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LETTER

Power spectra of self-organized critical sandpiles

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Abstract. We analyse the power spectra of avalanches in two classes of self-organized critical sandpile models, the Bak–Tang–Wiesenfeld model and the Manna model. We show that these decay with a $1/f^{\alpha}$ power law, where the exponent value α is significantly smaller than 2 and equals the scaling exponent relating the avalanche size to its duration. We discuss the basic ingredients behind this result, such as the scaling of the average avalanche shape.

Keywords: self-organized criticality (theory), sandpile models (theory)

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Sandpile models were introduced almost twenty years ago as a paradigmatic example of self-organized criticality (SOC) [1], the tendency of slowly driven dissipative systems to display a scale free avalanche response. Such ideas have had an enormous impact in different fields, ranging from magnetic systems [2], superconductors [3] and mechanics [4, 5], to geophysics and plasma physics, including in particular the magnetosphere [6, 8].[4] The influence also extends beyond physics, to for example biology [9], human (heart) physiology [10] and cognitive processes or neuroscience [11].

The reason for this success lies in the wide variety of non-equilibrium systems displaying an avalanche response to an external driving. One of the primary aims of SOC was, originally, to explain the wide occurrence of $1/f^\alpha$ noise in natural phenomena, through a direct relation between avalanche scaling and spectral properties [1]. This idea was soon refuted when two groups [12, 13] published works independently claiming that sandpile models should lead instead to a Lorentzian spectrum that is decaying as $1/f^2$ at large frequencies. The theoretical arguments were supported by numerical simulations on relatively small system sizes [12, 13]. A non-trivial $1/f^\alpha$ decay in power spectra has only been found in non-critical sandpiles [14], or with an alternative definition of the noise signal [15], but not in standard cases such as in the original Bak–Tang–Wiesenfeld (BTW) model [1] and the stochastic Manna model [16].

Sandpile models represent a useful idealization of avalanche propagation, capturing the main ingredients of this process: a slow external driving, a local threshold—or non-linearity—for the dynamics and a dissipation mechanism. While a complete exact solution of sandpile models is possible only in some particular cases [17], the origin of the scaling behaviour is now well understood in the realm of non-equilibrium critical phenomena [18]–[20]. Systems presenting a transition from an absorbing state to a moving phase, or similarly a depinning transition [21, 22], can be turned into SOC under a suitable combination of a driving and a dissipation mechanism [18]–[21]. Conversely, criticality in sandpile models can be related to an underlying depinning critical point [23, 24]. The scaling of the power spectrum (PS) in sandpile models can be contrasted to avalanche induced crackling noise, which is typically characterized by a power law distribution of amplitudes and by a non-trivial $1/f^\alpha$ spectrum [25]. The most studied condensed matter examples include Barkhausen noise in ferromagnets [2] and acoustic emission in fracture [4, 5] and plasticity [26].

In this letter, we show that, notwithstanding previous beliefs, classical sandpile models display non-trivial $1/f^\alpha$ spectra. $\alpha < 2$ depends on the model and dimensions. We compute by means of numerical simulations the avalanche spectrum of two classes of sandpile models: the original two-dimensional BTW sandpile model and the stochastic two-state Manna model, in one, two and three dimensions (1D, 2D, 3D). These two models are now known to be in different universality classes. A further difference between the two classes of models is that stochastic sandpiles obey finite size scaling while the BTW model displays multiscaling [27]. We find that the power spectrum decays as $P(f) \sim f^{-\alpha}$, with $\alpha = 1.59 \pm 0.05$ for BTW and $\alpha = 1.44 \pm 0.05$, $\alpha = 1.77 \pm 0.05$, $\alpha = 1.9 \pm 0.1$ for Manna in with dimensionality $d = 1, 2, 3$, respectively.

The central idea as regards why SOC models can exhibit varying α, with the details depending on the dimension and universality class, is based on self-affine fractal dynamics.

[4] The role of SOC in (laboratory) plasma, solar and magnetospheric physics has been discussed in [7], containing numerous articles that refer to modelling and empirical data, and also to power spectra.

