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REALIZATION OF GENERALIZED SOFT-AND-HARD BOUNDARY

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Abstract—The classical soft-and-hard surface boundary conditions have previously been generalized to the form $\mathbf{a} \cdot \mathbf{E} = 0$ and $\mathbf{b} \cdot \mathbf{H} = 0$ where $\mathbf{a}$ and $\mathbf{b}$ are two complex vectors tangential to the boundary. A realization for such a boundary is studied in terms of a slab of special wave-guiding anisotropic material. It is shown that analytic expressions can be found for the material parameters and thickness of the slab as functions of the complex vectors $\mathbf{a}$ and $\mathbf{b}$. Application of a generalized soft-and-hard boundary as a polarization transformer is studied in detail.

1. INTRODUCTION

Soft-and-hard surface (SHS) characterizes a certain class of electromagnetic boundary surfaces imposing symmetric conditions for the electric and magnetic fields of the form [1, 2]

$$\mathbf{v} \cdot \mathbf{E} = 0, \quad \mathbf{v} \cdot \mathbf{H} = 0. \tag{1}$$

Here $\mathbf{v}$ denotes a real unit vector tangential to the boundary surface, i.e., it satisfies $\mathbf{n} \cdot \mathbf{v} = 0$ when the unit vector normal to the boundary is denoted by $\mathbf{n}$. Originally, SHS was defined as a useful model approximating boundary structures with tuned metallic corrugations introduced in 1944 [3–5], which have found early applications in antenna design [6]. Later, interest in realizing the SHS boundary by other physical structures has been of interest.

The condition (1) also defines a subclass of ideal boundaries [7] defined by the property that the complex Poynting vector does not have a component normal to the boundary,

$$\mathbf{n} \cdot \mathbf{E} \times \mathbf{H}^* = 0. \tag{2}$$

The most general class of anisotropic ideal boundaries can be expressed in the form [7]

$$\mathbf{a} \cdot \mathbf{E} = 0, \quad \mathbf{a}^* \cdot \mathbf{H} = 0, \tag{3}$$

in terms of a tangential complex unit vector $\mathbf{a}$ satisfying $\mathbf{n} \cdot \mathbf{a} = 0$ and $\mathbf{a} \cdot \mathbf{a}^* = 1$. Because (3) generalizes the SHS condition (1), it appears natural to introduce the even more general condition

$$\mathbf{a} \cdot \mathbf{E} = 0, \quad \mathbf{b} \cdot \mathbf{H} = 0, \tag{4}$$

in terms of two complex vectors $\mathbf{a}, \mathbf{b}$ satisfying $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b} = 0$ and

$$\mathbf{a} \cdot \mathbf{b} = 1. \tag{5}$$

The class of media defined by (4) was labeled as that of generalized soft-and-hard surfaces (GSHS) in [8]. It turns out that media defined by (4) belong to the class of ideal media only in the special case $\mathbf{b} = \mathbf{a}^*$ [7]. In [7] it was demonstrated that the GSHS boundary can be tailored to change any given polarization of an incident plane wave to any other given polarization for the reflected field by choosing the vectors $\mathbf{a}$ and $\mathbf{b}$ properly. This and any other possible application in mind gives motivation for finding a realization of the GSHS boundary.

Since the realization of the basic SHS boundary applies a structure composed of metal corrugations along the real $\mathbf{v}$ direction on the surface, such a structure cannot be generalized to the case when $\mathbf{a}$ and $\mathbf{b}$ are complex vectors. In the present study, another form of realization is studied with a slab of wave-guiding structure introduced in [9] and applied to the general surface impedance dyadic in [10]. Because the GSHS boundary involves zero and infinite impedance parameters, it must be handled through a certain limiting process. The purpose of this study is to find how this can be done in the general theory given in [10] in order to find a structure which satisfies the GSHS conditions (4) at its boundary. The realization is different from that given in [11] where certain alternatives were discussed to obtain a boundary satisfying the condition (3). The structures proposed were, however, designed for (3) to be valid for normally incident plane waves only, which restricts their application. The present realization is valid for arbitrary plane waves or any other fields.

2. VECTOR BASES

The vectors $\mathbf{a}$ and $\mathbf{b}$ do not make a basis in the two-dimensional space of vectors transverse to $\mathbf{n}$. However, due to (5) the vector triples

$\mathbf{n}, \mathbf{a}, (\mathbf{n}\times\mathbf{b})$ and $\mathbf{n}, \mathbf{b}, (\mathbf{n}\times\mathbf{a})$ make two three-dimensional vector bases because we have

$$\mathbf{n}\cdot\mathbf{a}\times(\mathbf{n}\times\mathbf{b}) = \mathbf{n}\cdot\mathbf{b}\times(\mathbf{n}\times\mathbf{a}) = (\mathbf{n}\times\mathbf{a})\cdot(\mathbf{n}\times\mathbf{b}) = \mathbf{a}\cdot\mathbf{b} = 1. \quad (6)$$

The two vector bases $\mathbf{n}, \mathbf{a}, \mathbf{a}' = (\mathbf{n}\times\mathbf{b})$ and $\mathbf{n}, \mathbf{b}, \mathbf{b}' = (\mathbf{n}\times\mathbf{a})$ are not orthogonal in general, because $\mathbf{a}\cdot\mathbf{a}' = -\mathbf{b}\cdot\mathbf{b}' = -\mathbf{n}\cdot\mathbf{a}\times\mathbf{b}$ may take any value. Instead, they are bi-orthogonal because of

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}'\cdot\mathbf{b}' = 1, \quad \mathbf{a}\cdot\mathbf{b}' = \mathbf{b}\cdot\mathbf{a}' = 0. \quad (7)$$

This allows one to expand the unit dyadic as [12, 13, 8]

$$\overline{\overline{\mathsf{I}}} = \mathbf{nn} + \mathbf{ab} + (\mathbf{n}\times\mathbf{b})(\mathbf{n}\times\mathbf{a}) = \mathbf{nn} + \mathbf{ba} + (\mathbf{n}\times\mathbf{a})(\mathbf{n}\times\mathbf{b}). \quad (8)$$

