Publication X

Esa Ollila and Visa Koivunen. 2004. Generalized complex elliptical distributions. In: Proceedings of the 3rd IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM 2004). Sitges, Spain. 18-21 July 2004, pages 460-464.

GENERALIZED COMPLEX ELLIPTICAL DISTRIBUTIONS

Esa Ollila and Visa Koivunen

SMARAD CoE, Signal Processing Laboratory,
Helsinki University of Technology,
P.O.Box 3000, FIN-02015 HUT, Finland

ABSTRACT

We introduce a new class of distributions called generalized complex elliptically symmetric distributions. Several distributions commonly used in the literature, for example, the multivariate complex normal and Cauchy and the generalized complex normal distribution, are prominent members of this class. The treatment covers both *proper* and *improper* random vectors and goes beyond second-order concepts in defining the distribution model. Some properties of these distributions are studied and illustrative examples of their applications in multichannel signal processing are presented such as tests for circularity.

1. INTRODUCTION

In many signal processing applications, e.g. in spectral analysis, communications and sensor array processing, the multivariate data is conveniently modeled as being complex. The complex valued representation is compact and simpler in notations and for algebraic manipulations. It is convenient for calculations by computer and has intuitive representation in problems with complex data. Consequently there is a need for complex multivariate probability models.

The most widely used probability model for a complex random vectors (r.v.'s) is the complex normal (CN) distribution. The probability density function (p.d.f) of CN distribution takes the form familiar from the real case. Consequently, many of the properties of real normal r.v.'s have a direct analogue in the complex case. This is achieved by imposing an additional restriction on the correlation structure (circularity) of the complex normal random vector. Dropping this restriction yields the generalized complex normal (GCN) distribution introduced in [1]. In this paper, we introduce a class of distributions called generalized complex elliptically symmetric distributions (GCES) which include the GCN distribution and complex elliptically symmetric (CES) distributions [2] as special cases.

2. PRELIMINARIES

A complex matrix $C = A + jB \in \mathbb{C}^{k\times k}$, where $A, B \in \mathbb{R}^{k\times k}$ and $j = \sqrt{-1}$ is the imaginary unit, is termed *symmetric* if $C^T = C$ and *hermitian* if $C^H = C$. Superscripts *, T and H denote conjugate, transpose and conjugate transpose, respectively. We denote by $\mathrm{PDS}(k)$ the set of all positive definite symmetric $k \times k$ real matrices, $\mathrm{PDH}(k)$ the set of all complex positive definite hermitian $k \times k$ matrices, $\mathrm{CS}(k)$ the set of all complex symmetric $k \times k$ matrices, and I the identity matrix.

A complex random variable (r.va.) $Z = X + jY$ is comprised of pair of real r.va.'s X and Y, and its distribution is defined as the distribution of the composite real r.v. $\mathbf{V} = (X, Y)^T$:

$$F(z) = P(Z \le z) \stackrel{\text{def}}{=} P(X \le x, Y \le y) = F(\mathbf{v}),$$

where $\mathbf{v} = (x, y)^T$ and $z = x + jy$. Thus, $F(z)$ is simply a different algebraic form of $F(\mathbf{v})$. Similarly, a complex r.v. $\mathbf{Z} = \mathbf{X} + j\mathbf{Y}$ of $\mathbb{C}^k$ is a pair of real r.v.'s $\mathbf{X}$ and $\mathbf{Y}$ of $\mathbb{R}^k$, and its distribution is identified with the distribution of the composite real r.v. $\mathbf{V} = (\mathbf{X}^T, \mathbf{Y}^T)^T$ of $\mathbb{R}^{2k}$. The p.d.f. $f(\mathbf{z})$ and the characteristic function (c.f.) $\Phi(\mathbf{z})$ of $\mathbf{Z}$ are defined as $f(\mathbf{z}) = f(\mathbf{v})$ and

$$\Phi(\mathbf{z}) = \Phi(\mathbf{v}) = E[\exp\{j\mathbf{v}^T\mathbf{V}\}] = E[\exp\{j\mathrm{Re}(\mathbf{z}^H\mathbf{Z})\}]$$

where $f(\mathbf{v})$ and $\Phi(\mathbf{v})$ are the p.d.f. and c.f. of $\mathbf{V}$, respectively, and $\mathbf{v} = (\mathbf{x}^T, \mathbf{y}^T)^T$ and $\mathbf{z} = \mathbf{x} + j\mathbf{y}$. Re and Im stands for real and imaginary parts, respectively.

The *mean* of $\mathbf{Z}$ is defined as $E(\mathbf{Z}) = E(\mathbf{X}) + jE(\mathbf{Y})$ and the complex *covariance* between two complex r.v.'s $\mathbf{Z}_1$ and $\mathbf{Z}_2$ as

$$\mathrm{cov}(\mathbf{Z}_1, \mathbf{Z}_2) \stackrel{\text{def}}{=} E[\{\mathbf{Z}_1 - E(\mathbf{Z}_1)\}\{\mathbf{Z}_2 - E(\mathbf{Z}_2)\}^H]$$

and the complex *pseudo-covariance* [3] as

$$\mathrm{pcov}(\mathbf{Z}_1, \mathbf{Z}_2) \stackrel{\text{def}}{=} E[\{\mathbf{Z}_1 - E(\mathbf{Z}_1)\}\{\mathbf{Z}_2 - E(\mathbf{Z}_2)\}^T].$$

Then we define $\mathcal{C} = \mathrm{cov}(\mathbf{Z}) \stackrel{\text{def}}{=} \mathrm{cov}(\mathbf{Z}, \mathbf{Z})$ and $\mathcal{R} = \mathrm{pcov}(\mathbf{Z}) \stackrel{\text{def}}{=} \mathrm{pcov}(\mathbf{Z}, \mathbf{Z})$, and call them, respectively, the *covariance matrix* and the *pseudo-covariance matrix* of a r.v. $\mathbf{Z}$. Pseudo-covariance matrix is termed as *relation matrix* in [4] and *complementary covariance matrix* in [5].