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Consider the time series $V(t)$, which records the number of 'topplings' (local relaxation events) taking place in the sandpile during each parallel update of the whole lattice, one such update defining the unit of time. An avalanche is defined here as a connected sequence of non-zero values of $V(t)$. If the average size (i.e. the total number of topplings) of such avalanches of duration T scales as $\langle s(T) \rangle \sim T^{\gamma_{st}}$ and the dynamics is self-similar, then the average avalanche shape $V(T,t)$ of avalanches of duration T should follow

$$V(T,t) = T^{\gamma_{st}-1} f_{shape}(t/T), \tag{1}$$

where $f_{shape}(x)$ is a scaling function [28]. The stationary correlation function is defined as

$$C(\theta) = \int V(t)V(t+\theta)\,dt, \tag{2}$$

from which the total energy is obtained as the $\theta = 0$ component, $E = C(0)$. Next, consider the correlation function $C(\theta|s)$ of avalanches of a given size s, averaged over all such avalanches. The corresponding energy spectrum of avalanches of size s, $E(f|s)$, is obtained by cosine transformation,

$$E(f|s) = \int_0^\infty C(\theta|s)\cos(f\theta)\,d\theta. \tag{3}$$

This will scale as

$$E(f|s) = s^2 g_E(f^{\gamma_{st}} s), \tag{4}$$

where $g_E(x)$ is another scaling function [13, 28]. The form of the power spectrum is obtained by averaging $E(f|s)$ over the avalanche size probability distribution $D(s) \sim s^{-\tau}$, $P(f) = \int D(s)E(f|s)\,ds$. The integral is bounded by the upper cut-off s^*, so that

$$P(f) = f^{-\gamma_{st}(3-\tau)} \int^{s^* f^{\gamma_{st}}} dx\, x^{2-\tau} g_E(x). \tag{5}$$

If the integral in equation (5) is convergent, we obtain $\alpha = \gamma_{st}(3-\tau)$ (as originally derived in [29]). In the opposite case, the final result crucially depends on the asymptotic behaviour of $g_E(x)$. Kertesz and Kiss assumed $g_E(x) \propto 1/(1 + x^{2/\gamma_{st}})$, obtaining $\alpha = 2$ [13]. Jensen *et al* approximate the avalanche shape with a box function, which implies $g_E(x) \propto (1 - \cos(x^{1/\gamma_{st}}))/x^{2/\gamma_{st}}$, yielding again $\alpha = 2$ [12]. More recently, Kuntz and Sethna [28] noticed that if the toppling dynamics in the avalanche is a local process, the released energy is an extensive function of the size s, or $E(f|s) \sim s$. From equation (4) it thus follows that $g_E(x) \sim A/x$. This implies that for $\tau < 2$ (which is the case for sandpile models), the integral in equation (5) is dominated by the frequency dependent upper cut-off, yielding $\alpha = \gamma_{st}$.

Here we analyse numerically the above set of four test-cases. We measure the shape of the pulse associated with an avalanche in the models and the scaling behaviour of the avalanche size for a given duration and compute the power spectra. Sandpile models are defined on a d-dimensional hypercubic lattice. On each site i of the lattice the height is an integer variable z_i. At each step the system is driven; a grain is dropped on a randomly chosen site raising its height by one unit ($z_i \to z_i + 1$). When one of the sites reaches or exceeds a threshold z_c a 'toppling' occurs: $z_i = z_i - z_c$ and $z_j = z_j + 1$, where j represents the nearest neighbour sites for site i. In the BTW model $z_c = 2d$ and each nearest

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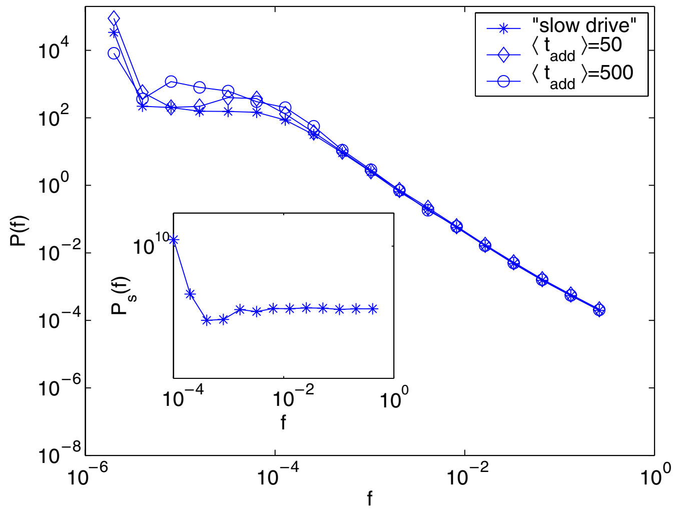

