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EXACT SOLUTIONS FOR SOME SPHERICAL ELECTROSTATIC SCATTERING PROBLEMS

Thesis for the degree of Doctor of Science in Technology

Mikko Pitkonen

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Abstract

The purpose of this thesis is to analyze some basic spherical structures in electrostatics using separable coordinate systems. The main emphasis is on a dielectric body immersed in a constant electric field. This setting gives rise to the concept of polarizability, which encapsulates the scattering properties of the dielectric body in a single matrix called a polarizability tensor. For simple structures, such as a sphere and ellipsoid, the polarizability tensor can be found in a closed-form. For more complicated geometries, where there is no separable coordinate system available, one usually must resort to numerical methods.

This thesis focuses on intersecting dielectric double spheres (both two- and three-dimensional). The coordinate system considered in three dimensions is a toroidal coordinate system (R-separable), which leads to an elegant numerical solution scheme (Neumann series) that can be implemented efficiently, for example, in a Java Applet. A special case of the toroidal coordinate system is the tangent sphere frame, representing spheres that intersect each other at a single point, in which the solution of the scattering problem is reduced to a second order linear ordinary differential equation with elementary coefficients. The two-dimensional double hemisphere (or double half-disk) is considered in the bipolar coordinate system, which leads to a closed form solution for the polarizability.

Keywords toroidal coordinate system, double sphere, polarizability, electrostatics, half-cylinder

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Tiivistelmä

Tässä väitöskirjassa analysoidaan joitakin pallosymmetrisiä sähköstatiikan perusrakenteita käyttäen separoituvia koordinaatistoja. Pääasiassa tarkastelu keskittyy dielektriseen kappaleeseen vakiosähkökentässä. Tässä asetelmassa polarisoituvuuden käsite, joka sisältää kappaleen sirontaan liittyvän informaation polarisoituvuustensorin muodossa osoittautuu oleelliseksi. Yksinkertaisille rakenteille (esim. ympyrä tai ellipsoidi) polarisoituvuustensorille on olemassa suljetun muodon lauseke, mutta monimutkaisimmille geometrioille, joille ei ole separoituvaa koordinaatistoa, joudutaan yleensä turvautumaan numeerisiin menetelmiin.

Tämä väitöskirja keskittyy erityisesti palloihin, jotka leikkaavat toisiaan. Koordinaatisto, jossa kolmedimensioiset toisiaan leikkaavat pallot ovat vakiokoordinaattipintoja, on toroidaalikoordinaatisto (R-separoituva), jossa potentiaali saa elegantin muodon Neumannin sarjana tietyn hyvin käyttäytyvän integraaliyhtälön ratkaisuna, jonka voi implementoida tehokkaasti esimerkiksi Java-sovelmana. Erikoistapaus toroidaalikoordinaatistosta on tangenttipallokoordinaatisto, joka esittää palloja, jotka koskettavat vain yhdessä pisteessä. Tässä tapauksessa sirontaongelma redusoituu tavalliseksi toisen asteen lineaariseksi differentiaaliyhtälöksi, jonka kerroinfunktiot ovat alkeisfunktioita. Kaksidimensioinen kaksoispallo-ongelmaa (puolikiekko) lähestytään bipolaarisesta koordinaatistosta käsin, mikä johtaa suljetun muodon lausekkeeseen polarisoituvuudelle.

Asiasanat toroidi koordinaatisto, kaksoispallo, polarisoituvuus, sähköstatiikka, puolisylinteri

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Preface

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Ari Sihvola: Supervisor and instructor of this thesis. He read and commented all the manuscripts of this work.

Henrik Wallén: Provided help with LaTeX and other computer related problems.

Katriina Nykänen: Helped with bureaucracy.

Reviewers: A number of reviewers improved the quality of my journal articles.

Freeware: Several open-source softwares were used in the process of composing this thesis:

- Debian Linux as an operating system.
- For typesetting and graphics LaTeX with packages concrete (text font), eulervm (math font), amsmath, amsfonts, graphicx, pstricks, hyperref, breakurl, calc.
- Gedit and Emacs as text editors.
- Java and related tools for building an easy access to the results of this research over the Internet.

Mathematica: The most important scientific tool in this work was Wolfram Mathematica, which was used for symbolic and numerical calculations as well as producing several visualizations. The cover was made with a modified version of http://demonstrations.wolfram.com/RandomCirclesWithPowerLawSizes/

pdf: http://www.htm2pdf.co.uk/ for HTML to pdf conversion and Adobe Acrobat Pro 8.0 for handling pdf files.

Contents

List of Publications

I M. Pitkonen: *Polarizability of the dielectric double-sphere*, J. Math. Phys. 47, 102901 (2006).

II M. Pitkonen: *An explicit solution for the electric potential of the asymmetric dielectric double sphere*, J. Phys. D: Appl. Phys. vol. 40, 1483-1488, 2007.