The expansions can be applied when expanding to a given vector $\mathbf{c}$ in either of the vector bases. For example, we can write

$$\mathbf{c} = \overline{\overline{\mathsf{I}}}\cdot\mathbf{c} = \mathbf{n}(\mathbf{n}\cdot\mathbf{c}) + \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) + (\mathbf{n}\times\mathbf{a})(\mathbf{n}\cdot\mathbf{b}\times\mathbf{c}). \quad (9)$$

3. BOUNDARY ADMITTANCE

Impedance-boundary condition is a mathematical restriction to ensure unique solutions for differential equations in a region terminated by the boundary. For electromagnetic fields it takes on the form of a linear relation between the electric and magnetic field components tangential to the boundary surface. Let us consider the form

$$\mathbf{n}\times\mathbf{H} = \overline{\overline{\mathsf{Y}}}_s\cdot\mathbf{E}, \quad (10)$$

in terms of a two-dimensional surface admittance dyadic $\overline{\overline{\mathsf{Y}}}_s$ satisfying

$$\mathbf{n}\cdot\overline{\overline{\mathsf{Y}}}_s = \overline{\overline{\mathsf{Y}}}_s\cdot\mathbf{n} = 0. \quad (11)$$

To approach the GSHS boundary, let us consider the following form of a normalized surface admittance dyadic:

$$j\eta_o\overline{\overline{\mathsf{Y}}}_s = \frac{A}{\delta}\mathbf{ba} + B\delta\mathbf{ab}{}^{\times}_{\times}\mathbf{nn}, \quad (12)$$

where $A \neq 0$ and $B \neq 0$ are two scalar admittance parameters and δ is a dimensionless scalar parameter. The normalizing factor $j\eta_o = j\sqrt{\mu_o/\epsilon_o}$ taken for convenience anticipates imaginary components for

$\overline{\overline{\mathsf{Y}}}_s$ when the boundary is lossless. The double-cross product and double-dot product between two dyads **ab** and **cd** are defined as [12, 13]

$$\mathbf{ab}{}^{\times}_{\times}\mathbf{cd} = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \times \mathbf{d}), \tag{13}$$

and

$$\mathbf{ab} : \mathbf{cd} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}). \tag{14}$$

Inserting (12) in (10) gives us the component equations

$$\mathbf{b} \cdot \mathbf{H} = (\mathbf{n} \times \mathbf{b}) \cdot (\mathbf{n} \times \mathbf{H}) = B\delta(\mathbf{n} \times \mathbf{b} \cdot \mathbf{E})/j\eta_o, \tag{15}$$

$$j\eta_o\delta\mathbf{a} \cdot (\mathbf{n} \times \mathbf{H}) = A(\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{E}) = A(\mathbf{a} \cdot \mathbf{E}). \tag{16}$$

If we now let $\delta \to 0$, from (15) we obtain $\mathbf{b} \cdot \mathbf{H} \to 0$ while from (16) we have $\mathbf{a} \cdot \mathbf{E} \to 0$. This means that, in the limit $\delta \to 0$, the admittance dyadic (12) will correspond to the GSHS boundary with conditions (4) valid for any nonzero values of A and B. Because this limit would imply zero and infinite components in (12) making it rather useless in analysis, it is better to work first with finite δ and take the limit at the end.

4. REALIZATION OF GSHS

In [9] a structure involving a layer of gyrotropic anisotropic material was introduced to realize a special impedance surface called the perfect electromagnetic conductor (PEMC). The material was generalized in [10] for the realization of the general impedance surface and the same structure will be considered here. Let us assume a slab of anisotropic medium defined by permittivity and permeability dyadics of the form

$$\overline{\overline{\epsilon}} = \epsilon_o(\overline{\overline{\epsilon}}_t + \epsilon_n\mathbf{nn}), \quad \overline{\overline{\mu}} = \mu_o(\overline{\overline{\mu}}_t + \mu_n\mathbf{nn}), \tag{17}$$

with $\epsilon_n \to \infty, \mu_n \to \infty$. The thickness of the slab is d and it is terminated by a PEC plane. The medium (17) with infinite normal components can be called wave guiding because a field of any functional dependence in the transverse plane is guided along the **n** direction with the same dependence, which resembles wave propagation along a collection of parallel waveguides. Such a medium can be realized by parallel PEC and PMC wires embedded in an anisotropic host medium and, approximately, by thin well conducting iron cylinders with high conductivity and permeability values [9]. It was shown in [10] that in the medium (17) the wave is split in two eigenwaves propagating along the normal direction with propagation factors β_1, β_2 depending on the medium dyadics as

$$\beta_{1,2}^2 = \frac{k_o^2}{2}\left(\overline{\overline{\epsilon}}_t {}^\times_\times \overline{\overline{\mu}}_t^T : \mathbf{nn} \pm \sqrt{(\overline{\overline{\epsilon}}_t {}^\times_\times \overline{\overline{\mu}}_t^T : \mathbf{nn})^2 - 4\Delta(\overline{\overline{\epsilon}}_t)\Delta(\overline{\overline{\mu}}_t)}\right). \tag{18}$$

Here Δ denotes determinant of a two-dimensional dyadic (2×2 matrix) which can be computed in terms of dyadic algebra as [12, 13]

$$\Delta(\overline{\overline{\epsilon}}_t) = \frac{1}{2}\overline{\overline{\epsilon}}_t {}^\times_\times \overline{\overline{\epsilon}}_t : \mathbf{nn}, \qquad \Delta(\overline{\overline{\mu}}_t) = \frac{1}{2}\overline{\overline{\mu}}_t {}^\times_\times \overline{\overline{\mu}}_t : \mathbf{nn}, \tag{19}$$

and

$$\overline{\overline{\epsilon}}_t {}^\times_\times \overline{\overline{\mu}}_t^T : \mathbf{nn} = \Delta(\overline{\overline{\epsilon}}_t + \overline{\overline{\mu}}_t^T) - \Delta(\overline{\overline{\epsilon}}_t) - \Delta(\overline{\overline{\mu}}_t). \tag{20}$$

