i)).

4. Summary

We have calculated the confinement potentials that maximize the wave functional overlap of a few electron droplet with the $\frac{1}{3}$ Laughlin state and the Moore–Read state at $\nu = \frac{1}{2}$ and $\frac{5}{2}$. The overlaps were found to increase clearly, particularly for

Table 1

The initial and optimized overlaps for the $\frac{1}{3}$ Laughlin state and the Moore–Read state at $\nu = \frac{1}{2}$ and $\frac{5}{2}$ with N electrons.

N	4	5	6
$ \langle \Psi_L \Psi_i \rangle $	0.979	0.985	0.982
$ \langle \Psi_L \Psi_f \rangle $	0.995	0.995	0.993
N	4	6	8
$ \langle \Psi_{M-R}^{LLL} \Psi_i \rangle $	0.922	0.790	0.586
$ \langle \Psi_{M-R}^{LLL} \Psi_f \rangle $	0.963	0.933	0.866
N	4	6	8
$ \langle \Psi_{M-R}^{SLL} \Psi_i \rangle $	0.970	0.849	0.730
$ \langle \Psi_{M-R}^{SLL} \Psi_f \rangle $	0.997	0.978	0.935

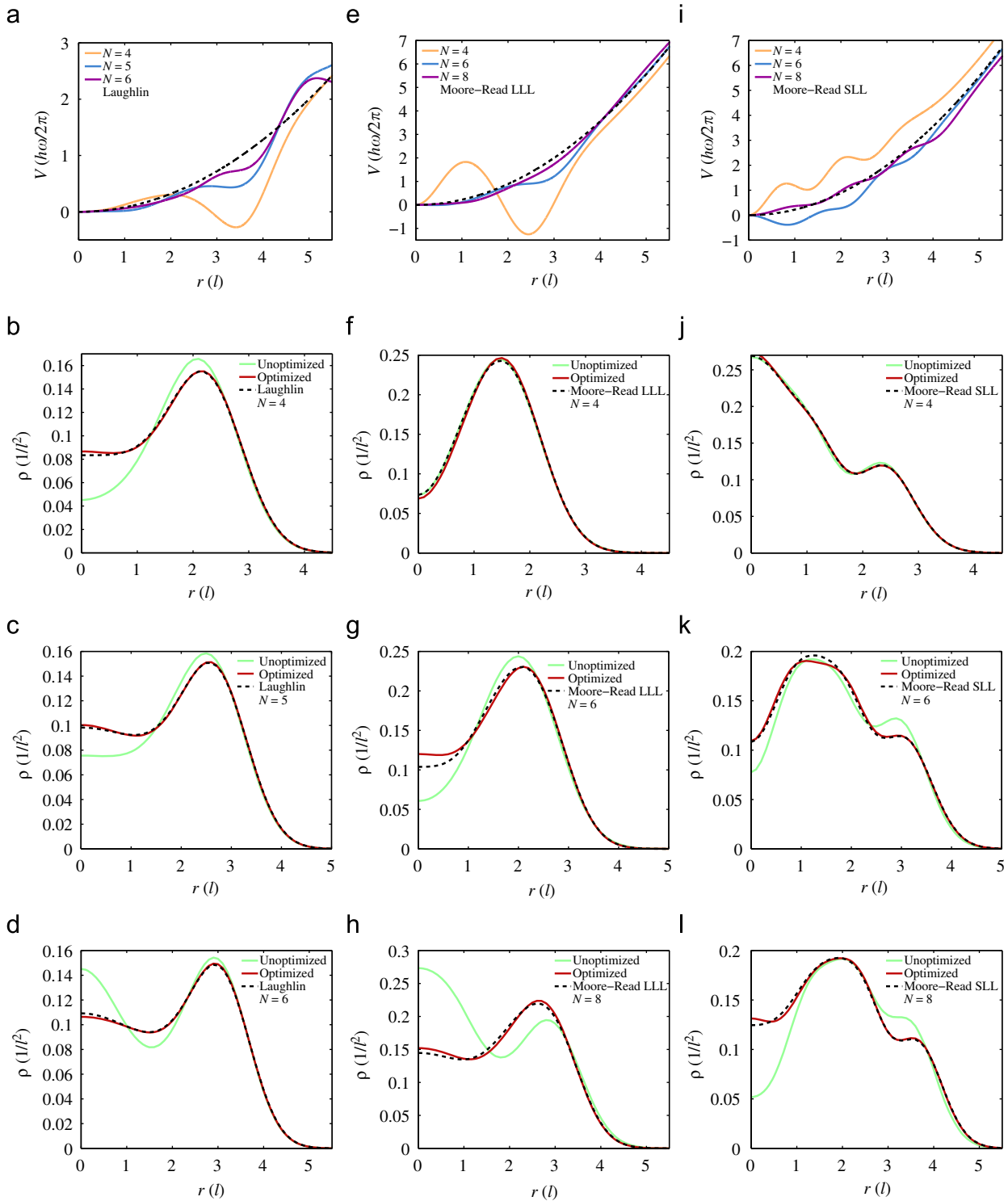


Fig. 1. The optimal confinement potential to maximize the overlap with (a) the Laughlin state with $\omega_0 = \frac{2}{5}$, (e) the Moore–Read state in the lowest Landau level, (i) the Moore–Read state in the second lowest Landau level with $\omega_0 = \frac{2}{5}$, where the dashed line corresponds to the parabolic potential, and the thus obtained electronic densities (b)–(d), (e)–(h), and (i)–(l), respectively, compared to the trial wave function and unoptimized potential.

the larger electron numbers, and the obtained electron densities matched well with the fluctuations of the trial densities. The Laughlin state and the Moore–Read state in lowest Landau level favored a sharpened potential near the edge of the droplet.

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Appendix A

In the lowest Landau level ($n = 0$), the matrix elements v_{mm}^n reduce to

$$v_{m'm}^0 = \delta_{m'm} \sum_k \frac{a_k}{2^{m+k+1}} \frac{(m+k)!}{m!k!}, \quad m \geq 0, k \geq 0. \quad (7)$$

In the second lowest Landau level ($n = 1$), we have

$$v_{m'm}^1 = \delta_{m'm} \sum_k h_{mk} a_k, \quad (8)$$

where we have $k \geq -1$ and $m \geq -1$, and the matrix h_{mk} is given by

$$h_{mk} = \frac{(4k^3 m - m^2 + 2km^2 + 6k^2 m^2 - 2m^3 - 4km^3 + m^4)(m+k)!}{2^{m+k+5} m!k!} \quad (9)$$

for $m+k \geq 0$, $h_{mk} = 1/8$ for $m+k = -1$, and $h_{mk} = 1/4$ for $m+k = -2$.

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