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# Perfectly anisotropic impedance boundary

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**Abstract:** The concept of a perfectly anisotropic boundary (PAB) is defined in terms of a surface impedance dyadic without an isotropic component. Realisation of PAB in terms of a slab of anisotropic metamaterial is suggested. It is shown that PAB can serve as a simple rotatable polarisation transformer, transforming a linearly polarised field to a field with given elliptic polarisation and handedness, or conversely. It is also shown that, unlike at a regular impedance boundary, a PAB can simultaneously support two orthogonally polarised surface waves propagating in certain directions along the boundary.

## 1 Introduction

The impedance boundary condition (IBC) was originally introduced by Shchukin and Leontovich in the 1940s [1–3], although it was anticipated earlier by Schelkunoff [4]. IBC expresses a linear relation between the electric and magnetic field components tangential to the boundary surface. Assuming a planar boundary  $z=0$  with unit normal  $\mathbf{u}_z$  pointing towards the surface, the condition can be represented by the linear relation

$$\mathbf{E}_t = -\bar{\bar{Z}}_s \cdot (\mathbf{u}_z \times \mathbf{H}), \quad -\mathbf{u}_z \times \mathbf{H} = \bar{\bar{Y}}_s \cdot \mathbf{E}_t \quad (1)$$

where  $\bar{\bar{Z}}_s$  is the surface impedance and  $\bar{\bar{Y}}_s$  the surface admittance dyadic and the subscript  $t$  denotes component transverse to  $\mathbf{u}_z$ . In the past, IBCs have been applied as approximations, when replacing physical structures by boundaries with simple conditions (1), where  $\bar{\bar{Z}}_s$  or  $\bar{\bar{Y}}_s$  are either algebraic or operator quantities [5–7]. More recently, a method for realising the surface admittance dyadic  $\bar{\bar{Y}}_s$  in terms of a layer of anisotropic medium was introduced, in which an exact analytic relation between the parameters of the layer and  $\bar{\bar{Y}}_s$  was found [8].

In the present study, a certain class of impedance boundaries is defined and some of its properties studied. The surface-impedance and admittance dyadics can be characterised as two-dimensional dyadics, because they satisfy

$$\mathbf{u}_z \cdot \bar{\bar{Z}}_s = \bar{\bar{Z}}_s \cdot \mathbf{u}_z = 0, \quad \mathbf{u}_z \cdot \bar{\bar{Y}}_s = \bar{\bar{Y}}_s \cdot \mathbf{u}_z = 0 \quad (2)$$

Two-dimensional dyadics can be expanded in terms of four basis dyadics built upon a two-dimensional basis of unit vectors  $\mathbf{u}_x, \mathbf{u}_y$  satisfying  $\mathbf{u}_x \times \mathbf{u}_y = \mathbf{u}_z$ . The basis dyadics are chosen as [9]

$$\bar{\bar{I}} = \mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y \quad (3)$$

$$\bar{\bar{J}} = \mathbf{u}_y \mathbf{u}_x - \mathbf{u}_x \mathbf{u}_y = \mathbf{u}_z \times \bar{\bar{I}}_t \quad (4)$$

$$\bar{\bar{K}} = \mathbf{u}_x \mathbf{u}_x - \mathbf{u}_y \mathbf{u}_y \quad (5)$$

$$\bar{\bar{L}} = \mathbf{u}_x \mathbf{u}_y + \mathbf{u}_y \mathbf{u}_x \quad (6)$$

### 1.1 Isotropic boundary

The four basic dyadics can be split in two groups. The transverse unit dyadic  $\bar{\bar{I}}_t$  and the antisymmetric dyadic  $\bar{\bar{J}}$  do not depend on the choice of the vector basis  $\mathbf{u}_x, \mathbf{u}_y$ . In fact, defining the transformation  $\mathbf{u}_x, \mathbf{u}_y \rightarrow \mathbf{u}'_x, \mathbf{u}'_y$  by

$$\begin{pmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \mathbf{u}'_x \\ \mathbf{u}'_y \end{pmatrix} \quad (7)$$

we can easily show that

$$\mathbf{u}_x \mathbf{u}_x + \mathbf{u}_y \mathbf{u}_y = \mathbf{u}'_x \mathbf{u}'_x + \mathbf{u}'_y \mathbf{u}'_y \quad (8)$$

$$\mathbf{u}_y \mathbf{u}_x - \mathbf{u}_x \mathbf{u}_y = \mathbf{u}'_y \mathbf{u}'_x - \mathbf{u}'_x \mathbf{u}'_y \quad (9)$$

As linear combination of  $\bar{\bar{I}}_t$  and  $\bar{\bar{J}}$  do not depend on any particular vector in the  $xy$  plane, they define a space of two-dimensional isotropic dyadics. Another way of defining the class of isotropic dyadics is through the condition [9]

$$\bar{\bar{D}} \times \mathbf{u}_z \mathbf{u}_z = \bar{\bar{D}} \quad (10)$$

where the double-cross product is defined by [10]

$$(\mathbf{a}\mathbf{b}) \times (\mathbf{c}\mathbf{d}) = (\mathbf{a} \times \mathbf{c})(\mathbf{b} \times \mathbf{d}) \quad (11)$$

It is easy to show that the inverse of an isotropic dyadic is isotropic. Also, as  $\bar{\bar{J}} \cdot \bar{\bar{J}} = -\bar{\bar{I}}_t$ , the dot product of isotropic dyadics is an isotropic dyadic.