In [10] a relation was derived between the transverse permittivity and permeability dyadics and thickness of the slab on one hand and the surface admittance dyadic at the interface of the slab on the other hand. The resulting analytic expression has the form

$$\begin{aligned}\overline{\overline{\mathsf{Y}}}_s \; = \; & \frac{\beta_1 \cot \beta_1 d - \beta_2 \cot \beta_2 d}{j k_o \eta_o (\beta_1^2 - \beta_2^2)} k_o^2 \overline{\overline{\epsilon}}_t \\ & + \frac{\beta_2 \beta_1^2 \cot \beta_2 d - \beta_1 \beta_2^2 \cot \beta_1 d}{j k_o \eta_o (\beta_1^2 - \beta_2^2)} (\overline{\overline{\mu}}_t^{-1} {}^\times_\times \mathbf{nn})\end{aligned} \tag{21}$$

when assuming $\beta_1^2 \neq \beta_2^2$.

The expression (21) is now applicable for realizing the GSHS boundary. The basic question is how to determine the medium dyadics $\overline{\overline{\epsilon}}_t, \overline{\overline{\mu}}_t$ and the thickness d of the slab to obtain the GSHS admittance (12) in the limit $\delta \to 0$. This appears possible because zero and infinite admittance components can be realized by certain resonances in the structure corresponding to zero and infinite values of the two cotangent functions. Let us assume that the thickness of the slab is depends on the parameter δ so that, originally the thickness is d' and approaches d as

$$d' = d(1+\delta) \to d \tag{22}$$

when $\delta \to 0$. Let us rewrite (21) as

$$\overline{\overline{\mathsf{Y}}}_s = \beta_1 \cot \beta_1 d' \frac{k_o^2 \overline{\overline{\epsilon}}_t - \beta_2^2 \overline{\overline{\mu}}_t^{-1} {}^\times_\times \mathbf{nn}}{j k_o \eta_o (\beta_1^2 - \beta_2^2)} - \beta_2 \cot \beta_2 d' \frac{k_o^2 \overline{\overline{\epsilon}}_t - \beta_1^2 \overline{\overline{\mu}}_t^{-1} {}^\times_\times \mathbf{nn}}{j k_o \eta_o (\beta_1^2 - \beta_2^2)} \tag{23}$$

and identify termwise with (12) as

$$A\mathbf{ba} \;=\; \beta_1\delta\cot\beta_1 d'\,\frac{k_o^2\overline{\overline{\epsilon}}_t - \beta_2^2\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn}}{k_o(\beta_1^2-\beta_2^2)}, \tag{24}$$

$$B\mathbf{ab}^{\times}_{\times}\mathbf{nn} \;=\; -\frac{\beta_2}{\delta}\cot\beta_2 d'\,\frac{k_o^2\overline{\overline{\epsilon}}_t - \beta_1^2\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn}}{k_o(\beta_1^2-\beta_2^2)}. \tag{25}$$

The limit $\delta \to 0$ can now be handled by requiring

$$\beta_1 d' = \pi(1+\delta) \to \pi, \quad \beta_2 d' = \frac{\pi}{2}(1+\delta) \to \frac{\pi}{2}, \tag{26}$$

whence

$$\beta_1 d = \pi, \quad \beta_2 d = \pi/2. \tag{27}$$

It is also easy to find finite limits for the two quantities in (24), (25) as

$$\delta\cot(\beta_1 d') \;=\; \delta\cot(\pi\delta) \to \frac{1}{\pi}, \tag{28}$$

$$\frac{1}{\delta}\cot(\beta_2 d') \;=\; -\frac{1}{\delta}\tan(\pi\delta/2) \to -\frac{\pi}{2}. \tag{29}$$

Solving the medium dyadics from (24) and (25) in the form

$$\overline{\overline{\epsilon}}_t \;=\; \frac{\beta_1/k_o}{\delta\cot\beta_1 d'}A\mathbf{ba} + \frac{\beta_2/k_o}{\cot\beta_2 d'/\delta}B\mathbf{ab}^{\times}_{\times}\mathbf{nn} \tag{30}$$

$$\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn} \;=\; \frac{k_o/\beta_1}{\delta\cot\beta_1 d'}A\mathbf{ba} + \frac{k_o/\beta_2}{\cot\beta_2 d'/\delta}B\mathbf{ab}^{\times}_{\times}\mathbf{nn} \tag{31}$$

and taking the limits (28), (29) with (27) we obtain

$$\overline{\overline{\epsilon}}_t \;=\; \frac{1}{k_o d}\left(\pi^2 A\mathbf{ba} - B\mathbf{ab}^{\times}_{\times}\mathbf{nn}\right) \tag{32}$$

$$\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn} \;=\; k_o d(A\mathbf{ba} - \frac{4}{\pi^2}B\mathbf{ab}^{\times}_{\times}\mathbf{nn}). \tag{33}$$

To find the dyadic $\overline{\overline{\mu}}_t$, we must form the determinant

$$\Delta(\overline{\overline{\mu}}_t^{-1}) = \Delta(\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn}) = -\frac{4}{\pi^2}AB(k_o d)^2 = 1/\Delta(\overline{\overline{\mu}}_t), \tag{34}$$

and apply the rule [13]

$$\overline{\overline{\mu}}_t^{-1T}{}^{\times}_{\times}\mathbf{nn} = \overline{\overline{\mu}}_t\Delta(\overline{\overline{\mu}}_t^{-1}), \tag{35}$$

which produces the permeability dyadic

$$\overline{\overline{\mu}}_t = \frac{1}{k_o d}\left(-\frac{\pi^2}{4}\frac{1}{B}\mathbf{ab} + \frac{1}{A}\mathbf{ba}{}^{\times}_{\times}\mathbf{nn}\right). \tag{36}$$

The expressions (32) and (36) can be considered as forming the solution for the realization problem. Because the GSHS boundary conditions are obtained for any values of the scalars A and B, their choice depends on the realizability of the medium dyadics $\overline{\overline{\epsilon}}_t$ and $\overline{\overline{\mu}}$.