III M. Pitkonen: *Polarizability of a pair of touching dielectric spheres*, J. Appl. Phys. 103, 104910 (2008).

IV M. Pitkonen: *A closed-form solution for the polarizability of a dielectric double half-cylinder*, J. of Electromagnetic Waves and Appl. (accepted for publication)

The motivation for calculating the polarizability of a dielectric double sphere (publication **I**) came from Prof. Ari Sihvola. The idea of using the toroidal coordinate system was from myself, inspired by references [13] and [14]. Then the natural generalization of the paper **I** led me to the article **II**. I wrote the article **III** to show that the previously proposed exact solution by Radchik *et al.* [21] was incorrect. Finally, the motivation for the last article **IV** was provided by Ari Sihvola, and I found a method that led to a closed-form solution.

1 Basic Theoretical Background

The physical world is represented as a four-dimensional continuum. If in this I adopt a Riemannian metric, and look for the simplest laws which such a metric can satisfy, I arrive at the relativistic gravitation theory of empty space. If I adopt in this space a vector field, or the antisymmetrical tensor field derived from it, and if I look for the simplest laws which such a field can satisfy, I arrive at the Maxwell equations for free space.

ALBERT EINSTEIN [Essays in Science, 1934]

What we are dealing with in this work, is electrostatics. To appreciate the very special nature of that, we start from the general setting, namely, the classical electrodynamics (or electromagnetics) and work our way down in a deductive manner.

1.1 Electrodynamics

Electrodynamics is a theory about interaction of electric charges. Charges interact through the electromagnetic force field $\boldsymbol{\Phi}$. Knowing the charge distribution and its movement (electric current), the force field is determined by the coordinate-free Maxwell's equations (using differential forms)

$$\begin{aligned} \mathbf{d}\boldsymbol{\Phi} &= 0 \\ \mathbf{d}\boldsymbol{\Psi} &= \boldsymbol{\gamma}, \end{aligned} \tag{1}$$

where $\boldsymbol{\gamma}$ is the source field (or form). Stating a local linear dependence between $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ is equivalent to choosing a metric for the space-time. This means that the equivalence of matter and geometry is inherent in the differential form setting. Adopting the Minkowskian metric

$$\mathbf{d}s^2 = -\mathbf{d}t^2 + \mathbf{d}x^2 + \mathbf{d}y^2 + \mathbf{d}z^2,$$

leads to electrodynamics in a flat empty space. In fact, this metric gives rise to the canonical operator $\star$ (Hodge star) related to the metric as

$$\begin{aligned} \star(\mathbf{d}t \wedge \mathbf{d}x) &= -\mathbf{d}y \wedge \mathbf{d}z \\ \star(\mathbf{d}t \wedge \mathbf{d}y) &= \mathbf{d}x \wedge \mathbf{d}z \\ \star(\mathbf{d}t \wedge \mathbf{d}z) &= -\mathbf{d}x \wedge \mathbf{d}y \\ \star(\mathbf{d}x \wedge \mathbf{d}y) &= \mathbf{d}t \wedge \mathbf{d}z \\ \star(\mathbf{d}x \wedge \mathbf{d}z) &= -\mathbf{d}t \wedge \mathbf{d}y \\ \star(\mathbf{d}y \wedge \mathbf{d}z) &= \mathbf{d}t \wedge \mathbf{d}x, \end{aligned}$$

where $\wedge$ denotes the wedge product. By choosing $\boldsymbol{\Psi} = \star\boldsymbol{\Phi}$ the theory is complete and, in particular, it turns out that waves propagate at the speed of 1 (the speed of light with this metric).

To obtain the more familiar set of equations that we usually consider as the Maxwell's equations, we have to choose a special time coordinate t (i.e. an inertial frame) and perform the 3D-decompositions with respect to it as

$$\begin{aligned}\boldsymbol{\Phi} &= \mathbf{B} - \mathbf{d}t \wedge \mathbf{E} \\ \boldsymbol{\Psi} &= \mathbf{D} + \mathbf{d}t \wedge \mathbf{H} \\ \boldsymbol{\gamma} &= \boldsymbol{\rho} + \mathbf{d}t \wedge \mathbf{J}\end{aligned}$$

(see [1]). This decomposition leads to the flat-space non-covariant Maxwell's equations

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= \rho\end{aligned} \tag{2}$$

with the material equations (provided by $\boldsymbol{\Psi} = \star\boldsymbol{\Phi}$) $\mathbf{D} = \mathbf{E}$ and $\mathbf{B} = \mathbf{H}$ in empty space. However, usually the metric of the space-time is chosen such that the material equations are $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$, therefore, we shall also adopt that convention. For more about geometry and electromagnetics, see the axiomatic approach in [2] or the "bible" of general relativity [3].