An impedance boundary can be called isotropic, if its impedance or admittance dyadic can be expressed in terms of a two-dimensional isotropic dyadic. The most general isotropic boundary can be represented by an admittance dyadic of the form

$$\bar{\bar{Y}}_s = Y_I \bar{\bar{I}}_t + Y_J \bar{\bar{J}} \quad (12)$$

for scalars  $Y_I$  and  $Y_J$ . The isotropic surface admittance  $\bar{\bar{Y}}_s = Y_I \bar{\bar{I}}_t$  can be obtained by a slab of transversely isotropic medium  $\bar{\bar{\epsilon}}_t = \epsilon_t \bar{\bar{I}}_t, \bar{\bar{\mu}}_t = \mu_t \bar{\bar{I}}_t$ , while the component  $\bar{\bar{Y}}_s = Y_J \bar{\bar{J}}$  requires gyrotropic material [11].

The general isotropic boundary will perform polarization rotation for a normally incident plane wave. In fact, the reflection dyadic for normal incidence can be easily shown to be of the same isotropic form, whence it can be expressed as

$$\bar{\bar{R}} = R_J \bar{\bar{I}}_t + R_J \bar{\bar{J}} = R(\bar{\bar{I}}_t \cos \theta + \bar{\bar{J}} \sin \theta) \quad (13)$$

Any linearly polarised transverse vector  $\mathbf{v}$  is mapped as

$$\bar{\bar{R}} \cdot \mathbf{v} = R(\mathbf{v} \cos \theta + \mathbf{u}_z \times \mathbf{v} \sin \theta) \quad (14)$$

Thus, the polarisation of the plane wave experiences a change in magnitude, through the factor  $R$ , as well as rotation of the polarisation, through the angle  $\theta$ , in reflection from an isotropic boundary.

## 1.2 Perfectly anisotropic boundary

In contrast to the isotropic dyadics  $\bar{\bar{I}}_t$  and  $\bar{\bar{J}}$ , the basis dyadics  $\bar{\bar{K}}$  and  $\bar{\bar{L}}$  depend on the chosen basis vectors  $\mathbf{u}_x, \mathbf{u}_y$ . A two-dimensional dyadic can be called anisotropic if it contains components on the dyadics  $\bar{\bar{K}}$  and/or  $\bar{\bar{L}}$ . Let us now define the class of dyadics which are linear combinations of  $\bar{\bar{K}}$  and  $\bar{\bar{L}}$  dyadics as that of perfectly anisotropic (PA) dyadics. Thus, any dyadic of the form

$$\bar{\bar{D}} = D_K \bar{\bar{K}} + D_L \bar{\bar{L}} \quad (15)$$

is perfectly anisotropic. Another way to define the space of two-dimensional PA dyadics is the condition [9]

$$\bar{\bar{D}} \times \mathbf{u}_z \mathbf{u}_z = -\bar{\bar{D}} \quad (16)$$

Making the transformation (7), we have

$$\bar{\bar{D}} = \bar{\bar{K}}'(D_K \cos 2\theta - D_L \sin 2\theta) + \bar{\bar{L}}'(D_L \cos 2\theta + D_K \sin 2\theta) \quad (17)$$

with

$$\bar{\bar{K}}' = \mathbf{u}'_x \mathbf{u}'_x - \mathbf{u}'_y \mathbf{u}'_y, \quad \bar{\bar{L}}' = \mathbf{u}'_x \mathbf{u}'_y + \mathbf{u}'_y \mathbf{u}'_x \quad (18)$$

Now choosing  $\theta$  so that

$$D_L \cos 2\theta + D_K \sin 2\theta = 0 \quad (19)$$

the dyadic  $\bar{\bar{D}}$  becomes a multiple of  $\bar{\bar{K}}'$ , while choosing

$$D_K \cos 2\theta - D_L \sin 2\theta = 0 \quad (20)$$

it becomes a multiple of  $\bar{\bar{L}}'$ . Thus any PA dyadic can be expressed as a multiple of the dyadic  $\bar{\bar{K}}$  or  $\bar{\bar{L}}$  alone, defined in a suitably chosen vector basis  $\mathbf{u}_x, \mathbf{u}_y$ . We must however note that the basis vectors so defined may become complex valued if the components  $D_K, D_L$  are complex. We can easily show that the inverse of a PA dyadic is perfectly anisotropic

$$\bar{\bar{D}}^{-1} = \frac{D_K \bar{\bar{K}} + D_L \bar{\bar{L}}}{D_K^2 + D_L^2} \quad (21)$$

but the dot-product of two PA dyadics is an isotropic dyadic because of the relations

$$\bar{\bar{K}} \cdot \bar{\bar{K}} = \bar{\bar{L}} \cdot \bar{\bar{L}} = \bar{\bar{I}}_t, \quad \bar{\bar{L}} \cdot \bar{\bar{K}} = -\bar{\bar{K}} \cdot \bar{\bar{L}} = \bar{\bar{J}} \quad (22)$$

In analogy to the definition of the isotropic boundary, let us call the boundary defined by the surface admittance of

the form

$$\bar{\bar{Y}}_s = Y_K \bar{\bar{K}} + Y_L \bar{\bar{L}} \quad (23)$$

a perfectly anisotropic boundary (PAB). Another way to define the PA boundary is by requiring the condition [9]

$$\bar{\bar{Y}}_s \times \mathbf{u}_z \mathbf{u}_z = -\bar{\bar{Y}}_s \quad (24)$$

to be valid. It is the purpose of this paper to study the basic electromagnetic properties of the PAB.