5. VERIFYING THE THEORY

To verify the previous theory let us assume that the transverse medium dyadics have the form

$$\overline{\overline{\epsilon}}_t = \epsilon_t(A\pi^2\mathbf{ba} - B\mathbf{ab}{}^{\times}_{\times}\mathbf{nn}), \tag{37}$$

$$\overline{\overline{\mu}}_t = \mu_t\left(\frac{\pi^2}{4B}\mathbf{ab} - \frac{1}{A}\mathbf{ba}{}^{\times}_{\times}\mathbf{nn}\right). \tag{38}$$

Forming the determinants

$$\Delta(\overline{\overline{\epsilon}}_t) = -\epsilon_t^2 AB\pi^2, \quad \Delta(\overline{\overline{\mu}}_t) = -\mu_t^2\frac{\pi^2}{4AB}, \tag{39}$$

from (35) we have

$$\overline{\overline{\mu}}_t^{-1}{}^{\times}_{\times}\mathbf{nn} = \frac{1}{\Delta(\overline{\overline{\mu}}_t)}\overline{\overline{\mu}}_t^T = \frac{1}{\mu_t}\left(-A\mathbf{ba} + \frac{4}{\pi^2}B\mathbf{ab}{}^{\times}_{\times}\mathbf{nn}\right). \tag{40}$$

To find the propagation constants we first must construct

$$\overline{\overline{\epsilon}}_t{}^{\times}_{\times}\overline{\overline{\mu}}_t^T : \mathbf{nn} = \frac{5\pi^2}{4}\mu_t\epsilon_t, \quad \Delta(\overline{\overline{\epsilon}}_t)\Delta(\overline{\overline{\mu}}_t) = \frac{\pi^4}{4}\mu_t^2\epsilon_t^2, \tag{41}$$

whence the two roots of (18) can be expressed as

$$\beta_1 = \pi k_t, \quad \beta_2 = \frac{\pi}{2}k_t, \tag{42}$$

when we denote

$$k_t = k_o\sqrt{\mu_t\epsilon_t}. \tag{43}$$

Now the thickness of the slab is determined by the two resonance conditions (27) which for the present medium can be represented by the single condition

$$k_t d = 1, \quad \Rightarrow \quad d = 1/k_t. \tag{44}$$

Inserting the above expressions in the surface admittance dyadic expression (23) with d replaced by $d' = d(1+\delta)$, after a number of algebraic steps the following limiting expression can be obtained:

$$\overline{\overline{\mathsf{Y}}}_s \to \frac{1}{j\eta_o\sqrt{\mu_t/\epsilon_t}}\left(\frac{1}{\delta}A\mathbf{ba} + \delta B\mathbf{ab}{}^{\times}_{\times}\mathbf{nn}\right). \tag{45}$$

Since this is of the form of the GSHS boundary admittance dyadic (12), the transverse medium dyadics of the form (37), (38) and the slab thickness determined by (44) will be sufficient for the realization of the generalized SHS boundary. Numerical values of the scalar parameters ϵ_t, μ_t, A, B do not affect the GSHS property but will determine its bandwidth.

6. APPLICATION: A POLARIZATION TRANSFORMER

The motivation for finding a realization for the GSHS boundary rests largely on the possible applications. One of the most interesting properties of the GSHS is its effect on the polarization of elliptically and linearly polarized plane waves upon reflection [7]. We will study a special case of the GSHS boundary and its application as a polarization transformer as well as the medium dyadics required to obtain such a boundary. Let us assume that the boundary coincides with the xy plane with $\mathbf{n} = \mathbf{u}_z$ and consider a special class of boundaries defined by

$$\begin{aligned} \mathbf{a} &= \mathbf{u}_x\cos\varphi + j\mathbf{u}_y\sin\varphi, \\ \mathbf{b} &= \mathbf{u}_x\cos\varphi - j\mathbf{u}_y\sin\varphi = \mathbf{a}^*, \end{aligned} \tag{46}$$

where the parameter φ is assumed to vary between $-\pi/2\cdots\pi/2$. The complex vectors $\mathbf{a}, \mathbf{b}$ define two similar elliptic polarizations with opposite handedness. For $\sin 2\varphi = 0$ ($\varphi = 0, \pm\pi/2$) the vectors are linearly polarized corresponding to the classical SHS boundary, for $\cos 2\varphi = 0$ ($\varphi = \pm\pi/4$) the two vectors are circularly polarized (see Fig. 1).

The reflected field $\mathbf{E}_r$ can be computed with the help of the reflection dyadic $\overline{\overline{\mathsf{R}}}$ as

$$\mathbf{E}_r = \overline{\overline{\mathsf{R}}}\cdot\mathbf{E}_i, \tag{47}$$

where the reflection dyadic of the GSHS surface is given by [8]

$$\overline{\overline{\mathsf{R}}} = -\frac{1}{J}\left(\frac{1}{k^2}[\mathbf{k}^r\times(\mathbf{b}\times\mathbf{k}^r)][\mathbf{k}^i\times(\mathbf{a}\times\mathbf{k}^i)] + (\mathbf{a}\times\mathbf{k}^r)(\mathbf{b}\times\mathbf{k}^i)\right), \tag{48}$$

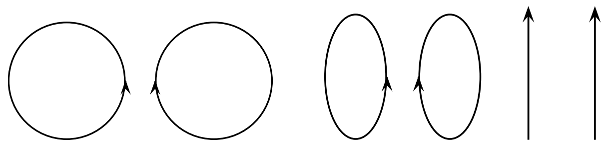

Figure 1. Different polarizations of vectors **a** and **b**, with (a) $\varphi = \pi/4$, (b) $0 < \varphi < \pi/4$, and (c) $\varphi = 0$.