1.2 Electrostatics

Electrostatics is by definition a situation in which fields do not depend on time. For Maxwell's equations, this means that all the time derivatives vanish, i.e.

$$\begin{aligned}\nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{D} &= \rho\end{aligned}$$

and since in this study we are only dealing with electric responses (no currents and magnetic fields), only equations

$$\begin{aligned}\nabla \times \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho\end{aligned} \tag{3}$$

together with the empty space condition $\mathbf{D} = \varepsilon_0 \mathbf{E}$ remain. Next, we put some matter into the empty space in order to introduce the notion of electromagnetic material, which is at the very center in scattering theory.

2 Electrostatic Scattering Problems

In this work, we consider situations that involve a dielectric body exposed to a constant electric field $\mathbf{E}_0$ (incident field). This field creates a dipole moment density $\mathbf{P}$, which creates a scattered field $\mathbf{E}_s$. The total field (the one that obeys Maxwell's equations) is the sum of these $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_0$. The reason why this kind of a decomposition of the total field $\mathbf{E}$ is often necessary lies under the fact that we have to solve differential equations with some boundary conditions, and for the total field one boundary condition would be that when we are far away from the dielectric body, the total field $\mathbf{E}$ approaches the incident field $\mathbf{E}_0$ asymptotically. Consequently, the information about the incident field encodes itself onto the boundary condition, as in the equation (7).

2.1 Dielectric Scatterers

Let us consider a dielectric body. What this means is that when the body is exposed to an incident electric field $\mathbf{E}_0$ a dipole moment density $\mathbf{P}$ is induced in the material. These dipoles are the source the electric potential

$$\phi_p(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \mathbf{P}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \, dV,$$

where the integration goes over the volume V of the body. We assume that $\mathbf{P}$ varies smoothly with respect to position inside the body, and vanishes outside. A little bit of vector analysis (see e.g. Pollack & Stump [4]) gives the result

$$\phi_p(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \oint_{\partial V} \frac{\mathbf{n}' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dS - \frac{1}{4\pi\varepsilon_0} \int_V \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, dV,$$

where the vector $\mathbf{n}'$ is a unit normal at point $\mathbf{r}'$ and ∂V is the boundary surface of the volume V (see Fig. 1). From this we see that there are two type of charges (internal charges) induced in the body, the surface charge

$$\rho_{ps}(\mathbf{r}) = \mathbf{n} \cdot \mathbf{P}(\mathbf{r}) \, \delta_{\partial V}(\mathbf{r}),$$

where $\delta_{\partial V}$ is the Dirac delta symbol for the surface ∂V that can be thought as the function

$$\delta_{\partial V}(\mathbf{r}) = \begin{cases} \infty, & \mathbf{r} \in \partial V, \\ 0, & \mathbf{r} \notin \partial V, \end{cases}$$

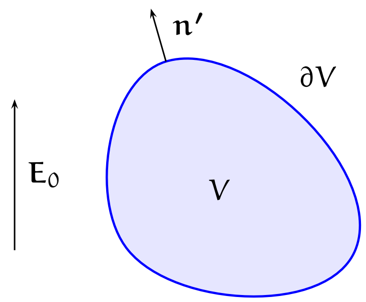

Fig. 1: Electrostatic scattering problem.

producing the surface integral over ∂V when integrated over the whole space, and the volume charge

$$\rho_p(\mathbf{r}) = \begin{cases} -\nabla \cdot \mathbf{P}(\mathbf{r}), & \mathbf{r} \in V, \\ 0, & \mathbf{r} \notin V. \end{cases}$$

Next, we get rid of the charges ρ_p and ρ_{ps} by considering the Gauss's law

$$\nabla \cdot \varepsilon_0 \mathbf{E}(\mathbf{r}) = \rho_p(\mathbf{r}) + \rho_{ps}(\mathbf{r}),$$

which, by defining a new quantity $\mathbf{D}$ (flux density), takes the form

$$\nabla \cdot \underbrace{\left(\varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})\right)}_{\mathbf{D}(\mathbf{r})} = 0.$$

Notice that the surface charge emerges due to the discontinuity in the dipole moment density $\mathbf{P}(\mathbf{r})$ when $\mathbf{r}$ lies on the surface of the body.

In this work, the dielectric body is assumed to be linear, isotropic and homogeneous, therefore, the relation between the electric field and the induced dipole moment can be written as $\mathbf{P}(\mathbf{r}) = \varepsilon_0 \chi \mathbf{E}(\mathbf{r})$ for some dimensionless constant χ (electric susceptibility) and, consequently, $\rho_p = 0$ and

$$\mathbf{D} = \varepsilon_0 \underbrace{(1+\chi)}_{\varepsilon} \mathbf{E},$$

where ε is the so-called relative permittivity. If the body were inhomogeneous, the volume charge would be $\rho_p(\mathbf{r}) = \nabla\chi(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$.