## 2 Wave-guiding medium

In a previous study [8], a method for the realisation of impedance boundaries with a layer of wave-guiding anisotropic material was suggested. The idea came from the realisation of a special case, the perfect electromagnetic conductor (PEMC) boundary [11]. The wave-guiding material is defined by medium dyadics of the form.

$$\bar{\bar{\epsilon}} = \epsilon_0(\bar{\bar{\epsilon}}_t + \mathbf{u}_z \mathbf{u}_z \epsilon_z) \quad (25)$$

$$\bar{\bar{\mu}} = \mu_0(\bar{\bar{\mu}}_t + \mathbf{u}_z \mathbf{u}_z \mu_z) \quad (26)$$

assuming  $\epsilon_z \rightarrow \infty, \mu_z \rightarrow \infty$ . In this case, the field in the medium becomes TEM with respect to the direction  $\mathbf{u}_z$ , i.e. it satisfies  $\mathbf{u}_z \cdot \mathbf{E} = 0$  and  $\mathbf{u}_z \cdot \mathbf{H} = 0$  so that  $\mathbf{u}_z \cdot \mathbf{D}$  and  $\mathbf{u}_z \cdot \mathbf{B}$  will have finite values. Assuming a wave of the form

$$\mathbf{E}(r) = \mathbf{E}(z)f(x,y), \quad \mathbf{E}(z) = \mathbf{E}e^{-j\beta z} \quad (27)$$

$$\mathbf{H}(r) = \mathbf{H}(z)f(x,y), \quad \mathbf{H}(z) = \mathbf{H}e^{-j\beta z} \quad (28)$$

in such a medium, the Maxwell equations outside the sources can be shown to be satisfied for any function  $f(x, y)$ . The transverse part of the Maxwell equations reduces to the following two-dimensional algebraic relations for the amplitude vectors  $\mathbf{E}, \mathbf{H}$ :

$$\bar{\bar{\epsilon}}_t \cdot \mathbf{E} = -\frac{\beta \eta_0}{k_0} \mathbf{u}_z \times \mathbf{H} \quad (29)$$

$$\bar{\bar{\mu}}_t \cdot \mathbf{H} = \frac{\beta}{k_0 \eta_0} \mathbf{u}_z \times \mathbf{E} \quad (30)$$

from which the propagation constant can be solved as [8]

$$\beta^2 = \frac{k_0^2}{2} \left( \bar{\bar{\epsilon}}_t \times \bar{\bar{\mu}}_t^T : \mathbf{u}_z \mathbf{u}_z \pm \sqrt{(\bar{\bar{\epsilon}}_t \times \bar{\bar{\mu}}_t^T : \mathbf{u}_z \mathbf{u}_z)^2 - 4\Delta(\bar{\bar{\mu}}_t)\Delta(\bar{\bar{\epsilon}}_t)} \right) \quad (31)$$

Here,  $\Delta(\bar{\bar{D}})$  denotes the two-dimensional determinant function [9]

$$\Delta(\bar{\bar{D}}) = \frac{1}{2} \bar{\bar{D}} \times \bar{\bar{D}} : \mathbf{u}_z \mathbf{u}_z \quad (32)$$

In the general case, (31) has two pairs of different roots  $\pm \beta_1 \pm \beta_2$ . The axial components of the Maxwell equation give rise to the equations

$$(\mathbf{u}_z \times \nabla f(x,y)) \cdot \mathbf{E} = -j\omega \mathbf{u}_z \cdot \mathbf{B}(x,y) \quad (33)$$

$$(\mathbf{u}_z \times \nabla f(x,y)) \cdot \mathbf{H} = -j\omega \mathbf{u}_z \cdot \mathbf{D}(x,y) \quad (34)$$

from which the finite axial components of  $\mathbf{D}$  and  $\mathbf{B}$  can be determined for any given function  $f(x, y)$ , after  $\mathbf{E}$  and  $\mathbf{H}$  have first been solved from (29) and (30). It is interesting to note that, while the transverse fields  $\mathbf{B}_t = \mu_0 \bar{\bar{\mu}}_t \cdot \mathbf{H}$  and  $\mathbf{D}_t = \epsilon_0 \bar{\bar{\epsilon}}_t \cdot \mathbf{E}$  depend on  $x$  and  $y$  as  $f(x, y)$ , the axial components  $\mathbf{u}_z \cdot \mathbf{B}$  and  $\mathbf{u}_z \cdot \mathbf{D}$  have another dependence, in general.

As any transverse field distribution  $f(x, y)$  propagates without change along the  $z$  axis, this resembles propagation in space filled with small parallel waveguides. This gives us the reason to call the material a wave-guiding medium. Power propagation in the  $\mathbf{u}_z$  direction can be expressed as

$$P_z = \frac{1}{2} \Re\{\mathbf{u}_z \cdot \mathbf{E} \times \mathbf{H}^*\} = \frac{k_o}{2|\beta|^2 \eta_o} \Re\{\beta \mathbf{E} \cdot \bar{\bar{\epsilon}}_t^* \cdot \mathbf{E}^*\} \quad (35)$$

For example, if  $\bar{\bar{\epsilon}}_t$  is a Hermitian and positive-definite dyadic,  $\mathbf{E} \cdot \bar{\bar{\epsilon}}_t^* \cdot \mathbf{E}^*$  is real and positive for any  $\mathbf{E} \neq 0$ . In this case,  $\Re\{\beta\} > 0$  corresponds to power propagation in the direction  $\mathbf{u}_z$  and  $\Re\{\beta\} < 0$  in the direction  $-\mathbf{u}_z$ .

