Department of Mathematics and Systems Analysis

Numerical Solution of the Real-linear Equations of Electrical Impedance Tomography for Nonsmooth Conductivities

Allan Perämäki

Aalto University

DOCTORAL DISSERTATIONS

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Allan Perämäki

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Abstract

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Abstract

This thesis studies the numerical solution and convergence of a certain discretized real-linear Beltrami equation. This equation arises in the uniqueness proof by Astala and Päivärinta for the two-dimensional electrical impedance tomography problem with nonsmooth conductivities. The real-linear matrix equation appearing after discretizing the Beltrami equation is found to have the form appropriate for the application of the real-linear Generalized Minimal Residual (GMRES) method published by Eirola, Huhtanen and von Pfaler.

The findings include a fast numerical solution method for the discretized real-linear Beltrami equation, and an implementation of a reconstruction method based on the Astala-Päivärinta uniqueness proof. The solution of the discretized Beltrami equation is shown to converge to the correct solution as the grid is refined, including a convergence rate estimate.

For the real-linear GMRES method, the norms of the residuals are bounded in terms of a polynomial approximation problem on the complex plane resembling the situation of classical GMRES. Moreover, complex symmetric matrices are shown to possess a mathematical framework analogous to the classical Hermitian Lanczos framework.

Keywords inverse problem, Beltrami equation, iterative methods, discrete convergence, orthogonal polynomials, Jacobi matrix, condiagonalizable

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Tekijä
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Tiivistelmä
Väitoskirjassa tutkitaan erään diskretoidun reaalilineaarisen Beltramin yhtälön numeerista ratkaisemista ja suppenemista. Tämä yhtälö esiintyy Astalan ja Päivärinnan kaksiulotteisen sähköisen impedanssitomografian ongelman yksikäsitteisyystodistuksessa epäsileille johtavuuksille. Diskretoitua Beltramin yhtälöä vastaava reaalilineaarinen matriisiyhtälön nähdään olevan soveltuvaa muotoa Eirolan, Huhtasen ja von Pfalerin reaalilineaarisen GMRES (Generalized Minimal Residual) -menetelmän käytölle.

Tulokset sisältävät nopean numeerisen ratkaisumenetelmän diskretoidulle reaalilineaariselle Beltramin yhtälölle ja Astalan-Päivärinnan todistukseen perustuvan menetelmän toteutuksen. Diskretoidun Beltramin yhtälön ratkaisun osoitetaan suppenevan oikeaan ratkaisuun hilaa tihennettäessä sisältäen myös suppenemisnopeuden arvion.

Reaalilineaarisen GMRES-menetelmän jäännösvektoreiden normeille osoitetaan yläraja-arvio kompleksitason polynomiapproksimaatiotehtävän suhteen muistuttaen klassisen GMRES-menetelmän tilannetta. Lisäksi kompleksisymmetrisille matriiseille osoitetaan klassista hermiittisten matriisien Lanczosin matemaattista viitekehystä vastaavan kehyksen olemassaolo.

Avainsanat inversio-ongelma, Beltramin yhtälö, iteratiiviset menetelmät, diskreetti suppeneminen, ortogonaaliset polynomit, Jacobin matriisi, kondiagonalisoituva

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Preface

This thesis consists of four publications, three of which are co-authored. My share results from the research I accomplished from late 2009 to late 2011 at the Department of Mathematics and Systems Analysis at Aalto University.

I thank my advisor and co-author Marko Huhtanen for interesting research problems, ideas and guidance from which I have greatly benefited. I am grateful to my supervisor, and director of the department, Olavi Nevanlinna for the opportunity and conducive research environment. I also thank the co-authors of the second publication for the collaboration, especially Samuli Siltanen for the well thoughtout framework. Further thanks to the pre-examiners Martin Hanke-Bourgeois and Lothar Reichel for reading and commenting on the manuscript resulting in slight revision and improvement.

The research was partly supported by the Academy of Finland.

Espoo, Otaniemi, July 12, 2012,

Allan Perämäki

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Marko Huhtanen and Allan Perämäki. Numerical solution of the $\mathbb{R}$-linear Beltrami equation. *Mathematics of Computation*, Volume 81, Number 277, pages 387–397, doi:10.1090/s0025-5718-2011-02541-x, 2012.

II Kari Astala, Jennifer L. Mueller, Lassi Päivärinta, Allan Perämäki and Samuli Siltanen. Direct electrical impedance tomography for nonsmooth conductivities. *Inverse Problems and Imaging*, Volume 5, Number 3, pages 531–549, doi:10.3934/ipi.2011.5.531, 2011.