with

$$J = \mathbf{a} \cdot \mathbf{k}^r \times (\mathbf{b} \times \mathbf{k}^r) \neq 0. \tag{49}$$

The reflection dyadic $\overline{\overline{\mathsf{R}}}$ is a function of the incident direction, which makes studying the polarizations of the reflected fields in analytic form quite difficult. The analysis is made easier if we restrict ourselves to some special direction, notably the normal direction. For normal incidence, with $\mathbf{k}^r = \mathbf{u}_z = -\mathbf{k}^i$, the reflection dyadic becomes

$$\overline{\overline{\mathsf{R}}} = \mathbf{ab}{}^{\times}_{\times}\mathbf{u}_z\mathbf{u}_z - \mathbf{ba} = \mathbf{ab}{}^{\times}_{\times}\mathbf{u}_z\mathbf{u}_z - (\mathbf{ab})^T. \tag{50}$$

The following dyadic products can be expanded as

$$\mathbf{ab} = \frac{1}{2}(\overline{\overline{\mathsf{I}}}_t + \overline{\overline{\mathsf{K}}}\cos 2\varphi + j\overline{\overline{\mathsf{J}}}\sin 2\varphi), \tag{51}$$

$$\mathbf{ab}{}^{\times}_{\times}\mathbf{u}_z\mathbf{u}_z = \frac{1}{2}(\overline{\overline{\mathsf{I}}}_t - \overline{\overline{\mathsf{K}}}\cos 2\varphi + j\overline{\overline{\mathsf{J}}}\sin 2\varphi), \tag{52}$$

with the two-dimensional basis dyadics defined as

$$\overline{\overline{\mathsf{I}}}_t = \mathbf{u}_x\mathbf{u}_x + \mathbf{u}_y\mathbf{u}_y, \quad \overline{\overline{\mathsf{K}}} = \mathbf{u}_x\mathbf{u}_x - \mathbf{u}_y\mathbf{u}_y, \quad \overline{\overline{\mathsf{J}}} = \mathbf{u}_z \times \overline{\overline{\mathsf{I}}} = \mathbf{u}_y\mathbf{u}_x - \mathbf{u}_x\mathbf{u}_y. \tag{53}$$

Writing the reflection dyadic for normal incidence (50) with the aid of (53), it then becomes simply

$$\overline{\overline{\mathsf{R}}} = -\overline{\overline{\mathsf{K}}}\cos 2\varphi + j\overline{\overline{\mathsf{J}}}\sin 2\varphi. \tag{54}$$

For a general elliptically polarized field,

$$\mathbf{E}_i = \cos\beta(\mathbf{u}_x\cos\alpha + \mathbf{u}_y\sin\alpha) + j\sin\beta(-\mathbf{u}_x\sin\alpha + \mathbf{u}_y\cos\alpha), \tag{55}$$

where $\beta = 0 \cdots \pi/2$, the reflected field vector can be written

$$\begin{aligned} \mathbf{E}_r = &- \mathbf{u}_x(\cos\alpha(\cos 2\varphi\cos\beta - \sin 2\varphi\sin\beta) \\ &\quad - j\sin\alpha(\cos 2\varphi\sin\beta - \sin 2\varphi\cos\beta)) \\ &+ \mathbf{u}_y(\sin\alpha(\cos 2\varphi\cos\beta + \sin 2\varphi\sin\beta) \\ &\quad + j\cos\alpha(\cos 2\varphi\sin\beta + \sin 2\varphi\cos\beta)), \end{aligned} \tag{56}$$

from which we can easily obtain the reflected field of a linearly polarized incident field, with $\beta = 0$,

$$\begin{aligned}\mathbf{E}_r = &- \mathbf{u}_x(\cos 2\varphi \cos\alpha + j \sin 2\varphi \sin\alpha) \\ &+ \mathbf{u}_y(\cos 2\varphi \sin\alpha + j \sin 2\varphi \cos\alpha),\end{aligned} \tag{57}$$

and similarly we can obtain the reflected field of a circularly polarized incident field with $\beta = \pi/4$ (omitting constant $1/\sqrt{2}$),

$$\begin{aligned}\mathbf{E}_r = &- \mathbf{u}_x(\cos 2\varphi - \sin 2\varphi)(\cos\alpha - j \sin\alpha) \\ &+ \mathbf{u}_y(\cos 2\varphi + \sin 2\varphi)(\sin\alpha + j \cos\alpha).\end{aligned} \tag{58}$$

Depending on the values of the parameters α and φ, the reflected field can be either elliptically, circularly or linearly polarized. For a linearly polarized incident field, with $\alpha = 0$ and $\varphi = \pi/8$, i.e., for $\cos 2\varphi = 1/\sqrt{2} = \sin 2\varphi$, which corresponds to elliptically polarized vectors **a** and **b**, it is easy to see that the reflected field is a circularly polarized field. For $\varphi = 0$, or $\varphi = \pi/4$, which correspond to linearly and circularly polarized vectors **a** and **b**, the terms $\sin 2\varphi$ or $\cos 2\varphi$ are zero, respectively, and the reflected field vector is linearly polarized for all values of α, i.e., the polarization type is not changed. For $\alpha = \pi/4$, the reflected field is linearly polarized for all values of φ.

Similarly we can see that for a circularly polarized incident field, the reflected field is linearly polarized for all values of α if $\cos 2\varphi - \sin 2\varphi = 0$, i.e., $\cos 2\varphi = 1/\sqrt{2} = \sin 2\varphi$, with $\varphi = \pi/8$. It is also easy to see that for $\varphi = 0, \pi/4$, the reflected field would be a circularly polarized field, so there would be no change in the polarization type.