In [8], the problem of a slab of wave-guiding medium was considered. It was shown that a slab of thickness  $d$  backed by a PEC plane can be exactly represented by an impedance boundary whose surface admittance dyadic depends on the parameters of the slab through the following expression

$$\bar{\bar{Y}}_s = \frac{\beta_1 \cot \beta_1 d - \beta_2 \cot \beta_2 d}{jk_o \eta_o (\beta_1^2 - \beta_2^2)} k_o^2 \bar{\bar{\epsilon}}_t + \frac{\beta_2 \beta_1^2 \cot \beta_2 d - \beta_1 \beta_2^2 \cot \beta_1 d}{jk_o \eta_o (\beta_1^2 - \beta_2^2)} \bar{\bar{\mu}}_t^{-1} \times \mathbf{u}_z \mathbf{u}_z \quad (36)$$

The inverse of a two-dimensional dyadic can be expressed as [9]

$$\bar{\bar{D}}^{-1} = \frac{1}{\Delta(\bar{\bar{D}})} \bar{\bar{D}}^T \times \mathbf{u}_z \mathbf{u}_z \quad (37)$$

provided the determinant  $\Delta(\bar{\bar{D}})$  does not vanish. Equation (36) can be used for finding the parameters of the slab to realise a given admittance dyadic  $\bar{\bar{Y}}_s$ . There is a lot of freedom for how to do this, as for 4 (complex) parameter of  $\bar{\bar{Y}}_s$ , there are 8 (complex) parameters of  $\bar{\bar{\epsilon}}_t$  and  $\bar{\bar{\mu}}_t$ , and the real parameter  $d$ . This means that the same admittance dyadic may be realised by many different material slabs.

The expression (36) is not applicable for the special case  $\beta_1^2 = \beta_2^2$ . However, by taking the limit  $\beta_1 \rightarrow \beta_2 \rightarrow \beta$ , the limiting case of (36) can be analytically determined as

$$\bar{\bar{Y}}_s = \frac{k_o}{2j\eta_o \beta} \left( \cos \beta d - \frac{\beta d}{\sin^2 \beta d} \right) \bar{\bar{\epsilon}}_t + \frac{\beta}{2j\eta_o k_o} \left( \cot \beta d + \frac{\beta d}{\sin^2 \beta d} \right) \bar{\bar{\mu}}_t^{-1} \times \mathbf{u}_z \mathbf{u}_z \quad (38)$$

The same expression is also valid in the limit  $\beta_1 \rightarrow -\beta_2 \rightarrow \beta$ , because  $\bar{\bar{Y}}_s$  is an even function of  $\beta$ .

### 3 Realisation of PAB

As the surface admittance dyadic of a perfectly anisotropic boundary is symmetric, PAB is always reciprocal [9]. To be lossless, the surface admittance dyadic must also be anti-Hermitian [9], which means that  $\bar{\bar{Y}}_s$  must be an imaginary dyadic. Let us consider a possible realisation of the lossless surface admittance dyadic

$$\bar{\bar{Y}}_s = jB_s \bar{\bar{K}} = jB_s (\mathbf{u}_x \mathbf{u}_x - \mathbf{u}_y \mathbf{u}_y) \quad (39)$$

where  $B_s$  is a real surface susceptance. As  $\bar{\bar{K}}$  is real, the basis vectors  $\mathbf{u}_x, \mathbf{u}_y$  are real. From (36), we can assume both  $\bar{\bar{\epsilon}}_t$  and  $\bar{\bar{\mu}}_t$  to be multiples of  $\bar{\bar{K}}$ , as

$$\bar{\bar{\epsilon}}_t = \epsilon_K \bar{\bar{K}} \quad \bar{\bar{\mu}}_t = -\mu_K \bar{\bar{K}} \quad (40)$$

The minus sign is added for future convenience without any assumptions on the real-valued parameters  $\epsilon_K$  and  $\mu_K$  being positive or negative. The two eigenwaves 1 and 2 in the wave-guiding medium have the respective  $\text{TE}_y$  and  $\text{TE}_x$  polarisations

$$\mathbf{E}_1 = \mathbf{u}_x E_1, \quad \mathbf{H}_1 = \mathbf{u}_y H_1 \quad (41)$$

$$\mathbf{E}_2 = \mathbf{u}_y E_2, \quad \mathbf{H}_2 = \mathbf{u}_x H_2 \quad (42)$$

because they are eigenvectors of the medium dyadics (40).

Applying the dyadic property  $\bar{\bar{K}} \times \bar{\bar{K}} = -2\mathbf{u}_z \mathbf{u}_z$  [9], the medium dyadics can be shown to obey the rules

$$\bar{\bar{\epsilon}}_t \times \bar{\bar{\mu}}_t^T : \mathbf{u}_z \mathbf{u}_z = 2\epsilon_K \mu_K, \quad \Delta(\bar{\bar{\epsilon}}_t) = -\epsilon_K^2, \\ \Delta(\bar{\bar{\mu}}_t) = -\mu_K^2 \quad (43)$$

whence, from (31), we have the eigenvalues

$$\beta_1^2 = \beta_2^2 = k_K^2, \quad k_K = k_o \sqrt{\mu_K \epsilon_K} \quad (44)$$

It is seen that the propagation factors  $\beta_{1,2}$  have real values only when  $\epsilon_K$  and  $\mu_K$  are both positive or negative, which explains the choice for the minus sign in (40). Considering the case  $\epsilon_K > 0$  and  $\mu_K > 0$ , the slab consists of anisotropic metamaterial defined by

$$\bar{\bar{\epsilon}} = \epsilon_K \mathbf{u}_x \mathbf{u}_x - \epsilon_K \mathbf{u}_y \mathbf{u}_y, \quad (45)$$

$$\bar{\bar{\mu}} = -\mu_K \mathbf{u}_x \mathbf{u}_x + \mu_K \mathbf{u}_y \mathbf{u}_y, \quad (46)$$