There is a discontinuity in the permittivity $\varepsilon(\mathbf{r})$ when $\mathbf{r}$ is on the boundary of the body, hence, the local Maxwell's equations (3) are not defined, therefore, we need to impose extra information to compensate for that. In particular, because we are considering homogeneous dielectric bodies, which have a discontinuity at the boundary of the body, we need some boundary (or interface) condition to know how the fields behave across that discontinuity. We of course could approximate the discontinuity by some smooth function and take the limit at the end, but a more elegant way is to generalize Maxwell's equations so that such discontinuities are no problem. This we shall do by using Gauss's and Stokes theorem to transform the equations (3) to integral form (assuming no external charges)

$$\oint_{\partial S} \mathbf{E} \cdot d\mathbf{r} = 0 \tag{4}$$

$$\oint_{\partial V} \mathbf{D} \cdot d\mathbf{S} = 0 \tag{5}$$

where S is a surface with boundary ∂S, and V is a volume with boundary ∂V. Then assume that these will hold also when the fields are not necessarily smooth.

Now, from (4) it follows that the tangential component of the electric field is continuous across the boundary, and (5) implies that the normal component of the flux density is continuous across the boundary.

2.2 Polarizability

The scattered field $\mathbf{E}_s$ contains an excessive amount of information. However, it can be expanded as a multipole series where the lowest order term, the so-called dipole moment

$$\mathbf{p} = \int_V \mathbf{P}(\mathbf{r}) dV,$$

contains for most purposes enough information because the higher order multipoles are negligible sufficiently far away from the scatterer. To separate from the dipole moment the effect of the incident field, we introduce the concept of polarizability, which is defined by

$$\mathbf{p} = \overline{\overline{\alpha}}_0 \cdot \mathbf{E}_0, \tag{6}$$

where the polarizability tensor (or dyadic) is $\overline{\overline{\alpha}}_0$. The polarizability defined as in (6) depends on the shape and size of the body. To get rid of the size dependency, we normalize the polarizability by dividing it by the volume of the body and empty space permittivity ε_0. This so-called normalized polarizability (dimensionless)

$$\overline{\overline{\alpha}} = \frac{\overline{\overline{\alpha}}_0}{V \varepsilon_0}$$

is what we shall use in the following.

It can be shown that the polarizability tensor is symmetric, which means that in some coordinate system it is diagonal. For example, if the dielectric body is axially symmetric (as it is in cases considered in this work), it can be written in dyadic notation as

$$\overline{\overline{\alpha}} = \alpha_l \mathbf{u}_l \mathbf{u}_l + \alpha_t (\overline{\overline{I}} - \mathbf{u}_l \mathbf{u}_l),$$

where α_l stands for longitudinal polarizability (along the symmetry axis), and α_t for transversal polarizability.

The notion of polarizability is an important one. For example, when building composite materials it is most often sufficient to know the polarizability tensor of the building blocks (see e.g. [5]). In these composite materials, some extraordinary properties may emerge — properties that were not present in the building blocks. This leads to the concept of metamaterial, which has attracted a lot of interest in recent electromagnetics research community (see e.g. the Journal METAMATERIALS by Elsevier[1]).

2.3 Electrostatic Resonances

There are two ways to define an *electrostatic resonance* (also the terms *surface plasmon*, *Fröhlich resonance*, *Mossotti Catastrophe* are being used):

1. There are solutions in the absence of incident field.

2. There polarizability is infinite.

The case 1 is more general than the case 2. This can be seen from the Fredholm alternative [6], which states that when a homogeneous equation has no solutions (besides the trivial solution 0), the corresponding inhomogeneous equation has a solution, and when the homogeneous equation has solutions (case 1), the corresponding inhomogeneous equation either has an infinite number of solutions or has no solution (case 2). This applies at least when the scatterer has a smooth boundary. More in-depth treatment of these electrostatic resonances can be found in [7] or [8], although, in these articles no mention about the necessary assumption of a smooth boundary has been made (this is necessary because otherwise the boundary integral operator is not bounded, let alone compact). In fact, for bodies having sharp edges, the resonance spectrum (type-1) is continuous as we shall see in the article **IV**.

[1] http://www.sciencedirect.com/science/journal/18731988

For example, the normalized polarizability of a dielectric sphere having relative permittivity ε is

$$3\frac{\varepsilon-1}{\varepsilon+2},$$

which is infinite when $\varepsilon = -2$, so that it is the only resonance according to definition 2. However, there are type-1 resonances for permittivity values $\varepsilon = 1 + 1/n$ for integers $n \geqslant 1$. These other resonance modes are the ones that cannot be excited by a constant incident field. For those, the solution exists but is nonunique.