Figure 1. Main figure: a comparison of the power spectra of the 2D Manna model, $L = 256$, with slow and continuous drives. In the continuous drive case, the time intervals t_{add} between successive grain additions are taken from a Poisson distribution with two different averages. Notice how the high frequency part of the PS is independent of the drive. Inset: the power spectrum of the avalanche size time series displaying white noise character.

neighbour receives a grain after the toppling of the site i. In the Manna model $z_{\mathrm{c}} = 2$ and therefore only two randomly chosen neighbouring sites receive a grain. A toppling can induce nearest neighbour sites to topple in their turn and so on, until all the lattice sites are below the critical threshold. This process defines an avalanche. We use parallel dynamics, meaning that every overcritical site topples when the lattice is updated, one such update defining the unit of time. The slow driving condition implies that grains are added only when all the sites are below the threshold. Grains can leave the system from the open boundaries. After a transient, sandpile models reach a steady state with avalanches of all sizes. Here we consider linear system sizes ranging from $L = 1024$ to 16 384 in 1D, from $L = 64$ to 2048 in 2D and $L = 32$ to 256 in 3D.

Power spectra are measured by computing the absolute square of the fast Fourier transform (FFT) of the signal produced by the number of toppling events $V(t)$ as a function of time. To check that $P(f)$ reflects only correlations within avalanches, we also compute the power spectrum $P_{\mathrm{s}}(f)$ of the avalanche size s time series, which indeed turns out to have a white noise character; see the inset of figure 1. Waiting times between avalanches have no effect on the scaling of the high frequency parts of the spectrum, reflecting the internal correlations of individual avalanches (i.e. these frequencies correspond to timescales smaller than that of the duration of the longest avalanche). We have checked this e.g. by inserting a constant number of zeros between every successive two avalanches, and by means of a continuous drive with Poisson-distributed time intervals between grain additions; see figure 1. Therefore in what follows, the spectra are computed from the case where the slow driving condition is applied. In figures 2 and 3 we display the power spectra $P(f)$ and $\langle s(1/T)\rangle$ of the Manna and BTW models for different system sizes. While for the smallest lattices the power spectrum might be fitted with a Lorentzian,