From (35), the power propagating in the  $\mathbf{u}_z$  direction becomes

$$P_z = \frac{k_o \epsilon_K}{2k_K^2 \eta_o} \beta (\mathbf{E} \cdot \bar{\bar{K}} \cdot \mathbf{E}^*) \quad (47)$$

which, for the two eigenwaves, reduces to

$$P_{1z} = \frac{k_o \epsilon_K}{2k_K^2 \eta_o} |\mathbf{E}_1|^2 \beta_1, \quad P_{2z} = -\frac{k_o \epsilon_K}{2k_K^2 \eta_o} |\mathbf{E}_2|^2 \beta_2 \quad (48)$$

From this we see that, for a wave with power propagating in the  $\mathbf{u}_z$  direction,  $\beta_1$  must be positive and  $\beta_2$  negative. Thus, the  $\text{TE}_x$  wave 2 in the wave-guiding medium is actually a backward wave and we can write the solutions of (44) as

$$\beta_1 = k_K, \quad \beta_2 = -k_K \quad (49)$$

Although we now have  $\beta_1^2 = \beta_2^2$ , the problem is not actually degenerate because  $\beta_1 \neq \beta_2$ . This is also indicated by the fact that the two eigenwaves have unique polarisation. However, (36) is not applicable in this case. Instead, we must use (38) for finding the corresponding surface admittance dyadic of the slab by replacing  $\beta$  either  $k_K$  or  $-k_K$  which lead to the same result

$$\bar{\bar{Y}}_s = jB_s \bar{\bar{K}} = \frac{k_o}{2j\eta_o k_K} \left( \cot k_K d - \frac{k_K d}{\sin^2 k_K d} \right) \epsilon_K \bar{\bar{K}} + \frac{k_K}{2j\eta_o k_o} \left( \cot k_K d + \frac{k_K d}{\sin^2 k_K d} \right) \frac{1}{\mu_K} \bar{\bar{K}} \quad (50)$$

from which we have the simple result

$$B_s = -\frac{1}{\eta_K} \cot k_K d, \quad \eta_K = \eta_o \sqrt{\frac{\mu_K}{\epsilon_K}} = \sqrt{\frac{\mu_o \mu_K}{\epsilon_o \epsilon_K}} \quad (51)$$

Thus, for the realisation of the lossless PAB surface, a slab of medium defined by (40) is required. The medium can be called an anisotropic metamaterial which cannot be

characterised as having double-negative parameters in some spatial direction. However, we could characterise it as being double-positive for TE<sub>y</sub> fields and double-negative for TE<sub>x</sub> fields. A similar medium has been called medium with indefinite permittivity and permeability in the past, and methods for its realisation in laboratory have been proposed [12, 13].

#### 4 Reflection from PAB plane

Let us consider a normally incident plane wave reflecting from a lossless PAB plane (39) at  $z = 0$ . The total fields satisfy the boundary condition

$$-\mathbf{u}_z \times (\mathbf{H}^i + \mathbf{H}^r) = \bar{\bar{\mathbf{Y}}}_s \cdot (\mathbf{E}^i + \mathbf{E}^r) \quad (52)$$

Inserting

$$\mathbf{H}^i = \frac{1}{\eta_o} \mathbf{u}_z \times \mathbf{E}^i, \quad \mathbf{H}^r = -\frac{1}{\eta_o} \mathbf{u}_z \times \mathbf{E}^r \quad (53)$$

we obtain

$$\mathbf{E}^r = \bar{\bar{\mathbf{R}}} \cdot \mathbf{E}^i \quad (54)$$

with the reflection dyadic defined by

$$\bar{\bar{\mathbf{R}}} = (\bar{\bar{\mathbf{I}}}_t + \eta_o \bar{\bar{\mathbf{Y}}}_s)^{-1} \cdot (\bar{\bar{\mathbf{I}}}_t - \eta_o \bar{\bar{\mathbf{Y}}}_s) = R_x \mathbf{u}_x \mathbf{u}_x + R_y \mathbf{u}_y \mathbf{u}_y \quad (55)$$

$$R_x = \frac{1 - j\eta_o B_s}{1 + j\eta_o B_s}, \quad R_y = \frac{1 + j\eta_o B_s}{1 - j\eta_o B_s}, \quad = 1/R_x \quad (56)$$

Here the real boundary susceptance  $B_s$  may have either sign. Denoting

$$\eta_o B_s = \tan \psi \quad (57)$$

we have

$$R_x = \frac{1 - j \tan \psi}{1 + j \tan \psi} = e^{-2j\psi}, \quad R_y = 1/R_x = e^{2j\psi} \quad (58)$$

The reflection dyadic now becomes

$$\bar{\bar{\mathbf{R}}} = e^{-2j\psi} \mathbf{u}_x \mathbf{u}_x + e^{2j\psi} \mathbf{u}_y \mathbf{u}_y = \cos 2\psi \bar{\bar{\mathbf{I}}}_t - j \sin 2\psi \bar{\bar{\mathbf{K}}} \quad (59)$$

The polarisation of the reflected field depends on the orientation of the polarisation of the incident field with respect to the axes of the PAB.