In addition, one can show (see [7]) that these resonances can take place only for the negative permittivity values. In addition, for smooth bodies, these resonances are discrete and depend only on the shape of the body (not size). The difference of these definitions gets more interesting when considering bodies having sharp edges. For instance, in the paper **III** when using the definition 2, we get no longitudinal resonances, but when using the definition 1, we get a continuous spectrum of resonant solutions.

The concept of electrostatic resonance is slightly misleading: how can something resonate if nothing is changing with time? It is apparent that in physical reality these electrostatic resonances cannot occur for static fields (that would lead to infinite energy for the scattered field), instead, they can occur in the quasistatic setting, where the field has a wavelength much larger than the scatterer. However, there are often significant losses when the permittivity is negative, which is a consequence of the Kramers–Kronig relations, so that the amplitude of the resonance is finite.

3 Laplace Equation

A good method to solve electrostatic scattering problems is to use a potential function ψ which is defined by

$$\mathbf{E} = -\nabla\psi.$$

The existence of the (global) potential function is guaranteed by the Faraday's law $\nabla \times \mathbf{E} = 0$ and the simple topological structure of the underlying space (euclidean). Using the potential, the equation $\nabla \times \mathbf{E}$ is fullfilled automatically, the Gauss's law states that $\nabla^2\psi = 0$ (Laplace equation) inside the body, and outside the body (not on the boundary). The dielectric boundary conditions (the continuity of the potential, and discontinuity of the normal derivative of the potential) take the form

$$\psi_1 = \psi_2$$
$$\mathbf{n}\cdot\nabla\psi_1 = \varepsilon\mathbf{n}\cdot\nabla\psi_2$$

where ψ_1 is the total potential outside the body, and ψ_2 the total potential inside the body. The scattered potentials, which we shall denote by ϕ_1 and ϕ_2, also satisfy the Laplace equation (because the sources of the incident field are assumed to be outside the body) and the boundary conditions for those are

$$\begin{aligned} \phi_1 &= \phi_2 \\ n \cdot \nabla(\phi_1 - \varepsilon\phi_2) &= (\varepsilon - 1) n \cdot \nabla\phi_0, \end{aligned} \tag{7}$$

where ϕ_0 is the potential of the incident field E_0 and the unit vector n is the boundary normal. (See Fig. 2)

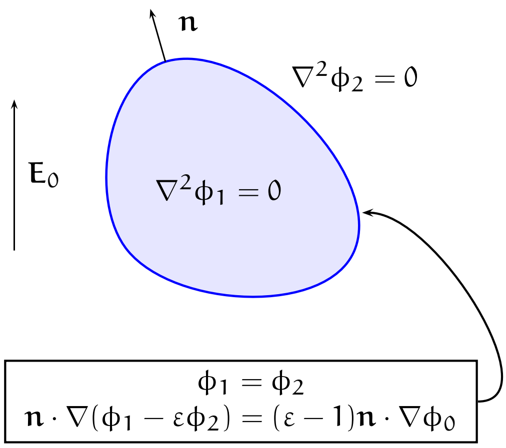

Fig. 2: Electrostatic scattering problem.

3.1 Separable Coordinate Systems

A simple separable coordinate system for Laplace equation is a coordinate system (x_1, x_2, x_3) in which the trial solution

$$\phi(x_1, x_2, x_3) = f(x_1)g(x_2)h(x_3), \tag{8}$$

separates the Laplace's equation into three ordinary differential equations for each function f, g and h. There is also a notion of R-separable (or conformally separable) coordinate systems (see [9]), which means that we relax the condition (8) by allowing the use of a conformal transformation when trying to separate the Laplace equation. In others words for some function R the trial solution

$$\phi(x_1, x_2, x_3) = R(x_1, x_2, x_3) f(x_1) g(x_2) h(x_3)$$

separates the Laplace equation into three ordinary differential equations for the functions f, g and h.

According to Morse and Feschbach [9] there are 11 simply separable coordinate systems and two additional R-separable coordinates (bispherical and toroidal). However, Moon and Spencer [11] claim that there are over 60 such R-separable coordinate systems.

In this work, the toroidal coordinate system is of special interest, so let us consider that in detail.

3.1.1. Toroidal Coordinate System

The toroidal coordinates (u, v, φ) are defined by formulae

$$x = \frac{\sinh u \, \cos\varphi}{\cosh u - \cos v}, \qquad y = \frac{\sinh u \, \sin\varphi}{\cosh u - \cos v}, \qquad z = \frac{\sin v}{\cosh u - \cos v},$$

where (x, y, z) are Cartesian coordinates. The toroidal coordinates (u, v, ϕ) have ranges $0 \leqslant u < \infty$, $-\pi \leqslant v < \pi$ and $-\pi \leqslant \varphi < \pi$, which span the whole space (see Fig. 3). The constant v surfaces are the boundaries of two intersecting spheres. The special cases include an ordinary sphere $v = \pi/2$ and the tangent sphere $v \to 0$. The constant u surfaces are tori, and if r is the radius of the tube and R is the radius of the torus (the distance from the center of the tube to the center of the torus), then we have the relation $\cosh(u) = R/r$.