Partition $\mathbf{Z} = \mathbf{X} + j\mathbf{Y} \in \mathbb{C}^k$ as $\mathbf{Z} = (\mathbf{Z}_1^H, \mathbf{Z}_2^H)^H$ with $\mathbf{Z}_1 = \mathbf{X}_1 + j\mathbf{Y}_1 \in \mathbb{C}^p$ and $\mathbf{Z}_2 = \mathbf{X}_2 + j\mathbf{Y}_2 \in \mathbb{C}^q$ $(p + q = k)$, so $\mathbf{X} = (\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ and $\mathbf{Y} = (\mathbf{Y}_1^T, \mathbf{Y}_2^T)^T$. The conditional distribution of $\mathbf{Z}_1$ given $\mathbf{Z}_2$, $P(\mathbf{Z}_1|\mathbf{Z}_2)$, is simply defined as the distribution $P(\mathbf{X}_1, \mathbf{Y}_1|\mathbf{X}_2, \mathbf{Y}_2)$ and the *conditional mean* is defined as $E(\mathbf{Z}_1|\mathbf{Z}_2) = E(\mathbf{X}_1|\mathbf{X}_2, \mathbf{Y}_2) + jE(\mathbf{Y}_1|\mathbf{X}_2, \mathbf{Y}_2)$.

3. GENERALIZED COMPLEX ELLIPTICAL DISTRIBUTIONS

Definition 1 *A r.v.* $\mathbf{Z} = \mathbf{X} + j\mathbf{Y} \in \mathbb{C}^k$ *is said to have a* generalized complex elliptically symmetric (GCES) *distribution iff* $\mathbf{V} = (\mathbf{X}^T, \mathbf{Y}^T)^T \in \mathbb{R}^{2k}$ *has a real elliptically symmetric (RES) distribution.*

The authors would like to thank Academy of Finland CoE program for financial support that made this research possible

Throughout the paper we write $m = 2k$. As the r.v. $\mathbf{V}$ in Definition 1 has a m-variate RES distribution, its p.d.f. is [6]:

$$f(\mathbf{v}) = c_{m,g} \det(\Gamma)^{-1/2} g\{(\mathbf{v}-\boldsymbol{\delta})^T \Gamma^{-1} (\mathbf{v}-\boldsymbol{\delta})\} \qquad (1)$$

where $g : [0,\infty) \to [0,\infty)$ is a fixed function, called the *density generator*, independent of the parameters $\boldsymbol{\delta} \in \mathbb{R}^m$ and $\Gamma \in \mathrm{PDS}(m)$. The functional form of g uniquely distinguishes different RES distributions from another. The normalizing constant $c_{m,g}$ equals $(s_m \mu_{m-1,g})^{-1}$, where s_m is the area of unit sphere surface in $\mathbb{R}^m$,

$$s_m = \frac{2\pi^{m/2}}{\Gamma(\frac{m}{2})} \quad \text{and} \quad \mu_{m,g} = \int_0^\infty t^m g(t^2) dt.$$

Naturally, $c_{m,g}$ could be absorbed into the function g, but with this notation g can be independent of m. We note that any non-negative function g can be a density generator iff $\mu_{m-1,g} < \infty$. We will use $\mathbf{V} \sim \mathrm{E}_m(\boldsymbol{\delta}, \Gamma, g)$ to denote that $\mathbf{V}$ has a m-variate RES distribution with parameter values $\boldsymbol{\delta}$ and Γ and density generator g. Note also that, if desired, RES distributions can be defined more generally without the existence of a density function. See [6] for an excellent review of RES distributions.

We decompose $\boldsymbol{\delta}$ and Γ according to $\mathbf{V} = (\mathbf{X}^T, \mathbf{Y}^T)^T$, so

$$\boldsymbol{\delta} = \begin{pmatrix} \boldsymbol{\delta}_{\mathrm{x}} \\ \boldsymbol{\delta}_{\mathrm{y}} \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} \Gamma_{\mathrm{xx}} & \Gamma_{\mathrm{xy}} \\ \Gamma_{\mathrm{yx}} & \Gamma_{\mathrm{yy}} \end{pmatrix},$$

where $\boldsymbol{\delta}_{\mathrm{x}} \in \mathbb{R}^k$ and $\Gamma_{\mathrm{xx}}, \Gamma_{\mathrm{yy}} \in \mathbb{R}^{k\times k}$. From these partitions, we build a vector $\boldsymbol{\mu}$ of $\mathbb{C}^k$ and matrices Σ and Ω of $\mathbb{C}^{k\times k}$ as follows

$$\boldsymbol{\mu} = \boldsymbol{\delta}_{\mathrm{x}} + j\boldsymbol{\delta}_{\mathrm{y}}, \qquad (2)$$

$$\Sigma = \Gamma_{\mathrm{xx}} + \Gamma_{\mathrm{yy}} + j(\Gamma_{\mathrm{yx}} - \Gamma_{\mathrm{xy}}), \qquad (3)$$

$$\Omega = \Gamma_{\mathrm{xx}} - \Gamma_{\mathrm{yy}} + j(\Gamma_{\mathrm{yx}} + \Gamma_{\mathrm{xy}}). \qquad (4)$$

Note that $k \times k$ complex matrices Σ and Ω carry all the information about $m \times m$ real matrix Γ since

$$\Gamma_{\mathrm{xx}} = \tfrac{1}{2}\mathrm{Re}(\Sigma + \Omega), \quad \Gamma_{\mathrm{xy}} = \tfrac{1}{2}\mathrm{Im}(-\Sigma + \Omega),$$
$$\Gamma_{\mathrm{yy}} = \tfrac{1}{2}\mathrm{Re}(\Sigma - \Omega), \quad \Gamma_{\mathrm{yx}} = \tfrac{1}{2}\mathrm{Im}(\Sigma + \Omega).$$