As an example, let us choose the surface susceptance to have the special value

$$\psi = \pi/8, \quad \eta_o B_s = \tan(\pi/8) = \sqrt{2} - 1 \quad (60)$$

in which case the reflection dyadic takes the simple form

$$\bar{\bar{\mathbf{R}}} = \frac{1}{\sqrt{2}} (\bar{\bar{\mathbf{I}}}_t - j\bar{\bar{\mathbf{K}}}) = \sqrt{-j} (\mathbf{u}_x \mathbf{u}_x + j\mathbf{u}_y \mathbf{u}_y) \quad (61)$$

A linearly polarised incident field defined by the real field vector making an angle  $\varphi$  with the  $x$  axis,

$$\mathbf{E}^i = E(\mathbf{u}_x \cos \varphi + \mathbf{u}_y \sin \varphi) \quad (62)$$

is now transformed to an elliptically polarised reflected field vector  $\mathbf{E}^r = \bar{\bar{\mathbf{R}}} \cdot \mathbf{E}^i$ . Its polarisation depends on the angle  $\varphi$  as

$$\begin{aligned} \mathbf{E}^r &= E \bar{\bar{\mathbf{R}}} \cdot (\mathbf{u}_x \cos \varphi + \mathbf{u}_y \sin \varphi) \\ &= \sqrt{-j} E (\mathbf{u}_x \cos \varphi + j\mathbf{u}_y \sin \varphi) \end{aligned} \quad (63)$$

The axes of the ellipse defined by  $\mathbf{E}^r$  lie along the  $x$  and  $y$  axes and the axial ratio equals  $|\tan \varphi|$ . It is obvious that all possible axial ratios can be reached by choosing the angle  $\varphi$  in the region  $-\pi/2 \leq \varphi \leq \pi/2$ . The geometric construction of the ellipse in Fig. 1 reveals the relation between incident and reflected field polarisations. The handedness of the reflected field  $\mathbf{E}^r$  depends on the sign of  $\varphi$  in the region  $-\pi/2 \leq \varphi \leq \pi/2$ , so that, for positive  $\varphi$ , the field is right-handed and, for negative  $\varphi$ , left-handed with respect to the direction of propagation  $-\mathbf{u}_z$ .

Obviously, the PAB surface defined by (57) and (60) can be applied for transforming the linear polarisation of an incident field to an arbitrarily elliptical one, with either handedness, by reflection. The axial ratio of the reflected field can be simply changed by rotating the PAB surface to change the angle  $\varphi$ . The transformation can also be reversed. Assuming an elliptically polarised incident field and PAB directed so that its  $x$  and  $y$  axes are parallel to the axes of the ellipse, it can be expressed as

$$\mathbf{E}^i = E(\mathbf{u}_x \cos \xi + j\mathbf{u}_y \sin \xi) \quad (64)$$

The reflected field now becomes

$$\mathbf{E}^r = \bar{\bar{\mathbf{R}}} \cdot \mathbf{E}^i = \sqrt{-j} E (\mathbf{u}_x \cos \xi - \mathbf{u}_y \sin \xi) \quad (65)$$

which is linearly polarised and makes the angle  $-\xi$  with the  $x$  axis. This angle depends on the axial ratio  $|\tan \xi|$  and handedness (sign of  $\xi$ ) of the incident field. Transformation from any elliptic to any other elliptic polarisation would require two reflections from two suitably oriented PAB surfaces.

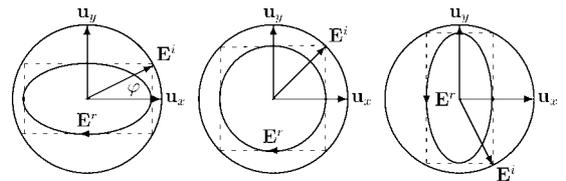
#### 5 Surface waves at PAB plane

As another problem involving the PAB surface, let us study the existence of surface waves in the half space  $z < 0$  above the PAB plane. Such fields are exponentially decaying for  $z \rightarrow -\infty$  and have the form

$$\mathbf{E}(r) = \mathbf{E} e^{\alpha z} e^{-j\gamma \mathbf{v} \cdot \mathbf{r}}, \quad \mathbf{H}(r) = \mathbf{H} e^{\alpha z} e^{-j\gamma \mathbf{v} \cdot \mathbf{r}} \quad (66)$$

Here  $\mathbf{v}$  is a unit vector satisfying  $\mathbf{v} \cdot \mathbf{u}_z = 0$  and both  $\alpha$  and  $\gamma$  are assumed real and positive for the lossless boundary. They are related by the plane-wave condition

$$k_o^2 = \gamma^2 - \alpha^2 \quad (67)$$



**Fig. 1** For normalised incident field  $\mathbf{E}_i$  (linearly polarised at angle  $\varphi$  to  $\mathbf{u}_x$ ) field  $\mathbf{E}^r$  reflected from PAB surface becomes elliptically polarised; the axial ratio of the reflected field equals  $|\tan \varphi|$  and the handedness, shown by the arrowheads, depends on the sign of  $\tan \varphi$

Inserting (66) in the Maxwell equations yields

$$\alpha \mathbf{u}_z \times \mathbf{E} - j\gamma \mathbf{v} \times \mathbf{E} = -jk_o \eta_o \mathbf{H} \quad (68)$$

$$\alpha \mathbf{u}_z \times \mathbf{H} - j\gamma \mathbf{v} \times \mathbf{H} = j(k_o/\eta_o) \mathbf{E} \quad (69)$$

which can be separated in axial and transverse components as

$$j\alpha \mathbf{u}_z \times \mathbf{E}_t + \gamma \mathbf{v} \times \mathbf{u}_z \mathbf{E}_z = k_o \eta_o \mathbf{H}_t \quad (70)$$

$$-j\alpha \eta_o \mathbf{u}_z \times \mathbf{H}_t - \gamma \eta_o \mathbf{v} \times \mathbf{u}_z \mathbf{H}_z = k_o \mathbf{E}_t \quad (71)$$

$$\gamma \mathbf{u}_z \cdot \mathbf{v} \times \mathbf{E}_t = k_o \eta_o \mathbf{H}_z \quad (72)$$

$$\gamma \eta_o \mathbf{u}_z \cdot \mathbf{v} \times \mathbf{H}_t = -k_o \mathbf{E}_z \quad (73)$$