Laplace equation in the toroidal coordinates reads as

$$\begin{aligned}\nabla^2\phi(u, v, \varphi) = \big(\cos(v) - \cosh(u)\big)\Big(&\sin(v)\phi^{(0,1,0)}(u, v, \varphi) \\ &+ \big(\cos(v) - \cosh(u)\big)\phi^{(2,0,0)}(u, v, \varphi) \\ &+ \big(\cos(v) - \cosh(u)\big)\big(\phi^{(0,2,0)}(u, v, \varphi) + \operatorname{csch}^2(u)\phi^{(0,0,2)}(u, v, \varphi)\big) \\ &+ \phi^{(1,0,0)}(u, v, \varphi)\big(\coth(u)\cos(v) - \operatorname{csch}(u)\big)\Big) = 0.\end{aligned}$$

Let us assume that the potential is independent of the coordinate φ and plug in the trial solution

$$\phi(u, v) = R(u, v) f(u) g(v)$$

and see what happens. The result after division by $f(u)g(v)(\cosh u - \cos v)$

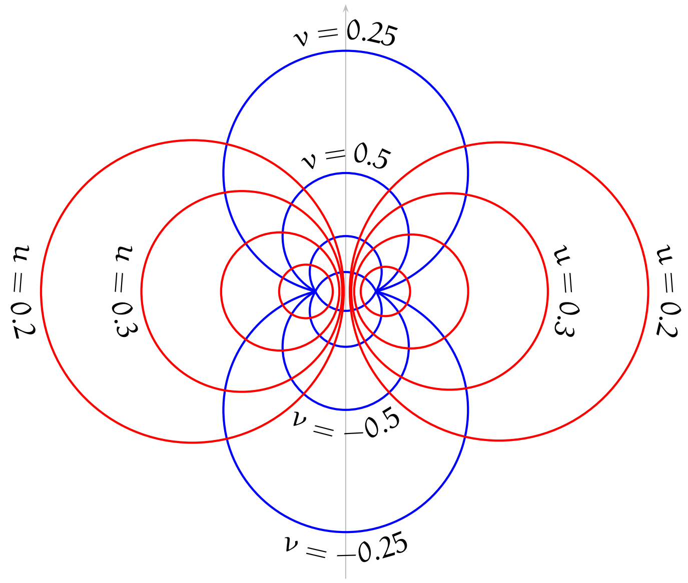

Fig. 3: Constant u (red color) and constant v (blue color) surfaces in the toroidal coordinate system. The rotational symmetry axis is in gray color.

and some rearranging of terms reads as

$$\begin{aligned}
&\frac{f'(u)}{f(u)}\Big(2R^{(1,0)}(u,v)\big(\cosh(u)-\cos(v)\big)+R(u,v)\big(\operatorname{csch}(u)-\coth(u)\cos(v)\big)\Big)\\
&+\frac{f''(u)}{f(u)}R(u,v)\big(\cosh(u)-\cos(v)\big)\\
&+\frac{g'(v)}{g(v)}\Big(2R^{(0,1)}(u,v)\big(\cosh(u)-\cos(v)\big)-\sin(v)R(u,v)\Big)\\
&+\frac{g''(v)}{g(v)}\Big(2R^{(0,1)}(u,v)R(u,v)\big(\cosh(u)-\cos(v)\big)\Big)\\
&-\sin(v)R^{(0,1)}(u,v)+R^{(0,2)}(u,v)\big(\cosh(u)-\cos(v)\big)\\
&+R^{(2,0)}(u,v)\big(\cosh(u)-\cos(v)\big)\\
&+R^{(1,0)}(u,v)\big(\operatorname{csch}(u)-\coth(u)\cos(v)\big)=0.
\end{aligned}$$

Next, we make an intelligent guess and choose the function R such that the coefficient of $g'(v)/g(v)$ vanishes. To accomplish that, R has to satisfy the following first order linear partial differential equation

$$2R^{(0,1)}(u,v)\big(\cosh(u)-\cos(v)\big)-\sin(v)R(u,v),$$

which is easy to solve (for example with the DSolve command of MATHEMATICA system [24]) giving solutions

$$R(u,v) = c(u)\sqrt{\cosh u - \cos v},$$

where c is an arbitrary function. Using this gives the result

$$\begin{aligned}
&\frac{f'(u)}{f(u)}\left(2c'(u) + c(u)\coth(u)\right)(\cosh(u) - \cos(v))^{3/2}\\
&+\frac{f''(u)}{f(u)}c(u)(\cosh(u) - \cos(v))^{3/2}\\
&+\frac{g''(v)}{g(v)}c(u)(\cosh(u) - \cos(v))^{3/2}\\
&+\frac{1}{4}\left(4\left(c''(u) + \coth(u)c'(u)\right) + c(u)\right)(\cosh(u) - \cos(v))^{3/2} = 0.
\end{aligned}$$