III Allan Perämäki. Convergence of a numerical solver for an $\mathbb{R}$-linear Beltrami equation. *BIT Numerical Mathematics*, Volume 52, Number 1, pages 155–178, doi:10.1007/s10543-011-0340-6, 2012.

IV Marko Huhtanen and Allan Perämäki. Orthogonal polynomials of the $\mathbb{R}$-linear generalized minimal residual method. *Submitted manuscript*, 19 pages, preprint: arXiv:1111.5167v2, December 14, 2011.

Author's Contribution

Publication I: "Numerical solution of the $\mathbb{R}$-linear Beltrami equation"

The author discovered the form of the equation yielding to fast numerical solution, the specific type of preconditioning applied in Section 5, carried out the numerical experiments and was responsible for writing Sections 2, 3 and 5.

Publication II: "Direct electrical impedance tomography for nonsmooth conductivities"

The author was involved in the Beltrami equation part of the work, wrote a large part of Section 5, wrote the corresponding `MATLAB` code and handled its integration with the boundary integral equation solver by Siltanen.

Publication III: "Convergence of a numerical solver for an $\mathbb{R}$-linear Beltrami equation"

This represents independent work by the author.

Publication IV: "Orthogonal polynomials of the $\mathbb{R}$-linear generalized minimal residual method"

The author is responsible for the lemmas, propositions, theorems and writing in Sections 3, 4.3 and 5.

1. Introduction

Large linear systems are ubiquitous in engineering and scientific computation, commonly arising from discretizing partial differential and integral equations in two or more dimensions. This thesis involves a particular type having its main application to a direct method of electrical impedancy tomography (EIT) in two dimensions. Although the EIT method is direct, it is an independent choice whether to solve the corresponding discretized linear systems by direct or iterative methods. The latter are called for systems too large to be solved by (direct) methods based on matrix factorization. Krylov subspace methods became popular in the 1970s and are indispensable tools these days. Given an initial guess for the solution, these methods then typically give an ever more accurate approximate solution after each (iteration) step of the method. An attractive feature is that they do not require explicit access to the coefficient matrix entries. Only the ability to compute matrix-vector products with the coefficient matrix is needed. Often such products can be computed fast, for example, when the matrix has (relatively) few nonzero entries or it has otherwise exploitable structure. The latter is the case in the EIT application. Provided the rate of convergence of the method is fair, vastly larger linear systems are numerically solvable compared to using Gaussian elimination.

To be practical, the rate of convergence and the computational cost of each iteration are of central importance. In many cases the linear systems arising in practice consist of real number coefficients and data. As a result, these (Krylov subspace) methods were initially formulated in terms of real numbers only and computer software was written accordingly. However, complex linear systems do occur in practice. In the early days they were solved by separating out the real and imaginary parts thereby rewriting the initially complex system as a real system. The rate of convergence was typically poor and complex systems were considered hard to solve by Krylov subspace methods. Eventually the importance of the complex viewpoint became appreciated for its faster convergence and this thesis continues its advocation. The complex equations arising in the direct EIT method pose an additional difficulty, specifically that the associated linear operator is homogeneous with respect to the real numbers only and not the full set of complex numbers.

Electrical impedance tomography is an imaging modality where the impedivity distribution inside an object is reconstructed by injecting currents and measuring voltages on the boundary of the object. Applications include geophysical prospection, industrial process monitoring, detecting cracks or impurities in materials, and medical imaging. Several medical applications have been proposed including lung function monitoring of intensive care patients and detection of breast cancer. While many experimental medical systems have been built since the 1980s, in recent years commercially available products intended for wide clinical use have been introduced as well.

In medicine, the injected currents are small and harmless to the patient. Almost real-time imaging with tens of frames per second can be achieved with devices small enough to reside by the bedside, or small enough to be handheld, and relocated at ease. This makes EIT attractive especially for constant monitoring. In contrast, X-ray computed tomography (CT) or magnetic resonance (MRI) imaging requires moving the patient to the radiology department and the number of obtainable images is limited. Additionally, CT is limited by irradiation concerns. EIT also has a potential application to early diagnosis of breast cancer since the impedivity of abnormal breast tissue differs compared to normal tissue. The standard screening technique is X-ray mammography which has a high detection rate, but up to 80% of positive tests turn out to test negative for malignancy in biopsies [25]. The main drawback of EIT is the rather poor resolution of the reconstructed images. For example, the commercially available two-dimensional EIT imaging device by Dräger Medical reconstructs a cross-sectional image of a patient's chest using 340 pixels in total after which post-processing is applied to improve visual presentation on the display [24]. EIT is an extremely ill-posed inverse problem and high resolution imaging would require unpractically accurate current injection, voltage measurement, low noise levels, and large number of electrodes attached to the boundary of the object. A localized, but relatively large in magnitude, change to the impedivity distribution may have an unmeasurably small effect to the voltages on the boundary. The accuracy of the positioning of the electrodes on the boundary is also a concern as well as that in two-dimensional imaging the injected currents are not confined to travel on the plane determined by the electrodes.