Substituting the axial field components from (72) and (73) in (70) and (71), we obtain

$$j\alpha k_o \mathbf{u}_z \times \mathbf{E}_t = (k_o^2 \mathbf{v}\mathbf{v} - \alpha^2 \mathbf{u}_z \mathbf{u}_z \times \mathbf{v}\mathbf{v}) \cdot \eta_o \mathbf{H}_t \quad (74)$$

$$-j\alpha k_o \eta_o \mathbf{u}_z \times \mathbf{H}_t = (k_o^2 \mathbf{v}\mathbf{v} - \alpha^2 \mathbf{u}_z \mathbf{u}_z \times \mathbf{v}\mathbf{v}) \cdot \mathbf{E}_t \quad (75)$$

We can show that these are actually the same equation, i.e. (74) can be rewritten in the form of (75), and conversely. As a second condition the PAB relation for the transverse fields is needed,

$$-\mathbf{u}_z \times \mathbf{H} = jB_s \bar{\bar{\mathbf{K}}} \cdot \mathbf{E} \quad (76)$$

where the susceptance  $B_s$  is again a real number. This inserted in (74) and (75) leaves us with

$$\left( \frac{\alpha k_o}{\eta_o B_s} \bar{\bar{\mathbf{K}}} + k_o^2 \mathbf{v}\mathbf{v} - \alpha^2 \mathbf{u}_z \mathbf{u}_z \times \mathbf{v}\mathbf{v} \right) \cdot \eta_o \mathbf{H}_t = 0 \quad (77)$$

$$(\alpha k_o \eta_o B_s \bar{\bar{\mathbf{K}}} + k_o^2 \mathbf{v}\mathbf{v} - \alpha^2 \mathbf{u}_z \mathbf{u}_z \times \mathbf{v}\mathbf{v}) \cdot \mathbf{E}_t = 0 \quad (78)$$

These describe two eigenproblems with eigenvalues  $\alpha$  and transverse eigenvectors  $\mathbf{E}_t$ ,  $\mathbf{H}_t$ . Both lead to the same characteristic equation by requiring that the determinant of the multiplying dyadic in (77) or (78) vanishes and, ignoring the case  $\alpha = 0$

$$\alpha^2 + \alpha \frac{k_o(\eta_o^2 B_s^2 + 1)}{\eta_o B_s (\bar{\bar{\mathbf{K}}}: \mathbf{v}\mathbf{v})} + k_o^2 = 0 \quad (79)$$

This has two solutions

$$\alpha_{\pm} = -\frac{k_o}{2\eta_o B_s \bar{\bar{\mathbf{K}}}: \mathbf{v}\mathbf{v}} \times \left( \eta_o^2 B_s^2 + 1 \pm \sqrt{(\eta_o^2 B_s^2 + 1)^2 - 4\eta_o^2 B_s^2 (\bar{\bar{\mathbf{K}}}: \mathbf{v}\mathbf{v})^2} \right) \quad (80)$$

Denoting by  $\varphi$  the angle between the wave direction  $\mathbf{v}$  and the axial direction  $\mathbf{u}_x$  of the PAB

$$\mathbf{v} \cdot \mathbf{u}_x = \cos \varphi \Rightarrow \bar{\bar{\mathbf{K}}}: \mathbf{v}\mathbf{v} = \cos^2 \varphi - \sin^2 \varphi = \cos 2\varphi \quad (81)$$

the eigenvalues (80) can be expressed as

$$\alpha_{\pm} = -\frac{k_o}{2\eta_o B_s \cos 2\varphi} \times \left( \eta_o^2 B_s^2 + 1 \pm \sqrt{\eta_o^4 B_s^4 - 2\eta_o^2 B_s^2 \cos 4\varphi + 1} \right) \quad (82)$$

Assuming  $B_s > 0$  without losing the generality and requiring that the wave be a proper surface wave,  $\alpha > 0$ , we arrive at the restriction  $\cos 2\varphi < 0$ . This means that

waves propagating in the sector  $\pi/4 < \varphi < 3\pi/4$  or the sector  $-3\pi/4 < \varphi < -\pi/4$  are proper surface waves while those propagating in the other sectors are leaky waves. Thus, surface waves are propagating in the sectors around the directions  $\mathbf{v} = \pm \mathbf{u}_y$ , while the leaky waves are propagating around the directions  $\mathbf{v} = \pm \mathbf{u}_x$ .