Now, choose $c(u) = 1$ and divide by $(\cosh u - \cos v)^{3/2}$ to get

$$\frac{f'(u)}{f(u)}\coth(u) + \frac{f''(u)}{f(u)} + \frac{1}{4} = -\frac{g''(v)}{g(v)}.$$

The left hand does not depend on v and right hand side does not depend on u, hence, we can say that both of them must be constant with respect to u and v. Let that constant be λ^2 (a positive real number). Now we have two separate equations

$$\begin{aligned}
&g''(v) = -\lambda^2\, g(v)\\
&f'(u)\coth(u) + f''(u) = (\lambda^2 - 1/4)\, f(u)
\end{aligned}$$

The solutions of these equations are

$$\begin{aligned}
g(v) &= a\sin(\lambda v) + b\cos(\lambda v)\\
f(u) &= c\, P_{-1/2+\lambda}(\cosh u) + d\, Q_{-1/2+\lambda}(\cosh u)
\end{aligned}$$

for arbitrary constants a, b, c and d, where P and Q are the Legendre functions of the first and second kind. In particular, the spectrum of solutions is continuous. The solutions for f are known as toroidal functions. These can be used for toroidal geometry. If we choose $-\lambda^2$ instead of λ^2, we get an another set of solutions

$$\begin{aligned}
g(v) &= a\sinh(\lambda v) + b\cosh(\lambda v)\\
f(u) &= c\, P_{-1/2+i\lambda}(\cosh u) + d\, Q_{-1/2+i\lambda}(\cosh u),
\end{aligned}$$

where i denotes the imaginary unit. The solutions for f are in this case called conical functions. These solutions we shall use for the intersecting double sphere geometry (articles **I** and **II**).

There are also systematic methods for determining whether a coordinate system is separable, and when it is, determining the right factor R, which we managed to derive with this kind of a brute force and a bit of luck method (*ad hoc* for those who like Latin words).

4 The Quality of a Solution

In this section, we describe the various notions of solutions for a problem, which, in this work, is a differential equation (with or without some boundary conditions), and a solution is a function that satisfies that equation. The following properties are by no means standard ones, usually defined in a very vague manner, but we shall adopt these definitions in order to make the terminology well defined.

- An exact solution is a solution of the problem. A finite process that leads to an exact solution is called an algorithm.
- An approximate solution is a solution which is in some sense close to the exact solution. This is of course relative to some metric d, which measures how close things are to each others. A process that creates approximative solutions u_n to an exact solution u, meaning that $d(u, u_n) \to 0$ when $n \to \infty$, is called a computational method.
- An implicit solution is a solution that involves equations. For example, the unit sphere is defined implicitly as the points (x, y) for which $x^2 + y^2 = 1$.
- An explicit solution is one that involves no equations. For example, the unit sphere is defined explicitly as the points $(\cos t, \sin t)$ where t goes from 0 to 2π.
- An elementary solution is a solution that involves only elementary functions. A function is elementary if it can be expressed as an arbitrary finite combination (in terms of arithmetic operations and function compositions) of complex numbers, exponents, logarithms and algebraic (root of a polynomial) functions. See [10] for more a rigorous definition.
- There is no generally accepted definition of a closed-form solution. We shall define a closed-form solution as an explicit solution that involves a finite combinations of elementary functions and their integrals (antiderivatives). For instance, the expression $\int \exp(-x^2)\,dx$ is closed-form, but not elementary. This is the so-called Liouvillian solution in the field of differential Galois theory.

- A solution by quadratures is a solution that has a finite number of integrals in it. More precisely, it is a finite arithmetic combination of functions that belong to the image of some integral operator having an elementary kernel. For example the Riemann zeta function[2] can be written in terms of the qudrature

$$\zeta(s) = \frac{2^{s-1}}{1-2^{1-s}} \int_0^\infty \frac{\cos(s\tan^{-1}(t))}{(t^2+1)^{s/2}\cosh(\pi t/2)} dt.$$

5 Java Applets

Java is an object oriented programming language developed by Sun Microsystems in the early 1990. It was developed as platform independent language to be used in web applications. In particular, the so-called Java Applets are programs that can be run inside the web browser in a secure manner (applets have a very restricted access to the users computer). Nowadays, the scope of the Java has expanded to include over 3.8 billion[3] systems involving Java in some way.

The platform independency was in the early days of Java a major cause of a bad performance. Since then, the Java virtual machine (VM), which interpretes the Java byte-code, has developed enormously, and the performance of Java[4] is nowadays similar to the compiled languages such as Fortran and C.

The polarizability applets constructed in this work, can be found in the web page (see Fig. 4)

http://www.tkk.fi/Yksikot/Sahkomagnetiikka/kurssit/animaatiot/dipolapplet.