The direct EIT reconstruction method given in this thesis is based on the uniqueness proof by Astala and Päivärinta [1]. The main motivation was in improving imaging quality, a reasonable objective considering the modest assumptions on the smoothness of the conductivity the proof requires compared to prior work. Unfortunately, it has appeared difficult to discern in what situations, if any, the hoped for quality improvement exists. On the computational side, the most demanding discretized equations of the EIT method are its real-linear Beltrami equations. The thesis gives a fast numerical solution method for them and studies their convergence, both from the point of view of discrete convergence theory and matrix computations.

The rest of the overview is organized as follows. Chapter 2 introduces $\mathbb{R}$-linear systems of equations and discusses their (iterative) numerical solution methods. Chapter 3 discusses the direct EIT reconstruction method of Publication II. Finally, Chapter 4 presents a summary of the findings in the included publications.

2. $\mathbb{R}$-linear systems of equations

After the equations involved in the direct EIT method have been discretized, we are facing a problem of the form

$$\kappa z + M_{\#}\overline{z} = b, \tag{2.1}$$

where $z \in \mathbb{C}^n$ is the unknown and $\kappa \in \mathbb{C}$, $M_{\#} \in \mathbb{C}^{n\times n}$, $b \in \mathbb{C}^n$ are given. For the EIT method we have $\kappa = 1$, but we retain the slightly more general form here in keeping with the publication that introduced a generalized method of minimal residual (GMRES) for its solution [8]. The method presented in [8] minimizes the norm of the residual $r = b - \kappa z - M_{\#}\overline{z}$ when z belongs to the following $\mathbb{C}$-linear subspace spanned by k vectors

$$\mathcal{K}_k(M_{\#}, b) = \text{span}\left\{b, M_{\#}\overline{b}, M_{\#}\overline{M_{\#}}b, M_{\#}\overline{M_{\#}}M_{\#}\overline{b}, \ldots\right\}. \tag{2.2}$$

Note that classical GMRES [22] is not immediately applicable to (2.1) due to the complex-conjugation. It is applicable after a reformulation as a real system.

Considering the real and imaginary parts of (2.1) separately, we arrive at the real system

$$\begin{bmatrix} \kappa_1 I + \operatorname{Re} M_{\#} & -\kappa_2 I + \operatorname{Im} M_{\#} \\ \kappa_2 I + \operatorname{Im} M_{\#} & \kappa_1 I - \operatorname{Re} M_{\#} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{2.3}$$

where $\kappa = \kappa_1 + i\kappa_2$, $b = b_1 + ib_2$, $z = x + iy$. We denote the coefficient matrix by A_κ. The spectrum of A_0 is

$$\sigma(A_0) = \left\{\lambda \in \mathbb{C} \,\middle|\, \lambda^2 \in \sigma(M_{\#}\overline{M_{\#}})\right\}, \tag{2.4}$$

and as such the symmetrization of the spectrum with respect to 0 is inevitable leading to poor convergence [11]. For A_1 the spectrum is right shifted by 1 and does not necessarily have much negative impact on convergence rate as numerically demonstrated in Publication I. Of course, the actual convergence behaviour of a particular problem instance can depend heavily on properties of the matrix other than the spectrum [6] [21].