We call the parameters $\boldsymbol{\mu}$, Σ and Ω, the *location vector*, the *scatter matrix* and the *pseudo-scatter matrix*, respectively. Using the fact that $\Gamma \in \mathrm{PDS}(m)$ and simple matrix algebra, one finds that

$$\Sigma \in \mathrm{PDH}(k) \quad \text{and} \quad \Omega \in \mathrm{CS}(k).$$

We now construct the p.d.f. $f(\mathbf{z})$ of $\mathbf{Z}$ by expressing p.d.f. $f(\mathbf{v})$ of $\mathbf{V}$ as a function of $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ and the introduced complex valued parameters $\boldsymbol{\mu}$, Σ and Ω. The following construction is similar to [1, 4]. Write

$$\tilde{\mathbf{z}} = \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^* \end{pmatrix}, \; \tilde{\boldsymbol{\mu}} = \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}^* \end{pmatrix} \quad \text{and} \quad \tilde{\Gamma} = \begin{pmatrix} \Sigma & \Omega \\ \Omega^* & \Sigma^* \end{pmatrix}, \qquad (5)$$

where $\mathbf{z} = \mathbf{x} + j\mathbf{y} \in \mathbb{C}^k$. Then it is easy to verify that

$$(\mathbf{v} - \boldsymbol{\delta}) = M(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) \quad \text{and} \quad \Gamma = M\tilde{\Gamma}M^H,$$

where $\mathbf{v} = (\mathbf{x}^T, \mathbf{y}^T)^T$ and

$$M = \frac{1}{2}\begin{pmatrix} I & I \\ -jI & jI \end{pmatrix} \quad \text{and} \quad M^{-1} = \begin{pmatrix} I & jI \\ I & -jI \end{pmatrix} \qquad (6)$$

are $m \times m$ complex matrices. First note that $\tilde{\Gamma}$ is hermitian. From $\tilde{\Gamma} = M^{-1}\Gamma M^{-H}$ we can infer that the eigenvalues of $\tilde{\Gamma}$ are twice the eigenvalues of $\Gamma \in \mathrm{PDS}(m)$ and thus positive. Thereby, $\tilde{\Gamma} \in \mathrm{PDH}(m)$. With these notations, the quadratic form in (1) may be expressed as

$$(\mathbf{v}-\boldsymbol{\delta})^T \Gamma^{-1} (\mathbf{v}-\boldsymbol{\delta}) = (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \tilde{\Gamma}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}),$$

and the determinant in (1) becomes

$$\det(\Gamma) = \det(M)\det(\tilde{\Gamma})\det(M^H) = 2^{-m}\det(\tilde{\Gamma})$$

due to rule of determinant of product of matrices and the result that $\det(M^H) = 2^{-k} j^k (-1)^k$ and $\det(M) = 2^{-k} j^k$ which readily follows using (6). Combining these results, the p.d.f. (1) can be written in terms of $\mathbf{Z}$ and its parameters as follows

$$f(\mathbf{z}) = c_{m,g} 2^k \det(\tilde{\Gamma})^{-1/2} g\{(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \tilde{\Gamma}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})\}. \qquad (7)$$

We will use $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, \Omega, g)$ to denote that $\mathbf{Z}$ has a GCES distribution with density generator g and parameter values $\boldsymbol{\mu} \in \mathbb{C}^k$, $\Sigma \in \mathrm{PDH}(k)$ and $\Omega \in \mathrm{CS}(k)$.

Since $\mathbf{V} \sim \mathrm{E}_m(\boldsymbol{\delta}, \Gamma, g)$, its c.f. is of the form [6]:

$$\Phi(\mathbf{v}) = \exp(j\mathbf{v}^T\boldsymbol{\delta})\phi(\mathbf{v}^T\Gamma\mathbf{v}),$$

where ϕ is some fixed function of a scalar variable, called *characteristic generator*. Note that $\mathbf{v}^T\boldsymbol{\delta} = \mathrm{Re}(\mathbf{z}^H\boldsymbol{\mu})$ and using $\mathbf{v} = M\tilde{\mathbf{z}}$, $\Gamma = M\tilde{\Gamma}M^H$, $M^H M = (1/2)I$ we get

$$\mathbf{v}^T\Gamma\mathbf{v} = \tfrac{1}{4}\tilde{\mathbf{z}}^H\tilde{\Gamma}\tilde{\mathbf{z}} = \tfrac{1}{2}\{\mathbf{z}^H\Sigma\mathbf{z} + \mathrm{Re}(\mathbf{z}^H\Omega\mathbf{z}^*)\},$$

where the latter equality follows using the partitions (5) and simple matrix algebra. We may now write the c.f. as follows

$$\Phi(\mathbf{z}) = \exp\{j\mathrm{Re}(\mathbf{z}^H\boldsymbol{\mu})\}\phi\big(\tfrac{1}{2}\{\mathbf{z}^H\Sigma\mathbf{z} + \mathrm{Re}[\mathbf{z}^H\Omega\mathbf{z}^*]\}\big). \qquad (8)$$

Thus unlike the p.d.f (7), the c.f. (8) has a rather simple form in terms of $\mathbf{z} = \mathbf{x} + j\mathbf{y}$ and the parameters $\boldsymbol{\mu}$, Σ and Ω.

Next we derive an alternative expression for the p.d.f (7). Using the well-known result for the inverse of a partitioned matrix and adopting the notation from [4], we get

$$\tilde{\Gamma}^{-1} = \begin{pmatrix} P^{-*} & -R^H P^{-1} \\ -P^{-1}R & P^{-1} \end{pmatrix}, \qquad (9)$$

where P and R are defined as

$$P = \Sigma^* - \Omega^H\Sigma^{-1}\Omega, \qquad (10)$$

$$R = \Omega^H\Sigma^{-1}. \qquad (11)$$

Notation P^{-*} means $(P^{-1})^*$. $\tilde{\Gamma} \in \mathrm{PDH}(m)$ implies that $\tilde{\Gamma}^{-1}$ and P^{-1} exist and are hermitian and positive definite matrices as well. Furthermore,

$$\det(\tilde{\Gamma}) = \det(\Sigma)\det(P) \qquad (12)$$

$$= \det(P)^2\det(I - R^H R^T)^{-1}. \qquad (13)$$

Equation (12) follows using the result for the determinant of a partitioned matrix. From (10) we can solve Σ as a function of P and R, yielding $\Sigma = P^*(I - R^H R^T)^{-1}$. When this expression for Σ is plugged in (12) and using that $\det(P^*) = \det(P)$ as