For a general elliptical incident field finding the values α and φ which would yield a desired polarisation is not quite as straightforward as for linearly or circularly polarized incident fields. For example, to obtain a linearly polarized reflected field for $\alpha = 0$, requiring that $\cos 2\varphi \cos\beta - \sin 2\varphi \sin\beta = 0$ gives us to the condition

$$\varphi = \frac{1}{2} \arctan(\cot\beta), \tag{59}$$

which, marking $\beta = \pi/2 - \beta'$ and using $\cot(\pi/2 - \beta') = \tan\beta'$, can be reduced to

$$\varphi = \frac{\pi}{4} - \frac{\beta}{2}. \tag{60}$$

Similarly, for $\alpha = 0$, requiring that

$$\cos 2\varphi \cos\beta - \sin 2\varphi \sin\beta = \cos 2\varphi \sin\beta + \sin 2\varphi \cos\beta, \tag{61}$$

which gives us a circularly polarized reflected field, leads to the condition

$$\varphi = \frac{1}{2}\arctan\left(\frac{\cos\beta - \sin\beta}{\cos\beta + \sin\beta}\right), \tag{62}$$

which, marking $\beta = \beta' + \pi/4$ and using addition formulas for trigonometric functions, can be reduced to

$$\varphi = \frac{\pi}{8} - \frac{\beta}{2}. \tag{63}$$

As an example, for an elliptically polarized incident field with $\beta = \pi/8$, the values of φ which yield a linearly polarized reflected field and a circularly polarized reflected field are $\varphi = 3\pi/16$, and $\varphi = \pi/16$, respectively. One must also bear in mind that the handedness of the reflected wave may have changed, or the reflected field may be rotated, compared to the incident field, even though the polarization type may not have changed. The change of polarization of the reflected field compared to the incident field can be studied using the polarization match factor [13]. In the Fig. 2 are plotted the ellipticities of the reflected fields for linearly ($\beta = 0$), circularly ($\beta = \pi/4$), and elliptically ($\beta = \pi/8$) polarized incident fields, for $\varphi = 0 \cdots \pi/2$, $\alpha = 0$.

It is possible to compute the analytic expressions of the reflected field for any incident direction but these expressions are quite complicated. It is easier to study the polarization properties of the reflected fields numerically, if one wishes to consider any arbitrary direction. In the Figures 3, 4, and 5 are plotted the ellipticities of the reflected fields for linearly, elliptically ($\beta = \pi/8$), and circularly polarized incident fields for the angles $(\theta, \phi) = (\pi/8, \pi/4)$, $(\pi/4, \pi/4)$, and $(3\pi/8, \pi/4)$, with $\varphi = 0 \cdots \pi/2$. As can be seen from the figures, the dependency of the reflected field polarization on the incident direction is quite strong for all incident polarization types.

7. MEDIUM DYADICS FOR SOME SPECIAL CASES

As seen by previous examples, interesting choises for the parameter φ for the vectors vectors **a** and **b** defined in (46) seem to be $\varphi = 0$, $\pi/4$, and $\pi/8$, which correspond to linearly, circularly, or elliptically polarized vectors, respectively. The medium dyadics for the GSHS boundary defined by vectors **a**, **b**, obtained from (37) and (38), are

$$\overline{\overline{\epsilon}}_t = \frac{\epsilon_t}{2}\left((A\pi^2 - B)\overline{\overline{\mathsf{I}}}_t + (A\pi^2 + B)(\overline{\overline{\mathsf{K}}}\cos 2\varphi - j\overline{\overline{\mathsf{J}}}\sin 2\varphi)\right), \tag{64}$$

$$\overline{\overline{\mu}}_t = \frac{\mu_t}{2}\left((\frac{\pi^2}{4B} - \frac{1}{A})\overline{\overline{\mathsf{I}}}_t + (\frac{\pi^2}{4B} + \frac{1}{A})(\overline{\overline{\mathsf{K}}}\cos 2\varphi + j\overline{\overline{\mathsf{J}}}\sin 2\varphi)\right). \tag{65}$$

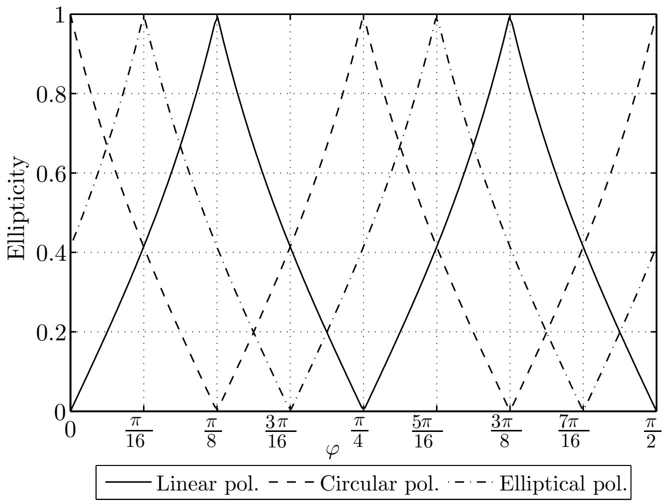

Figure 2. Ellipticities of the reflected fields of linearly, circularly, and elliptically ($\beta = \pi/8$) polarized normal incident fields as a function of $\varphi = 0 \cdots \pi/2$, with $\alpha = 0$.