2.1 Preconditioned $\mathbb{R}$-linear system

For simplicity, we now let $\kappa = 1$. Publication I proposes solving (2.1) by preconditioning it (from the right) by substituting $z = w - M_{\#}\overline{w}$ resulting in the $\mathbb{C}$-linear system

$$(I - M_{\#}\overline{M_{\#}})w = b, \qquad z = w - M_{\#}\overline{w}. \tag{2.5}$$

Classical GMRES is now applicable. By inspecting the spectra of the coefficient matrices in (2.3) and (2.5), two cases can be identified where we expect (2.5) to result in fewer iterations:

1. The norm of $M_{\#}$ is small,
2. $M_{\#}$ is complex skew-symmetric.

The first case is clear considering $\|A_0\| = \|M_{\#}\|$ and $\|M_{\#}\overline{M_{\#}}\| \leq \|M_{\#}\|^2$ so that we expect GMRES applied to (2.5) to take approximately half the number of iterations of GMRES applied to (2.3). Small $\|M_{\#}\|$ is in fact responsible for the numerical results in Publication I and the reason to applying the preconditioning (2.5) in Publication II as well. In the EIT application it would be possible to consider extremely high-contrast conductivities resulting in large $\|M_{\#}\|$, although impractical at least in medical imaging.

In the second case we have $I - M_{\#}\overline{M_{\#}} = I + M_{\#}M_{\#}^*$. The system (2.5) is therefore Hermitian positive-definite with the spectrum contained in $[1, \infty)$. The conjugate-gradient method is thus well-suited to solve the problem. For further insight to convergence, we compare the convergence rate of GMRES applied to the systems (2.3) and (2.5). (Of course, in the latter system GMRES reduces to the minimal residual method MINRES.) We do this by means of estimated asymptotic convergence factors [6] for both methods. For a compact $S \subset \mathbb{C}$, where $0 \notin S$, the estimated asymptotic convergence factor is defined as the number

$$\rho = \lim_{k\to\infty} \left(\min_{\substack{p\in\mathbb{P}_k \\ p(0)=1}} \max_{\lambda\in S} |p(\lambda)| \right)^{1/k} \leq 1,$$

where $\mathbb{P}_k$ is the set of polynomials of at most degree k. Denote by r_k the residual vector resulting from (iteration) step k of GMRES. Assuming S approximates the spectrum of the coefficient matrix, we have the estimate

$$\frac{\|r_k\|}{\|r_0\|} \approx \rho^k.$$

Suppose that the smallest and largest eigenvalues of $-M_\#\overline{M_\#}$ are λ_0 and λ_1, respectively, with λ_0 non-negative. Denoting $B = I - M_\#\overline{M_\#}$ we have $\sigma(B) \subset [1+\lambda_0, 1+\lambda_1]$ and the asymptotic convergence factor associated with the interval is (cf. [6])

$$\rho_B = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}, \qquad \kappa = \frac{1+\lambda_1}{1+\lambda_0}, \tag{2.6}$$

which is a number also well-known in conjugate-gradient convergence rate estimates.

On the other hand, by (2.4) the spectrum of A_1 is symmetric with respect to the real axis and contained within two intervals on the vertical line through 1

$$\sigma(A_1) \subset [1+i\sqrt{\lambda_0}, 1+i\sqrt{\lambda_1}] \cup [1-i\sqrt{\lambda_0}, 1-i\sqrt{\lambda_1}].$$

It follows from [9, Theorem 6] that now the asymptotic convergence factor is the square root of (2.6)

$$\rho_{A_1} = \sqrt{\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}}. \tag{2.7}$$

Related to the potential theoretic method [6] leading to (2.6), numerical conformal map methods exist involving several intervals [10].

Comparing (2.6) and (2.7) we again expect GMRES applied to (2.5) to take approximately half the number of iterations of GMRES applied to (2.3). Moreover, we can expect this benefit even when $M_\#$ is not skew-symmetric, but the corresponding spectra considered above are highly eccentric ellipses approximating the intervals.

2.2 The $\mathbb{R}$-linear GMRES polynomial minimization problem

Although by using the preconditioned system (2.5) we may achieve the same accuracy in half the number of iterations compared to using the real system (2.3) (according to the previous section) and also compared to using (2.1) with $\mathbb{R}$-linear GMRES (as demonstrated in Publication I), this is the best of circumstances. In terms of matrix-vector products by $M_\#$ the $\mathbb{R}$-linear GMRES method always takes at most the same number of matrix-vector products as classical GMRES applied to (2.5) to reach the same accuracy [7]. Moreover, $\mathbb{R}$-linear GMRES is always at least as fast as classical GMRES applied to the real system (2.3) [8].

Publication IV presents a polynomial approximation problem for evaluating the convergence rate of the $\mathbb{R}$-linear GMRES method. The polynomials are not holomorphic, but rather a special type of two-variable real polynomials. Hence known approximation theory results are not immediately applicable. The problem remains unsolved, but we can readily identify a case of expected poor convergence.