6 Summary of the Publications

The first paper **I** is about a polarizability of a symmetric intersecting dielectric double sphere. The conducting case $\varepsilon = \infty$ has been solved by quadratures in 1949 by Schiffer and Szegö (see [12]) and reinvented later in 2000 by Felderhof and Palaniappan [13]. Also the special case of orthogonal intersection has been treated with image theory in [15]. Radchik *et al.* [17] tried to solve the dielectric case, but, as shown by Felderhof and

[2] If you can prove that this has zeros in the right half-plane of the complex plane only on a line $s = 1/2$, you have solved the famous Riemann Hypothesis, which is one of the seven Clay Mathematics Institute Millennium Prize Problems each awarded with one million dollar prize.

[3] http://gcn.com/articles/2006/11/13/sun-opens-java.aspx

[4] http://en.wikipedia.org/wiki/Java_performance

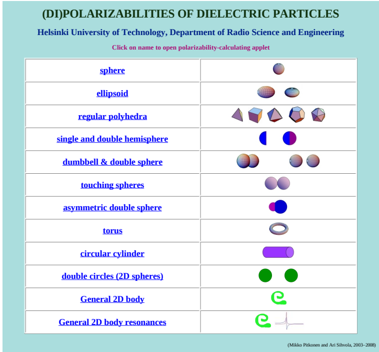

Fig. 4: The Applet page.

Palaniappan [13], their solution was incorrect. The paper **I** solves the potential (and polarizability) in a toroidal coordinate system as a Neumann series, which at the time I thought to be very special and a result of the use of toroidal coordinate system, but the work of Alexander G. Ramm [16] showed that there is a Neumann series solution for arbitrary shaped bodies. Nevertheless, the toroidal coordinate system gives a very smooth kernel for the integral operator, hence, removing the difficulties concerned with the sharp edges (at the intersection) of the scatterer. In addition, the applet

`http://www.tkk.fi/Yksikot/Sahkomagnetiikka/kurssit/animaatiot/mpitkone/Kaksoispallo/Kaksoispallo.html`

that calculates the polarizability of the double sphere for a user given permittivity value based on this method was built

The second paper **II** is a generalization of the work **I** to allow the spheres to have different sizes and different permittivities. Due to the apparent asymmetry, there was an additional difficulty concerning the vanishing of the total charge, which was automatically satisfied in the symmetric case. In this general setting too, the solution was described as a Neumann series. One could also use a spherical coordinate system to tackle this geometry. This approach was taken by Kettunen *et al.* in [18] in which an infinite series solution having the coefficients of which being a solution of an infinite linear system is obtained. There is also an approximate methods available: for instance, the one by Poladian ([19] and [20]) using multipole expansions leading to asymptotic approximations for large values of the permittivity ε. Also, the applet

`http://www.tkk.fi/Yksikot/Sahkomagnetiikka/kurssit/animaatiot/mpitkone/doubleSphere/DoubleSphere.html`

solving the polarizability of this body was built.

The third paper **III** is about the asymptotic limit of the double sphere geometry (as in **I**) when the spheres touch each other at a single point. The coordinate system satisfying this geometry is the so-called tangent sphere frame, which is a R-separable coordinate system and is usually considered as a special case of the toroidal coordinate system. Once again, the Radchik and others [21] had introduced a solution to the problem, and the purpose of this article **III** was to show that it was incorrect. Again, an applet was built:

http://www.tkk.fi/Yksikot/Sahkomagnetiikka/kurssit/animaatiot/mpitkone/TangentSphere/TangentSphere.html.

The tangent sphere frame allowed to go further than with in the case of **I**, and it turned out that the problem reduced to an ordinary differential equation having elementary coefficients; whether or not it has an explicit solution of some kind, remains an open problem.[5]

The last paper **IV** is actually the only closed-form solution of the ones considered in this thesis. It is about a two-dimensional dielectric hemisphere (or half-disk) and, in particular, its polarizability. The coordinate system of this geometry is a two-dimensional projection of the toroidal coordinate system. But now, there is a crucial difference: the Laplace equation is simply separable! This leads to the solution by quadratures for the potential and a solution for the polarizability in terms of dilogarithmic functions, which have a closed-form expression

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}\,dt.$$

Also the peculiar properties of solutions of scatterer having sharp edges is now seen in a new light when the nonuniqueness of the solutions in the certain range of negative permittivity values is proven. The heavy calculations involved were performed with MATHEMATICA [24]. (No applets were built.)

After having found the solution, I found also the articles of Alù and Engheta ([22] and [23]) that treat the same subject by very sophisticated methods, but there is no sight of any explicit expressions for potentials nor polarizabilities.

[5] which I proposed to the problem section of the Electronic Journal of Differential Equations http://math.uc.edu/ode/odesols/p2008.htm

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