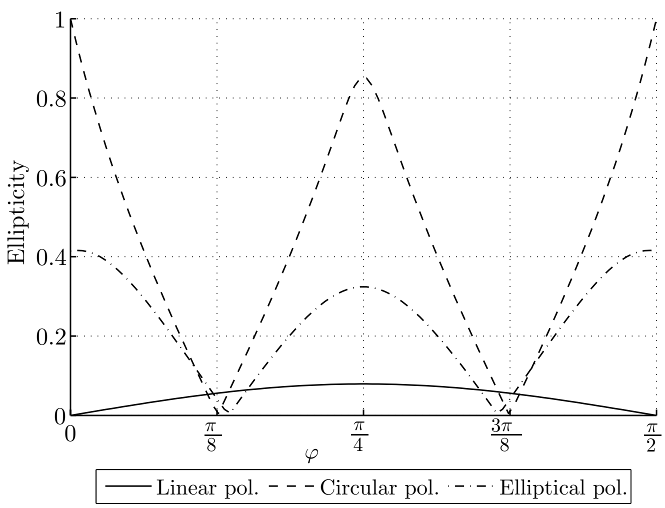

Figure 3. Ellipticity of the reflected field for linearly, circularly, and elliptically polarized ($\beta = \pi/8$) incident fields with $\theta = \pi/8$, $\phi = \pi/4$, and $\varphi = 0 \cdots \pi/2$.

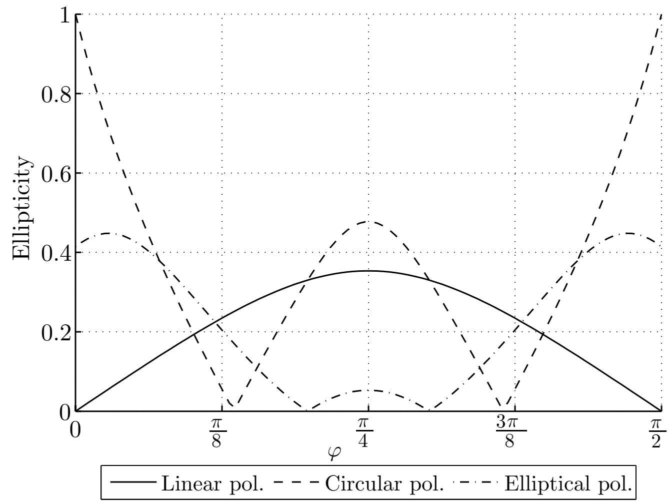

Figure 4. Ellipticity of the reflected field for linearly, circularly, and elliptically polarized ($\beta = \pi/8$) polarized incident fields, with $\theta = \pi/4$, $\phi = \pi/4$, and $\varphi = 0 \cdots \pi/2$.

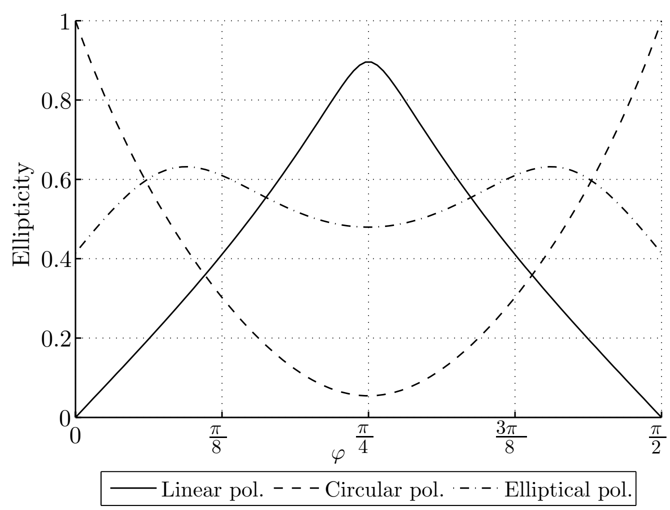

Figure 5. Ellipticity of the reflected field for linearly, circularly, and elliptically polarized ($\beta = \pi/8$) incident fields, with $\theta = 3\pi/8$, $\phi = \pi/4$, and $\varphi = 0 \cdots \pi/2$.

By choosing the parameters A and B in a suitable manner it is possible to make either $\overline{\overline{\epsilon}}_t$ or $\overline{\overline{\mu}}_t$ transversely isotropic, i.e., a multiple of $\overline{\overline{\mathsf{I}}}_t$. In fact, choosing $B = -A\pi^2$ we have $\overline{\overline{\epsilon}}_t = A\pi^2\epsilon_t\overline{\overline{\mathsf{I}}}_t$ while for $A = -4B/\pi^2$ we have $\overline{\overline{\mu}}_t = (\pi^2\mu_t/4B)\overline{\overline{\mathsf{I}}}_t$. Another simplification is obtained for the choice $B = \pi^2 A/2$:

$$\overline{\overline{\epsilon}}_t = \epsilon_t'(\overline{\overline{\mathsf{I}}}_t + 3\overline{\overline{\mathsf{K}}}\cos 2\varphi - 3j\overline{\overline{\mathsf{J}}}\sin 2\varphi), \quad \epsilon_t' = \frac{\pi^2 A\epsilon_t}{4}, \tag{66}$$

$$\overline{\overline{\mu}}_t = \mu_t'(\overline{\overline{\mathsf{I}}}_t - 3\overline{\overline{\mathsf{K}}}\cos 2\varphi - 3j\overline{\overline{\mathsf{J}}}\sin 2\varphi), \quad \mu_t' = -\frac{\mu_t}{4A}, \tag{67}$$

in which case the two medium dyadics have some similarity.

7.1. Linearly Polarized a and b

Let us now consider the case $\sin 2\varphi = 0$, corresponding to real $\mathbf{a}$ and $\mathbf{b}$ vectors. The classical SHS boundary is obtained by choosing $\mathbf{a}, \mathbf{b}$ as

$$\mathbf{a} = \mathbf{b} = \mathbf{u}_x, \quad \mathbf{u}_z \times \mathbf{a} = \mathbf{u}_z \times \mathbf{b} = \mathbf{u}_y. \tag{68}$$

The medium dyadics obtained from (37) and (38) are then of the simple form

$$\overline{\overline{\epsilon}}_t = \epsilon_x\mathbf{u}_x\mathbf{u}_x + \epsilon_y\mathbf{u}_y\mathbf{u}_y, \tag{69}$$

$$\overline{\overline{\mu}}_t = \mu_x\mathbf{u}_x\mathbf{u}_x + \mu_y\mathbf{u}_y\mathbf{u}_y, \tag{70}$$

with

$$\epsilon_x = \epsilon_t\pi^2 A, \quad \epsilon_y = -\epsilon_t B, \quad \mu_x = \mu_t\frac{\pi^2}{4B}, \quad \mu_y = -\frac{\mu_t}{A}. \tag{71}$$