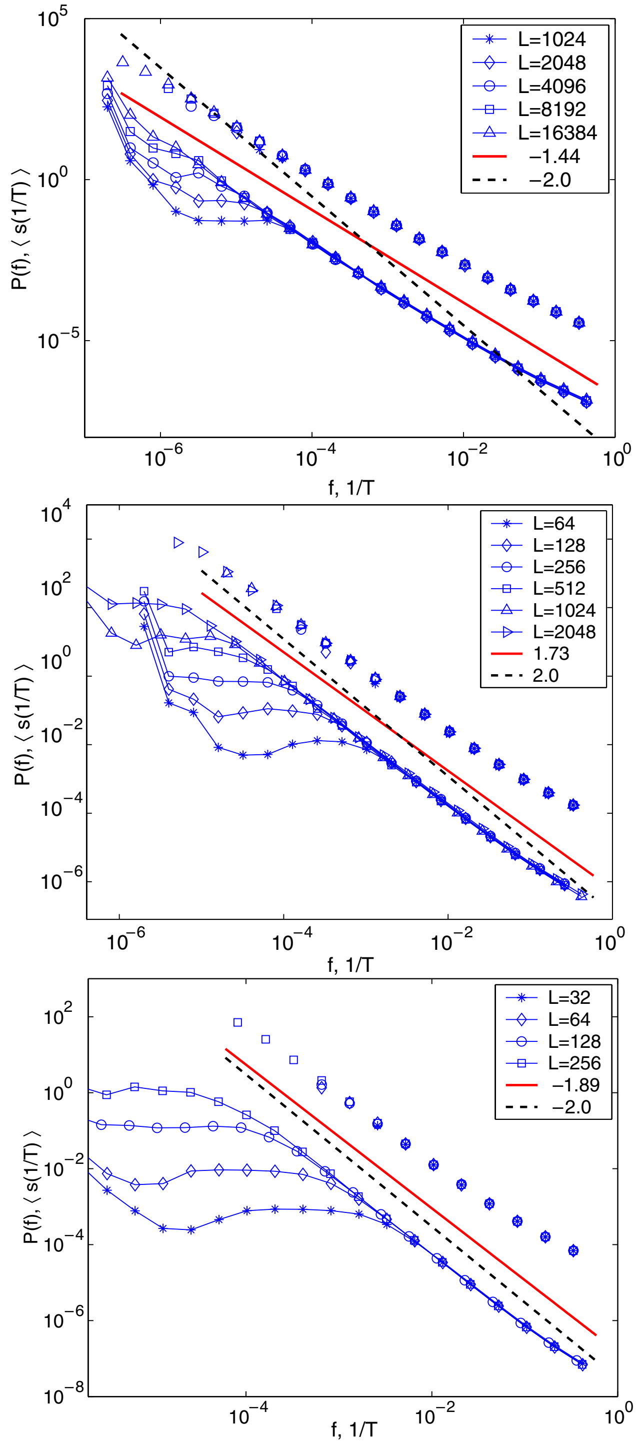

Figure 2. The scaling of the power spectra (symbols connected by a line) and $\langle s(1/T) \rangle$ (symbols without a connecting line) of the Manna sandpile model for different system sizes, Manna 1D (top), 2D (middle) and 3D (bottom). In all three cases, the slope of the spectrum is significantly different from 2 (dashed line).

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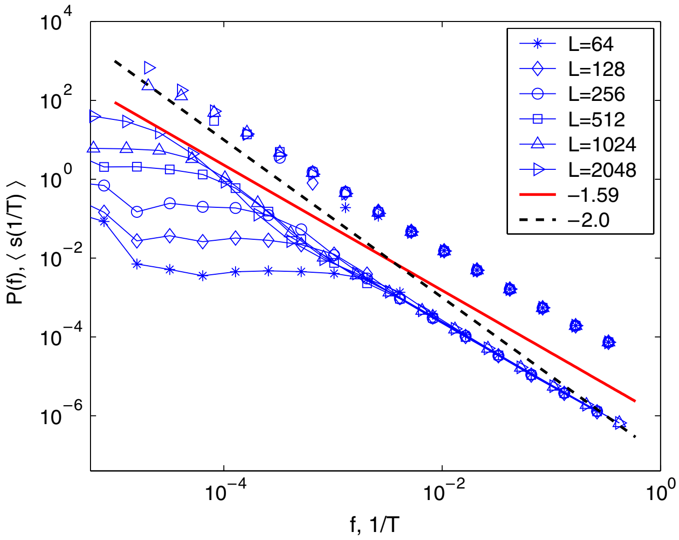

Figure 3. The scaling of the power spectrum (symbols connected by a line) and $\langle s(1/T)\rangle$ (symbols without a connecting line) of the BTW model in 2D for different system sizes. The slope of the high frequency part of the power spectrum is again significantly different from 2 (dashed line).

for larger system sizes the tails are definitely not scaling as $1/f^2$. Instead, by fitting to the scaling parts of the power spectra (frequencies higher than the one corresponding to the duration of the longest avalanche and lower than the inverse of a crossover time after which the avalanches will have a self-similar structure), we find $\alpha = 1.59 \pm 0.05$ for BTW and $\alpha = 1.44 \pm 0.05$, $\alpha = 1.77 \pm 0.05$ and $\alpha = 1.9 \pm 0.1$ for Manna in 1D, 2D and 3D, respectively. Note that the results are contrary to those in [12, 13], whose results were obscured by the small system sizes reachable at the epoch. Instead, at least for the Manna model, the scaling of the power spectra follows quite nicely that of $\langle s(1/T)\rangle \sim (1/T)^{-\gamma_{\rm st}}$, with $\gamma_{\rm st} = 1.44\pm0.05$, $\gamma_{\rm st} = 1.73\pm0.05$ and $\gamma_{\rm st} = 1.9\pm0.1$ in 1D, 2D and 3D, respectively[5]. The PS of the BTW has a 'bump' for small frequencies (figure 3), which shifts to still smaller ones with increasing L. Furthermore, for BTW, $\langle s(1/T)\rangle$ exhibits slight curvature even for large avalanches, but still the agreement is fair.