Let us assume  $\eta_o B_s \neq \pm 1$  and consider the case  $\varphi = \pi/2$ , or wave propagating in the direction  $\mathbf{u}_y$ , as  $\mathbf{v} = \mathbf{u}_y$ . From (80) we obtain the solutions

$$\alpha_+ = k_o \eta_o B_s, \quad \alpha_- = k_o / \eta_o B_s \quad (83)$$

for the rate of exponential decay of the field along the negative  $z$  axis. The corresponding two eigenfields denoted by the respective subscript satisfy, from (78), (77), (72) and (73), the conditions

$$\begin{aligned} \mathbf{u}_x \cdot \mathbf{H}_+ &= 0, & \mathbf{u}_y \cdot \mathbf{E}_+ &= 0, \\ H_{+z} &= -\frac{\gamma_+}{k_o \eta_o} \mathbf{u}_x \cdot \mathbf{E}_+, & E_{+z} &= 0 \end{aligned} \quad (84)$$

$$\begin{aligned} \mathbf{u}_y \cdot \mathbf{H}_- &= 0, & \mathbf{u}_x \cdot \mathbf{E}_- &= 0, \\ H_{-z} &= 0, & E_{-z} &= \frac{\gamma_- \eta_o}{k_o} \mathbf{u}_x \cdot \mathbf{H}_- \end{aligned} \quad (85)$$

Let us denote the two eigenwaves as  $\text{TE}_z$  and  $\text{TM}_z$  waves corresponding to the respective  $+$  and  $-$  subscripts. The total fields are

$$\mathbf{E}_+ = \mathbf{u}_x \mathbf{E}_+, \quad \mathbf{H}_+ = \mathbf{u}_y j B_s \mathbf{E}_+ - \mathbf{u}_z \frac{\gamma_+}{k_o \eta_o} \mathbf{E}_+, \quad \text{TE}_z \quad (86)$$

$$\mathbf{H}_- = \mathbf{u}_x \mathbf{H}_-, \quad \mathbf{E}_- = \mathbf{u}_y j \frac{H_-}{B_s} + \mathbf{u}_z \frac{\gamma_- \eta_o}{k_o} \mathbf{H}_-, \quad \text{TM}_z \quad (87)$$

and they satisfy the orthogonality conditions

$$\mathbf{E}_+ \cdot \mathbf{H}_+ = 0, \quad \mathbf{E}_- \cdot \mathbf{H}_- = 0, \quad \mathbf{E}_+ \cdot \mathbf{E}_- = 0, \quad \mathbf{H}_+ \cdot \mathbf{H}_- = 0 \quad (88)$$

The propagating power has no  $z$  component. This is seen by expanding

$$\begin{aligned} \mathbf{u}_z \cdot \mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) &= -\mathbf{E} \cdot (\mathbf{u}_z \times \mathbf{H})^* e^{2\alpha z} \\ &= \mathbf{E} \cdot \bar{\bar{\mathbf{Y}}}_s^* \cdot \mathbf{E}^* e^{2\alpha z} = -j B_s (\mathbf{E} \cdot \bar{\bar{\mathbf{K}}} \cdot \mathbf{E}^*) e^{2\alpha z} \end{aligned} \quad (89)$$

whence the real part of the  $z$  component of the complex Poynting vector vanishes for any surface wave. The expressions for the power propagating in the eigenwaves are

$$\begin{aligned} P_+ &= \frac{1}{2} \Re \{ \mathbf{E}_+ \times \mathbf{H}_+^* \} = \frac{1}{2} \Re \{ \mathbf{u}_x \mathbf{E}_+ \times \mathbf{u}_z \mathbf{H}_{+z}^* \} \\ &= \mathbf{u}_y \frac{\gamma_+}{2k_o \eta_o} |\mathbf{E}_+|^2 \end{aligned} \quad (90)$$

$$\begin{aligned} P_- &= \frac{1}{2} \Re \{ \mathbf{E}_- \times \mathbf{H}_-^* \} = \frac{1}{2} \Re \{ \mathbf{u}_z \mathbf{E}_{-z} \times \mathbf{u}_x \mathbf{H}_-^* \} \\ &= \mathbf{u}_y \frac{\gamma_- \eta_o}{2k_o} |\mathbf{H}_-|^2 \end{aligned} \quad (91)$$

and their directions coincide with those of the corresponding propagation vectors  $\mathbf{u}_y \gamma_{\pm}$ . There is no power coupling between the two eigenwaves.

## 6 Conclusion and discussion

A novel class of impedance boundaries, labelled as that of perfect anisotropic boundary (PAB), was introduced as based on a classification of two-dimensional dyadics. Realisation of PAB admittance in terms of a slab of

wave-guiding metamaterial was also suggested. As characteristic properties of PAB, polarisation transformation in reflection and existence of double surface waves were described.

Polarisation transformation appears to be an inherent property of the PAB surface. By a proper choice of the PAB susceptance, a linearly polarised field can be transformed to an elliptically polarised field which has axial ratio and handedness that can be continuously changed to any value by rotating the PAB structure. As any elliptic polarisation can be similarly transformed to linear polarisation, it can be further transformed to any other elliptic polarisation after reflection from a second PAB surface.

The second interesting property of the PAB surface is its capability of supporting two simultaneous surface waves. The surface waves are restricted to propagate in two sectors on the surface, while, in the other two sectors, there exist two leaky waves. In contrast, the conventional impedance boundary supports one surface wave and one leaky wave to every direction of propagation. This property may create some interesting applications in antenna design. For example, as the two surface waves have orthogonal polarisations, it appears possible to change the polarisation of radiation from a surface-wave antenna by a rotation in the feeding structure. Similar property can be applied to a leaky-wave antenna. Also, choosing  $\eta_0 B_s = 1$ , in which case the two eigenwaves have coinciding eigenvalues  $\alpha_+ = \alpha_- = k_o$  and propagation factors  $\gamma_+ = \gamma_- = 2k_o$ , the propagating field can be any combination of  $TE_z$  and  $TM_z$  fields. This fact could give

more freedom in tailoring polarisation properties of the radiated fields.

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