$P \in \mathrm{PDH}(k)$, we obtain (13). The quadratic form in p.d.f (7), denoted by $d(\mathbf{z})$, can also be expressed through $\mathbf{z}$ and $\boldsymbol{\mu}$, P and R:

$$\begin{aligned} d(\mathbf{z}) &= (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \tilde{\Gamma}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) \\ &= 2\mathbf{z}_0^H P^{-*} \mathbf{z}_0 - 2\mathrm{Re}\{\mathbf{z}_0^T P^{-1} R \mathbf{z}_0\} \\ &= 2\mathrm{Re}\{\mathbf{z}_0^H P^{-*} (\mathbf{z}_0 - R^* \mathbf{z}_0^*)\} \\ &= 2\mathbf{z}_0^H \Sigma^{-1} \mathbf{z}_0 + 2\mathrm{Re}\{(R\mathbf{z}_0)^H P^{-1} (R\mathbf{z}_0 - \mathbf{z}_0^*)\}. \end{aligned}$$

where $\mathbf{z}_0 = \mathbf{z} - \boldsymbol{\mu}$. The second identity above follows using partition (9) and simple matrix algebra, and third identity follows using $\mathbf{z}_0^H P^{-*} \mathbf{z}_0 = \mathrm{Re}(\mathbf{z}_0^H P^{-*} \mathbf{z}_0)$ (since P^{-*} is positive definite) and the result that $\mathrm{Re}(a^H) = \mathrm{Re}(a^*) = \mathrm{Re}(a)$ for all $a \in \mathbb{C}$. The last identity then results using that $P^{-*} = \Sigma^{-1} + R^H P^{-1} R$ which follows using the well-known matrix inversion lemma, see e.g. [7]. Which of the above expressions for $d(\mathbf{z})$ is the simplest is a matter of taste. Combining these results, the p.d.f. (7) can be written solely using parameters $\boldsymbol{\mu}$, P and R as follows

$$f(\mathbf{z}) = c_{m,g} 2^k \det(P)^{-1} \det(I - R^H R^T)^{1/2} g(d(\mathbf{z})). \quad (14)$$

As (7) and (14) demonstrate, we can parametrize the GCES distribution either with $(\boldsymbol{\mu}, \Sigma, \Omega)$ or with $(\boldsymbol{\mu}, P, R)$. The choice of parametrization is not relevant in case of Maximum Likelihood (ML-)theory since ML-estimates (MLE's) and the likelihood function are invariant under the one-to-one parameter transformation such as (10) and (11). This means, for example, that if (Σ_n, Ω_n) denote MLE's of (Σ, Ω), then MLE's of P and R are obtained by plugging in Σ_n and Ω_n in place of Σ and Ω in (10) and (11), respectively.

A third possible parametrization of GCES distributions is given next. Since $\Sigma \in \mathrm{PDH}(k)$ and $\Omega \in \mathrm{CS}(k)$, there exist a nonsingular A of $C^{k \times k}$ such that (Corollary 4.6.12(b) in [7]):

$$\Sigma = AA^H \quad \text{and} \quad \Omega = A \Lambda A^T, \quad (15)$$

where Λ is a real diagonal $k \times k$ matrix with non-negative diagonal entries. This parametrization was proposed in [8] for the covariance matrix and the pseudo-covariance matrix of a complex normal random vector. Thus, if we write the p.d.f. (7) using parameters A and Λ we obtain, after some calculations, that

$$\det(\tilde{\Gamma}) = \det(AA^H)^2 \det(I - \Lambda^2)$$

and

$$\begin{aligned} d(\mathbf{z}) &= 2\mathbf{u}^H (I - \Lambda^2)^{-1} \mathbf{u} - 2\mathrm{Re}[\mathbf{u}^H \Lambda (I - \Lambda^2)^{-1} \mathbf{u}^*] \\ &= 2\|\mathbf{u}\|^2 + 2\mathrm{Re}\{\mathbf{u}^H \Lambda (I - \Lambda^2)^{-1} (\Lambda \mathbf{u} - \mathbf{u}^*)\}, \end{aligned}$$

where $\mathbf{u} = A^{-1}(\mathbf{z} - \boldsymbol{\mu})$ and $\|\mathbf{u}\|^2 = \mathbf{u}^H \mathbf{u}$. So, the p.d.f. (7) can be written solely using parameters A and Λ as follows

$$f(\mathbf{z}) = c_{m,g} 2^k \det(AA^H)^{-1} \det(I - \Lambda^2)^{-1/2} g(d(\mathbf{z})). \quad (16)$$

With this parametrization we use the notation $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, A, \Lambda, g)$ instead of $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, \Omega, g)$.

Next we show that affine transformation $B\mathbf{Z} + \mathbf{b}$ with $\mathbf{b} \in \mathbb{C}^k$ and nonsingular $B \in \mathbb{C}^{k \times k}$, induces parameter transformation $\boldsymbol{\mu} \mapsto B\boldsymbol{\mu} + \mathbf{b}$, $\Sigma \mapsto B \Sigma B^H$ and $\Omega \mapsto B \Omega B^T$, whereas the functional form of the p.d.f remains.

Theorem 1 *Let* $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, \Omega, g)$. *Then* $B\mathbf{Z} + \mathbf{b} \sim \mathrm{CE}_k(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B^H, B\Omega B^T, g)$ *for all* $\mathbf{b} \in \mathbb{C}^k$ *and nonsingular* $B \in \mathbb{C}^{k \times k}$.

Proof. Denote the c.f. (8) of $\mathbf{Z}$ by $\Phi_{\mathrm{z}}(\mathbf{z})$ and the c.f. of $\mathbf{W} = B\mathbf{Z} + \mathbf{b}$ by $\Phi_{\mathrm{w}}(\mathbf{z})$. Then

$$\begin{aligned} \Phi_{\mathrm{w}}(\mathbf{z}) &= E\big[\exp\{j\mathrm{Re}(\mathbf{z}^H \mathbf{W})\}\big] \\ &= E\big[\exp\{j\mathrm{Re}(\mathbf{z}^H B\mathbf{Z})\}\big] \exp\{j\mathrm{Re}(\mathbf{z}^H \mathbf{b})\} \\ &= \Phi_{\mathrm{z}}(B^H \mathbf{z}) \exp\{j\mathrm{Re}(\mathbf{z}^H \mathbf{b})\} \\ &= \exp\{j\mathrm{Re}[\mathbf{z}^H (B\boldsymbol{\mu} + \mathbf{b})]\} \\ &\quad \cdot \phi\big(\tfrac{1}{2}\{\mathbf{z}^H (B\Sigma B^H)\mathbf{z} + \mathrm{Re}[\mathbf{z}^H (B\Omega B^T)\mathbf{z}^*]\}\big). \end{aligned}$$

This is the c.f. of a r.v. whose p.d.f is (7) with $\boldsymbol{\mu}$, Σ and Ω replaced by $B\boldsymbol{\mu} + \mathbf{b}$, $B\Sigma B^H$ and $B\Omega B^T$, respectively. □

Using parametrization (A, Λ) instead of (Σ, Ω), Theorem 1 states that if $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, A, \Lambda, g)$, then $B\mathbf{Z} + \mathbf{b} \sim \mathrm{CE}_k(B\boldsymbol{\mu} + \mathbf{b}, BA, \Lambda, g)$ for all $\mathbf{b} \in \mathbb{C}^k$ and nonsingular $B \in \mathbb{C}^{k \times k}$. Thus, parameter Λ is invariant under affine transformations.