In order to check that the observed results follow directly from the derivation outlined above, we compute the energy spectrum $E(f|s)$. The results reported in figure 4 confirm the scaling behaviour predicted by equation (4) with $g_E \sim 1/x$. For the avalanche shape, equation (1), the models show slightly different properties. Originally, [12] employed a simple box function to approximate $f_{\rm shape}$. This form is very far from the correct one, as is shown in the inset of figure 4. While the avalanche shape of the Manna model is symmetric, a more detailed look at the BTW model reveals that its avalanche shape cannot be rescaled as a function of duration T. The avalanches slowly develop an asymmetry, which could be related to the observations of multiscaling in the BTW model [27]. For the stochastic Manna sandpiles, in 1D the assumption of scale invariance holds nicely while in the 2D and 3D cases relatively small avalanches show a crossover behaviour so that

[5] One can derive from other measured quantities approximative values for $\gamma_{\rm st}$ using published data [30], and our values are in reasonable agreement.

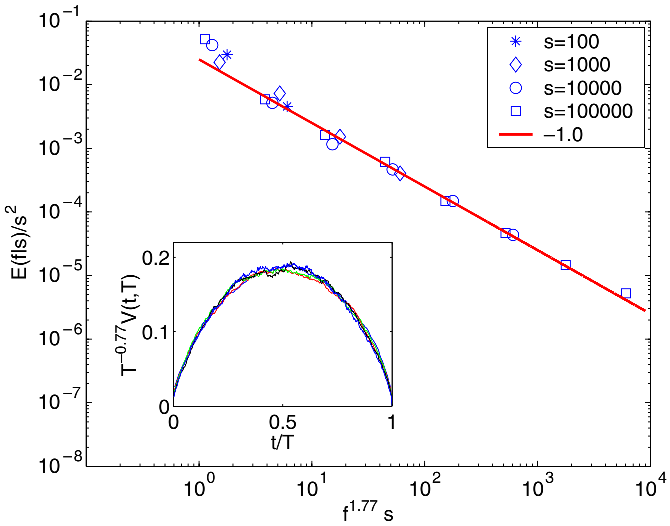

Figure 4. Main figure: the energy spectrum for different sizes s is collapsed according to equation (4), and the scaling function decays as $1/x$. Inset: the collapse of the average avalanche shape for avalanche durations ranging from $T = 200$ to 500. All data are for the 2D Manna model.

only around $T \sim 100$ is the scaling regime reached. This naturally implies corrections to scaling, but nevertheless $\alpha = \gamma_{\rm st}$ holds rather well (and in particular $\alpha < 2$).

It is theoretically important and intriguing that such a relation can be established between the sandpile critical exponents and the PS one. In spite of the relation between SOC and non-equilibrium phase transitions, we still lack analytical predictions for the critical exponents for $d < 4$, $d = 4$ being the upper critical dimension [20]. Thus, for $d \geq 4$ mean-field exponents are valid, so $\gamma_{\rm st} = \alpha = 2$. In two or three dimensions, however, we would generally expect $\alpha < 2$, and thus in many real physical systems the expectation would be the same.

To summarize, self-organized criticality leads rather generally to power spectra that exhibit $1/f^{\alpha}$ noise with $\alpha < 2$. This calls, perhaps, for a re-evaluation of experimental results in many cases, ranging from large systems met in solar and astrophysics to those in the laboratory and carrying over to the understanding of brain dynamics [11], biology and so forth. In other words, $\alpha < 2$ does not imply the absence of SOC, but instead may indicate exactly the contrary. There are also many theoretical issues that open up, including higher order power spectra [31]. In any case, we may safely conclude that the central point of the original paper by Bak, Tang and Wiesenfeld (i.e. the relation between avalanche scaling and a non-trivial power spectrum) was ultimately correct, contrary to what has been believed to be true from almost the beginning of the research on SOC and its applications to various phenomena.

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