It is seen that the parameters of the slab making a SHS boundary must satisfy the condition

$$4\epsilon_y/\epsilon_x = \mu_y/\mu_x, \tag{72}$$

which reproduces the result obtained in [10]. Because we have

$$k_t^2 = \omega^2\mu_t\epsilon_t = -\frac{1}{\pi^2}\omega^2\mu_y\epsilon_x = -\frac{4}{\pi^2}\mu_x\epsilon_y = 1/d^2, \tag{73}$$

it appears that two of the medium parameters of the wave-guiding medium $\epsilon_x, \epsilon_y, \mu_x, \mu_y$ must have positive values and the other two negative values for (73) to be satisfied for real d.

7.2. Circularly Polarized a and b

Let us next consider the GSHS boundary for $\cos 2\varphi = 0$ corresponding to circularly polarized $\mathbf{a}$ and $\mathbf{b}$ of opposite handedness:

$$\mathbf{a} = \frac{1}{\sqrt{2}}(\mathbf{u}_x + j\mathbf{u}_y), \quad \mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{u}_x - j\mathbf{u}_y), \tag{74}$$

so that $\mathbf{a}\cdot\mathbf{b} = 1$ is satisfied. Now the normalized GSHS admittance dyadic (12) becomes

$$j\eta_o\overline{\overline{\mathsf{Y}}}_s = \frac{A}{\delta}\,\mathbf{ba} + B\delta\,\mathbf{ab} = A\delta\,\mathbf{a}^*\mathbf{a} + B\delta\,\mathbf{aa}^* \tag{75}$$

$$= \left(\frac{A}{\delta\sqrt{2}} + \frac{B\delta}{\sqrt{2}}\right)\overline{\overline{\mathsf{I}}}_t + j\left(\frac{A}{\delta\sqrt{2}} - \frac{B\delta}{\sqrt{2}}\right)\overline{\overline{\mathsf{J}}}. \tag{76}$$

To achieve this, we need the material parameter dyadics

$$\overline{\overline{\epsilon}}_t = \frac{\epsilon_t}{2}\left[(\pi^2 A - B)\,\overline{\overline{\mathsf{I}}}_t + j(\pi^2 A + B)\,\overline{\overline{\mathsf{J}}}\right] \tag{77}$$

$$\overline{\overline{\mu}}_t = \frac{\mu_t}{2}\left[\left(\frac{\pi^2}{4B} - \frac{1}{A}\right)\overline{\overline{\mathsf{I}}}_t + j\left(\frac{\pi^2}{4B} + \frac{1}{A}\right)\overline{\overline{\mathsf{J}}}\right] \tag{78}$$

Choosing $B = -\pi^2 A$, we can realize the required conditions with isotropic permittivity, $\overline{\overline{\epsilon}}_t = \epsilon_t\pi^2 A\,\overline{\overline{\mathsf{I}}}_t$, and gyrotropic (ferrite-like) permeability: $\overline{\overline{\mu}}_t = \mu_t(-5\overline{\overline{\mathsf{I}}}_t + j3\mathbf{u}_z \times \overline{\overline{\mathsf{I}}})/(8A)$.

7.3. Elliptically Polarized a and b

An interesting choise for elliptically polarized $\mathbf{a}$ and $\mathbf{b}$, in view of the previous examples, seems to be the case $\varphi = \pi/8$. The medium dyadics (64), (65) of the GSHS boundary for this case, with $\cos 2\varphi = 1/\sqrt{2} = \sin 2\varphi$, are:

$$\overline{\overline{\epsilon}}_t = \frac{\epsilon_t}{2}\left((A\pi^2 - B)\overline{\overline{\mathsf{I}}}_t + \frac{1}{\sqrt{2}}(A\pi^2 + B)(\overline{\overline{\mathsf{K}}} - j\overline{\overline{\mathsf{J}}})\right), \tag{79}$$

$$\overline{\overline{\mu}}_t = \frac{\mu_t}{2}\left((\frac{\pi^2}{4B} - \frac{1}{A})\overline{\overline{\mathsf{I}}}_t + \frac{1}{\sqrt{2}}(\frac{\pi^2}{4B} + \frac{1}{A})(\overline{\overline{\mathsf{K}}} + j\overline{\overline{\mathsf{J}}})\right). \tag{80}$$

Again, by choosing $B = -\pi^2 A$, the medium dyadics are reduced to

$$\overline{\overline{\epsilon}}_t = \epsilon_t\pi^2 A\overline{\overline{\mathsf{I}}}_t, \tag{81}$$

$$\overline{\overline{\mu}}_t = \frac{\mu_t}{8A}\left(-5\overline{\overline{\mathsf{I}}}_t + \frac{3}{\sqrt{2}}(\overline{\overline{\mathsf{K}}} + j\overline{\overline{\mathsf{J}}})\right), \tag{82}$$

i.e., the permittivity is again isotropic, but the permeability dyadic is more complicated.

8. CONCLUSION

In this paper, a realization for the GSHS boundary in terms of a slab of special wave-guiding anisotropic material has been studied. Analytic expressions for the material parameters and thickness of the slab have been derived in terms of the vectors **a**, **b**, using a limiting process. With these analytic expressions, it is in theory possible to construct any GSHS boundary defined by (5).

Also the polarization transforming properties of the GSHS boundary have been studied. We have shown that the GSHS boundary is able to transform any polarization to another given polarization for normal incidence, if the parameters defining the boundary are chosen correctly. The choise of these parameters and the material parameters required by the boundary have been studied in detail. This ability makes the GSHS boundary an interesting research subject, since these polarization transforming properties have numerous real-world applications in radio and antenna technology.

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