Partition r.v. $\mathbf{Z} = (\mathbf{Z}_1^H, \mathbf{Z}_2^H)^H \in \mathbb{C}^k$ with $\mathbf{Z}_1 \in \mathbb{C}^p$ and $\mathbf{Z}_2 \in \mathbb{C}^q$ $(p + q = k)$ and partition the parameters conformably:

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \ \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \ \Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}. \quad (17)$$

Note that $\Sigma_{11} \in \mathrm{PDH}(p)$ and $\Sigma_{22} \in \mathrm{PDH}(q)$ since $\Sigma \in \mathrm{PDH}(k)$, and that $\Omega_{11} \in \mathrm{CS}(p)$ and $\Omega_{22} \in \mathrm{CS}(q)$ since $\Omega \in \mathrm{CS}(k)$. Next result states that all the marginal distributions of $\mathbf{Z}$ are also complex elliptical and so is the conditional distribution $\mathbf{Z}_1 | \mathbf{Z}_2$.

Theorem 2 *Let* $\mathbf{Z} = (\mathbf{Z}_1^H, \mathbf{Z}_2^H)^H \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, \Omega, g)$. *Then* $\mathbf{Z}_1 \sim \mathrm{CE}(\boldsymbol{\mu}_1, \Sigma_{11}, \Omega_{11}, g)$. *Furthermore,* $\mathbf{Z}_1 | \mathbf{Z}_2$ *has p-variate GCES distribution.*

Proof. The c.f. of $\mathbf{Z}_1$ is

$$\begin{aligned} \Phi_{\mathrm{z}_1}(\mathbf{z}_1) &= E\big[\exp\{j\mathrm{Re}(\mathbf{z}_1^H \mathbf{Z}_1)\}\big] = \Phi_{\mathrm{z}}\big((\mathbf{z}_1^H, \mathbf{0}^H)^H\big) \\ &= \exp\{j\mathrm{Re}(\mathbf{z}_1^H \boldsymbol{\mu}_1)\} \phi\big(\tfrac{1}{2}\{\mathbf{z}_1^H \Sigma_{11} \mathbf{z}_1 + \mathrm{Re}[\mathbf{z}_1^H \Omega_{11} \mathbf{z}_1^*]\}\big). \end{aligned}$$

This is the c.f. of a $p \times 1$ r.v. whose p.d.f. is (7) with $\boldsymbol{\mu}$, Σ and Ω replaced by $\boldsymbol{\mu}_1$, Σ_{11} and Ω_{11}, respectively.

As $P(\mathbf{Z}_1 | \mathbf{Z}_2)$ is the distribution $P(\mathbf{X}_1, \mathbf{Y}_1 | \mathbf{X}_2, \mathbf{Y}_2)$ which is known to have RES distribution due to result of r.v.'s with RES distribution and their conditional distributions (Theorem 2.18 in [6]). Then, by definition, $\mathbf{Z}_1 | \mathbf{Z}_2$ has GCES distribution. □

Finally, the next Theorem gives the interpretation for the parameters $\boldsymbol{\mu}$, Σ and Ω.

Theorem 3 *Let* $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, \Omega, g)$ *and write* $m = 2k$. *If* $\mu_{m+1,g} < \infty$ *then* $E(\mathbf{Z}) = \boldsymbol{\mu}$, $\mathcal{C} = c \cdot \Sigma$ *and* $\mathcal{R} = c \cdot \Omega$, *where* c *is positive real valued scalar,* $c = E(R^2)/m$, *and* R *is non-negative real random variable with p.d.f*

$$h(r) = s_m r^{m-1} g(r^2).$$

Proof. The mean and covariance matrix of $\mathbf{V} = (\mathbf{X}^T, \mathbf{Y}^T)^T \sim \mathrm{E}_m(\boldsymbol{\delta}, \Gamma, g)$ are (Theorem 2.17 of [6]) $E(\mathbf{V}) = (\boldsymbol{\delta}_{\mathrm{x}}^T, \boldsymbol{\delta}_{\mathrm{y}}^T)^T$ and

$$\mathrm{cov}(\mathbf{V}) = \begin{pmatrix} \mathrm{cov}(\mathbf{X}) & \mathrm{cov}(\mathbf{X}, \mathbf{Y}) \\ \mathrm{cov}(\mathbf{Y}, \mathbf{X}) & \mathrm{cov}(\mathbf{Y}) \end{pmatrix} = c \cdot \Gamma, \quad (18)$$

where c is positive real valued scalar, $c = E(R^2)/m$, and R is non-negative real r.va. with p.d.f $h(r) = s_m r^{m-1} g(r^2)$. The mean and covariance matrix of $\mathbf{V}$ exists iff $\mathrm{E}(R^2) < \infty$, or equivalently, iff $\mu_{m+1,g} < \infty$. The mean of $\mathbf{Z}$ is thereby $E(\mathbf{Z}) = E(\mathbf{X}) + jE(\mathbf{Y}) = \boldsymbol{\delta}_{\mathrm{x}} + \boldsymbol{\delta}_{\mathrm{y}} = \boldsymbol{\mu}$ and the covariance matrix of $\mathbf{Z}$,

$$\mathcal{C} = \{\mathrm{cov}(\mathbf{X}) + \mathrm{cov}(\mathbf{Y})\} + j\{\mathrm{cov}(\mathbf{Y}, \mathbf{X}) - \mathrm{cov}(\mathbf{X}, \mathbf{Y})\},$$

can be written, due to (18) and (3), as $\mathcal{C} = c \cdot \Sigma$. Similarly, we obtain that $\mathcal{R} = c \cdot \Omega$. □

In other words, Theorem 3 states that if the density generator g satisfies $\mu_{2k+1,g} < \infty$, then the covariance matrix $\mathcal{C}$ and the pseudo-covariance matrix $\mathcal{R}$ exist and the parameters, the scatter matrix Σ and the pseudo-scatter matrix Ω, are proportional to $\mathcal{C}$ and $\mathcal{R}$, respectively.