Take $\kappa = 0$ in (2.1) and let $M_\#$ be complex symmetric. By Takagi's factorization [15] there exists a unitary matrix U and a real nonnegative diagonal matrix $\Sigma_\#$ such that $M_\# = U\Sigma_\# U^T$. Substituting $v = U^* z$ and denoting $c = U^* b$ we find (2.1) in the form $\Sigma_\# \overline{v} = c$. Suppose Θ is a real diagonal matrix such that $\rho = e^{i\Theta} c$ is a real vector. Making a further substitution $w = e^{i\Theta} v$, $D_\# = e^{i\Theta}\Sigma_\# e^{i\Theta}$ we find (2.1) in the form $D_\# \overline{w} = \rho$. The convenience here is that the real vector ρ is unaffected by the complex-conjugations in (2.2). The $\mathbb{R}$-linear GMRES method then minimizes the norm of the residual vector r_k according to

$$\|r_k\| = \min_{\substack{p \in \mathcal{P}_k \\ p(0)=1}} \|p(D_\#)\rho\|, \qquad (2.8)$$

where $\mathcal{P}_k$ is the space of $\mathbb{C}$-linear combinations of the first $k+1$ of

$$1, \lambda, |\lambda|^2, \lambda|\lambda|^2, |\lambda|^4, \ldots$$

Assume now that $\sigma(D_\#)$ contains no real numbers and $\overline{\sigma(D_\#)} = \sigma(D_\#)$. If the polynomial p solves the problem (2.8) we find that the polynomial $\overline{p}$ with the coefficients of p conjugated solves the problem as well. By uniqueness p has real coefficients. It follows that in this case $\mathbb{R}$-linear GMRES makes use of only $\mathbb{R}$-linear combinations for $\mathcal{K}_k(D_\#, \rho)$ in (2.2). Hence it is equivalent to classical GMRES applied to a real system of the form (2.3). Due to the symmetry of the spectrum (2.4) we expect classical GMRES to make little to no progress on minimizing $\|r_k\|$ on every other iteration (with the ideal GMRES making no progress at all). Although we assumed that $\sigma(D_\#)$ is symmetric with respect to the real axis, we could have assumed symmetry with respect to any line through the origin. While such an assumption seems quite restrictive, the last numerical example of Publication IV nevertheless shows similar behaviour in a more general situation.

3. Electrical impedance tomography

In electrical impedance tomography the impedivity distribution inside an object is reconstructed by first injecting prescribed currents to the object through electrodes attached to its boundary and measuring the resulting voltages on the boundary (possibly using the same electrodes). The impedivity distribution is then computed and subsequently represented as an image to be displayed. Several computational methods [4, 19] are known with the $\overline{\partial}$-method in two dimensions concerning us the most as the immediate ancestor of the method presented in Publication II. The motivation to seek new methods is evident considering the low spatial resolution achieved by all existing EIT systems. (In contrast, temporal resolution can be excellent.) One hopes to find a method capable of utilizing the measurement accuracy of the physical apparatus to the fullest in order to obtain higher quality images. Originating from Nachman's constructive uniqueness proof [20], the first numerical $\overline{\partial}$-method implementation was presented by Siltanen *et al.* [23].

3.1 Mathematical formulation of EIT

The formulation employed here involves the reciprocal of impedivity, the admittivity, with the assumption that the object consists of linear media and has isotropic admittivity. EIT systems usually utilize alternating currents and voltages providing information about the permittivity of the material in addition to the conductivity. Alternating currents are also appropriate in medical applications to avoid electrolytic effects. However, for some commonly applied current frequencies ($\approx 10 - 100$ kHz) the permittivity can be considered negligible [12, 13] [1]. Hence our formulation

[1] An internet resource for the calculation of the dielectric properties of body tissues in the frequency range 10 Hz - 100 GHz: http://niremf.ifac.cnr.it/tissprop/

assumes no permittivity (no displacement currents). Due to the low frequencies, the (electro)quasi-static approximation to Maxwell's equations is valid. In this case the formulation equals that of direct current steady state measurements. Note that ignoring permittivity may limit EIT applicability to e.g. breast cancer detection since malignant breast tumours not only have differing conductivity compared to normal tissue, but differing permittivity as well [16]. Incidentally, a recent preprint describes a direct reconstruction algorithm for nonzero permittivities in two dimensions using the $\overline{\partial}$-method [14].