4. EXAMPLES OF GCES DISTRIBUTIONS

4.1. CES distributions

Definition 2 *If a r.v.* $\mathbf{Z} \in \mathbb{C}^k$ *has a GCES distribution with* $\Omega = 0$*, i.e.* $\mathbf{Z} \sim \mathrm{CE}_k(\boldsymbol{\mu}, \Sigma, 0, g)$*, then* $\mathbf{Z}$ *is said to have a* complex elliptically symmetric (CES) *distribution.*

In fact, the CES distributions of Definition 2, are the class of distributions introduced and studied in [2]. Thus Definition 1 is a generalization of CES distributions which is simply a special case. One may verify that when $\Omega = 0$ (or equivalently $\Lambda = 0$), the c.f. (8) and p.d.f. (7) (or equivalently (16)) take the forms familiar from the real case:

$$\Phi(\mathbf{z}) = \exp\{j\mathrm{Re}(\mathbf{z}^H\boldsymbol{\mu})\}\phi\big(\tfrac{1}{2}\mathbf{z}^H\Sigma\mathbf{z}\big),$$
$$f(\mathbf{z}) = c_{m,g}2^k\det(\Sigma)^{-1}g\{2(\mathbf{z}-\boldsymbol{\mu})^H\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\}.$$

Hence the regions of constant contours of CES distributions are ellipsoids in $\mathbb{C}^k$. Recall that a r.v. $\mathbf{Z}$ is called *circular* if $\mathbf{Z}$ and $\exp(j\theta)\mathbf{Z}$ have the same distribution for any $\theta \in \mathbb{R}$. It is then easy to verify from the c.f. or the p.d.f. above that CES distributions with location vector $\boldsymbol{\mu} = \mathbf{0}$ satisfy "circularity".

From (4) we see that $\Omega = 0$ implies that $\Gamma_{\mathrm{xx}} = \Gamma_{\mathrm{yy}}$ and $\Gamma_{\mathrm{xy}} = -\Gamma_{\mathrm{yx}}$, which means that $\Sigma = 2(\Gamma_{\mathrm{xx}} + j\Gamma_{\mathrm{yx}})$ and that Γ is of a special form

$$\Gamma = \begin{pmatrix}\Gamma_{\mathrm{xx}} & -\Gamma_{\mathrm{yx}} \\ \Gamma_{\mathrm{yx}} & \Gamma_{\mathrm{xx}}\end{pmatrix}.$$

Due to Theorem 3, $\Omega = 0$ also implies that the pseudo-covariance matrix $\mathcal{R} = 0$ (if exists). Thus, the second-order behaviour of $\mathbf{Z}$ with a CES distribution is completely described by the covariance matrix $\mathcal{C} = 2\{\mathrm{cov}(\mathbf{X}) + j\mathrm{cov}(\mathbf{Y}, \mathbf{X})\}$. Note that a r.v. with $\mathcal{R} = 0$ is called *proper* [3, 5]. Thus, their definition is a second-order concept while ours is not: For a r.v. $\mathbf{Z}$ with a CES distribution, the second-order moments may not exist (i.e. if $\mu_{2k+1,g}$ is not integrable). Such an example is the complex Cauchy distribution studied in [9] for which $g(t) = (1+t)^{-(2k+1)/2}$.

4.2. Generalized Complex Normal Distribution

Definition 3 *A r.v.* $\mathbf{Z} = \mathbf{X} + j\mathbf{Y} \in \mathbb{C}^k$ *is said to have a* generalized complex normal (GCN) *distribution iff* $\mathbf{V} = (\mathbf{X}^T, \mathbf{Y}^T)^T \in \mathbb{R}^{2k}$ *has normal distribution.*

Since normal distribution is a member RES distributions it follows that GCN distribution is a member of GCES distributions. Its characteristic and density generator is $\phi(t) = g(t) = \exp(-t/2)$ and we write $\mathbf{Z} \sim \mathbb{C}N_k(\boldsymbol{\mu}, \Sigma, \Omega)$. It is easy to verify that the normalizing constant $c_{m,g}$ equals $(2\pi)^{-k}$ in the normal case. Thus, the c.f. (8) becomes

$$\Phi_{\mathrm{GCN}}(\mathbf{z}) = \exp\left\{j\mathrm{Re}(\mathbf{z}^H\boldsymbol{\mu}) - \tfrac{1}{4}[\mathbf{z}^H\Sigma\mathbf{z} + \mathrm{Re}(\mathbf{z}^H\Omega\mathbf{z}^*)]\right\}$$

and the p.d.f can be expressed, using (7), (14) and (16) corresponding to three different parametrizations respectively, as follows

$$\begin{aligned} f_{\mathrm{CCN}}(\mathbf{z}) &= \pi^{-k}\det(\tilde{\Gamma})^{-1/2}\exp\{-\tfrac{1}{2}(\tilde{\mathbf{z}}-\tilde{\boldsymbol{\mu}})^H\tilde{\Gamma}^{-1}(\tilde{\mathbf{z}}-\tilde{\boldsymbol{\mu}})\} \\ &= \pi^{-k}\det(P)^{-1}\det(I - R^HR^T)^{1/2} \\ &\quad \cdot \exp\{-\mathbf{z}_0^H\Sigma^{-1}\mathbf{z}_0 + \mathrm{Re}[(R\mathbf{z}_0)^HP^{-1}(\mathbf{z}_0^* - R\mathbf{z}_0)]\} \\ &= \pi^{-k}\det(AA^H)^{-1}\det(I-\Lambda^2)^{-1/2} \\ &\quad \cdot \exp\{-\|\mathbf{u}\|^2 + \mathrm{Re}[\mathbf{u}^H(I-\Lambda^2)^{-1}\Lambda(\mathbf{u}^* - \Lambda\mathbf{u})]\}. \end{aligned}$$

The last parametrization, i.e. using (A, Λ), for a complex normal r.v. $\mathbf{Z}$ was advocated in [8]. Since $\mu_{2k+1,g}$ is integrable, the mean, covariance and pseudo-covariance matrix exists and c of Theorem 3 can be shown to be $c = 1$. Thus, in the normal case, we have that $\boldsymbol{\mu} = E(\mathbf{Z})$, $\Sigma = \mathrm{cov}(\mathbf{Z})$ and $\Omega = \mathrm{pcov}(\mathbf{Z})$.

Definition 4 *A r.v.* $\mathbf{Z}$ *is said to have a* complex normal (CN) *distribution iff* $\mathbf{Z}$ *has GCN distribution and* $\Omega = \mathrm{pcov}(\mathbf{Z}) = \mathbf{0}$.

Thus, CN distribution is simply a special case of GCN distribution, and also a member of CES distributions. CN distribution has been widely employed and studied in the literature and due time this special case has become "generally accepted" complex normal distribution. See discussions and references in [1]. CN distribution is also sometimes called circular complex normal [4], or, proper complex normal [3].

The c.f. and p.d.f of CN distribution now take the form familar from the real case:

$$\Phi_{\mathrm{CN}}(\mathbf{z}) = \exp\{j\mathrm{Re}(\mathbf{z}^H\boldsymbol{\mu}) - \tfrac{1}{4}\mathbf{z}^H\Sigma\mathbf{z}\},$$
$$f_{\mathrm{CN}}(\mathbf{z}) = \pi^{-k}\det(\Sigma)^{-1}\exp\{-(\mathbf{z}-\boldsymbol{\mu})^H\Sigma^{-1}(\mathbf{z}-\boldsymbol{\mu})\}.$$

Is is now evident that the c.f. of CN distribution appears in the expression of the c.f. of GCN distribution as follows

$$\Phi_{\mathrm{CGN}}(\mathbf{z}) = \Phi_{\mathrm{CN}}(\mathbf{z})\exp\{-\tfrac{1}{4}\mathrm{Re}(\mathbf{z}^H\Omega\mathbf{z}^*)\}.$$

The p.d.f. of GCN distribution can also be factorized in a product of the p.d.f of CN distribution and a function depending on $\boldsymbol{\mu}$ and matrices (Σ, Ω) through matrices (P, R) or (A, Λ) as follows:

$$\begin{aligned} f_{\mathrm{CGN}}(\mathbf{z}) &= f_{\mathrm{CN}}(\mathbf{z})\det(I - R^HR^T)^{-1/2} \\ &\quad \cdot \exp\{\mathrm{Re}[(R\mathbf{z}_0)^HP^{-1}(\mathbf{z}_0^* - R\mathbf{z}_0)]\} \\ &= f_{\mathrm{CN}}(\mathbf{z})\det(I-\Lambda^2)^{-1/2} \\ &\quad \cdot \exp\{\mathrm{Re}[\mathbf{u}^H(I-\Lambda^2)^{-1}\Lambda(\mathbf{u}^* - \Lambda\mathbf{u})]\}. \end{aligned}$$

See also [4].

5. CONDITIONAL MEAN ESTIMATOR

Let $\mathbf{Z} = \mathbf{X} + j\mathbf{Y} \in \mathbb{C}^k$ be partitioned to subvectors $\mathbf{Z}_1 = \mathbf{X}_1 + j\mathbf{Y}_1 \in \mathbb{C}^p$ and $\mathbf{Z}_2 = \mathbf{X}_2 + j\mathbf{Y}_2 \in \mathbb{C}^q$ $(p+q=k)$ as in Theorem 2. We assume that $\mathbf{Z}$ has a GCES distribution with $\boldsymbol{\mu} = \mathbf{0}$ and let the parameters Σ and Ω be partitioned as in (17). Assume that the density generator g of the model is such that $\mu_{2k+1,g} < \infty$. This guarantees by Theorem 3 that the covariance and the pseudo-covariance matrix of $\mathbf{Z}$ exist. Now recall that the conditional mean is $E(\mathbf{Z}_1|\mathbf{Z}_2) = E(\mathbf{X}_1|\mathbf{X}_2, \mathbf{Y}_2) + jE(\mathbf{Y}_1|\mathbf{X}_2, \mathbf{Y}_2)$.

Due to Theorem 2.18 of [6], $E(\mathbf{X}_1|\mathbf{X}_2, \mathbf{Y}_2) = L_1\mathbf{X}_2 + L_2\mathbf{Y}_2$ and $E(\mathbf{Y}_1|\mathbf{X}_2, \mathbf{Y}_2) = L_3\mathbf{X}_2 + L_4\mathbf{Y}_2$, where the real $p \times q$ matrices L_1, L_2, L_3 and L_4 are known but rather complicated functions of the submatrices of the parameter Γ of $\mathbf{V}$ (or equivalently of submatrices of parameters Σ and Ω of $\mathbf{Z}$). It then follows that conditional mean can be expressed explicitly through $\mathbf{Z}_2$ and $\mathbf{Z}_2^*$ as follows

$$E(\mathbf{Z}_1|\mathbf{Z}_2) = G\mathbf{Z}_2 + H\mathbf{Z}_2^*,$$

with $p \times q$ complex matrices $G = \frac{1}{2}(L_1 + L_4) + j\frac{1}{2}(L_3 - L_2)$ and $H = \frac{1}{2}(L_1 - L_4) + j\frac{1}{2}(L_2 + L_3)$. Thereby, conditional mean is *widely linear* [4, 5] in $\mathbf{Z}_2$. Then note that error between $\mathbf{Z}_1$ and its conditional mean, i.e. $\mathbf{Z}_1 - E(\mathbf{Z}_1|\mathbf{Z}_2)$, is uncorrelated with $\mathbf{Z}_2$ and $\mathbf{Z}_2^*$. This result together with the result that Σ and Ω are proportional to $\mathcal{C}$ and $\mathcal{R}$ (Theorem 3) yield the pair of equations

$$G\Sigma_{22} + H\Omega_{22}^* = \Sigma_{12}, \quad G\Omega_{22} + H\Sigma_{22}^* = \Omega_{12},$$

from which G and H can be solved as functions of complex matrices Σ_{ij} and Ω_{ij}, $i, j = 1, 2$.

6. TESTS FOR CIRCULARITY

Circularity is commonly assumed e.g. in many communications and array processing problems. Thus, one may want to validate this assumption.

6.1. A Likelihood Ratio (LR-)test

Assume that $\mathbf{z}$ is a realization from $\mathbf{Z} \sim \mathbb{C}N_k(\mathbf{0}, \Sigma, \Omega)$, and consider the following binary hypothesis:

$$H_0 : \Omega = 0 \qquad H_1 : \Omega = \Omega_0. \tag{19}$$

In other words, we wish test the null hypothesis H_0 which states that $\mathbf{Z}$ is a zero mean r.v. from CN distribution with *known* covariance Σ against the alternative hypothesis H_1 which states $\mathbf{Z}$ is a zero mean r.v. from GCN distribution with known covariance matrix Σ and known pseudo-covariance matrix $\Omega = \Omega_0$. The *likelihood ratio* (LR) $l(\mathbf{z}) = f_{\text{GCN}}(\mathbf{z})/f_{\text{CN}}(\mathbf{z})$ gives the optimal decision rule. Using the p.d.f's given in Section 4.2, LR becomes

$$\begin{aligned} l(\mathbf{z}) &= \det(I - R^H R^T)^{-1/2} \exp\{\text{Re}[(R\mathbf{z})^H P^{-1}(\mathbf{z}^* - R\mathbf{z})]\} \\ &= \det(I - \Lambda^2)^{-1/2} \exp\{\text{Re}[\mathbf{u}^H (I - \Lambda^2)^{-1}\Lambda(\mathbf{u}^* - \Lambda\mathbf{u})]\}, \end{aligned}$$

where $\mathbf{u} = A^{-1}\mathbf{z}$.

6.2. A Generalized Likelihood Ratio (GLR-)test

Let $\mathbf{z}_1, \ldots, \mathbf{z}_n$ be a random sample from $\mathbb{C}N_k(\mathbf{0}, \Sigma, \Omega)$ and write $Z_n = (\mathbf{z}_1 \cdots \mathbf{z}_n)$. Again we wish to test the hypothesis (19) of circularity. The likelihood function for Z_n under H_0 is

$$f_0(Z_n; \Sigma) = \pi^{-kn} \det(\Sigma)^{-n} \exp\{-n\text{Tr}(\Sigma^{-1}\Sigma_n)\},$$

where $\Sigma_n = \frac{1}{n}\sum_{i=1}^n \mathbf{z}_i\mathbf{z}_i^H$ is the *sample covariance matrix* and $\text{Tr}(\cdot)$ stands for trace. The likelihood function under H_1 becomes

$$f_1(Z_n; \tilde{\Gamma}) = \pi^{-kn} \det(\tilde{\Gamma})^{-n/2} \exp\{-\tfrac{n}{2}\text{Tr}(\tilde{\Gamma}^{-1}\tilde{\Gamma}_n)\},$$

where

$$\tilde{\Gamma}_n = \begin{pmatrix} \Sigma_n & \Omega_n \\ \Omega_n^* & \Sigma_n^* \end{pmatrix} = \frac{1}{n}\sum_{i=1}^n \begin{pmatrix} \mathbf{z}_i\mathbf{z}_i^H & \mathbf{z}_i\mathbf{z}_i^T \\ \mathbf{z}_i^*\mathbf{z}_i^H & \mathbf{z}_i^*\mathbf{z}_i^T \end{pmatrix}.$$

In practice, the parameters Σ and Ω are often unknown, and the LR-test can not be employed. Thereby we resort to *Generalized LR (GLR-)test approach* where we maximize $f_0(\cdot)$ and $f_1(\cdot)$ over Σ and $\tilde{\Gamma}$, respectively, and use the resulting (maximized) LR as a decision statistic. It is well known that MLE of Σ under H_0 is the sample covariance matrix Σ_n. Then, using complex matrix differentiation rules, we may differentiate log $f_1(Z_n; \tilde{\Gamma})$ w.r.t the matrix parameter $\tilde{\Gamma}$. Then setting the obtained differential equal to zero and solving for $\tilde{\Gamma}$, we find that the MLE of $\tilde{\Gamma}$ is $\tilde{\Gamma}_n$. Thereby, the GLR-test decision statistic becomes

$$\begin{aligned} \hat{l}(Z_n) &= \frac{f_1(Z_n; \tilde{\Gamma}_n)}{f_0(Z_n; \Sigma_n)} = \frac{\det(\Sigma_n)^n}{\det(\tilde{\Gamma})^{n/2}} = \det(I - R_n^H R_n^T)^{-n/2} \\ &= \det(I - \Lambda_n^2)^{-n/2}, \end{aligned}$$

where $R_n = \Omega_n^H \Sigma_n^{-1}$ and Λ_n are MLE's of parameters R and Λ, respectively. Λ_n is a diagonal matrix with positive diagonal elements which satisfy the factorization (recall (15)) $\Sigma_n = A_n A_n^H$ and $\Omega_n = A_n \Lambda_n A_n^T$, where A_n is non-singular MLE of the parameter A. The sampling and asymptotic distribution of the decision statistic under the null and under sequence of alternative hypothesis will be a subject of a separate paper.

7. CONCLUSIONS

We introduced a new class of distributions called generalized complex elliptically symmetric distributions. Various important distributional characteristics of this class were derived. As examples, we derived the conditional mean and LR-tests for circularity.

8. REFERENCES

[1] A. van den Bos, "The multivariate complex normal distribution - a generalization," *IEEE Trans. Inform. Theory*, vol. 41, no. 2, pp. 537–539, 1995.

[2] P. R. Krishnaiah and J. Lin, "Complex elliptically symmetric distributions," *Comm. Statist. - Th. and Meth.*, vol. 15, pp. 3693–3718, 1986.

[3] F. D Neeser and J. L. Massey, "Proper complex random processed with applications to information theory," *IEEE Trans. Inform. Theory*, vol. 39, no. 4, pp. 1293–1302, 1993.

[4] B. Picinbono, "Second order complex random vectors and normal distributions," *IEEE Trans. Signal Processing*, vol. 44, no. 10, pp. 2637–2640, 1996.

[5] P. J. Schreier and L. L. Scharf, "Second-order analysis of improper complex random vectors and processes," *IEEE Trans. Signal Processing*, vol. 51, no. 3, pp. 714–725, 2003.

[6] K-T Fang, S. Kotz, and K. W. Ng, *Symmetric multivariate and related distributions*, Chapman and hall, London, 1990.

[7] R. A. Horn and C. A. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 1985.

[8] J. Eriksson and V. Koivunen, "Identifiability, separability and uniqueness of complex ICA models," *HUT Signal Processing Laboratory Techical Report no:44*, 2004.

[9] E. Ollila and V. Koivunen, "Robust antenna array processing using M-estimators of pseudo-covariance," in *Proc. IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC'03)*, Beijing, China, Sept. 7–10, 2003, pp. 2659–2663.