In lieu of prescribing the currents, the mathematical formulation of EIT, in the generality we are concerned, was given by Calderón [5] in terms of prescribed voltages on the boundary and measuring the power required to maintain these voltages. Below we give the mathematically equivalent formulation more common in literature which prescribes the voltages on the boundary and, in fact, measures the induced currents. While all three mentioned formulations are mathematically equivalent, the first mentioned is the one used in practical systems due to advantage in signal-to-noise ratio.

For an open set $\Omega \subset \mathbb{R}^2$ we denote by $W^{k,p}(\Omega)$ the Sobolev space of functions on Ω having distributional derivates up to order k in $L^p(\Omega)$, let $H^k(\Omega) := W^{k,2}(\Omega)$, and define $W_0^{k,p}(\Omega)$ as the closure in $W^{k,p}(\Omega)$ of the set of compactly supported smooth functions on Ω.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded open set. We define the space of boundary functions (traces) as the quotient space $H^{1/2}(\partial\Omega) := H^1(\Omega)/H_0^1(\Omega)$. Suppose $\sigma : \Omega \to \mathbb{R}$ is a measurable function (the conductivity) such that $c \leq \sigma(x) \leq C$ for almost every $x \in \Omega$, where c and C are positive constants. Let $v \in H^1(\Omega)$ be given and consider the boundary value problem

$$\begin{aligned} \nabla \cdot \sigma \nabla u &= 0, \\ u - v &\in H_0^1(\Omega). \end{aligned} \tag{3.1}$$

Define the functional $\Lambda_\sigma v : H^{1/2}(\partial\Omega) \to \mathbb{R}$ by

$$\langle \Lambda_\sigma v, w \rangle = \int_\Omega \sigma \nabla u \cdot \nabla w \; dx, \tag{3.2}$$

where u is the weak solution to (3.1). Then $v \mapsto \Lambda_\sigma v$ defines a bounded linear operator $\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)^*$ called the Dirichlet-Neumann map. We point out that the mapping $\sigma \mapsto \Lambda_\sigma$, however, is nonlinear. The formula (3.2) is the weak formulation for the normal derivative of u. In fact, with appropriate smoothness assumptions the Gauss-Green theorem yields the interpretation $\Lambda_\sigma v = \sigma \frac{\partial u}{\partial \nu}\big|_{\partial\Omega}$, where ν is the outer unit normal to $\partial\Omega$.

Under these circumstances Astala and Päivärinta [1] proved that, if $\widetilde{\sigma}$ satisfies the same assumptions as σ and $\Lambda_\sigma = \Lambda_{\widetilde{\sigma}}$, then $\sigma(x) = \widetilde{\sigma}(x)$ for almost every $x \in \Omega$ settling, in two dimensions, the question of uniqueness posed by Calderón [5]. The proof is constructive yielding a procedure to reconstruct the conductivity σ from the knowledge of Λ_σ. A numerical implementation is described in Publication II, building on the foundation of the similar reconstruction method based on Nachman's [20] uniqueness proof.

3.2 Direct $\overline{\partial}$-methods in two dimensions

The methods we consider are called direct methods in contrast with iterative methods which refine an initial guess for the conductivity at each step.

The Nachman proof based reconstruction method starts by solving integral equations on the boundary $\partial\Omega$ involving the Dirichlet-Neumann map. The goal is to compute an intermediate object known as the scattering transform $t : \mathbb{C} \to \mathbb{C}, k \mapsto t(k)$ (also called nonlinear Fourier transform). To compute one value $t(k)$, one boundary integral equation is solved involving $k \in \mathbb{C}$ as a parameter. In practice the Dirichlet-Neumann map is known only approximately since only a finite number of (noisy) boundary measurements can be performed. The integral equation is then numerically solved only for small k, typically $|k| \lesssim 6$. The dimension of the discretized (complex) matrix system depends on the number of electrodes in the measurement apparatus, typical dimension then being $16 - 64$. After (an approximation of) t has been computed, the conductivity at a point $z \in \Omega$ (regarded as a parameter) is obtained by solving the so-called $\overline{\partial}$-equation involving the scattering transform t in the k-variable. The dimension of the discretized ($\mathbb{R}$-linear) system depends on the k-grid spatial step size used when t was computed. Since such an equation needs to be solved for each $z \in \Omega$, this part of the reconstruction method is the most computationally demanding. Fair reconstructions are obtained when the dimension is mere thousands. Even if shrimpy, these dense (but structured) systems are faster to solve by an iterative method such as GMRES than by Gaussian elimination.

The Astala-Päivärinta proof [1] based reconstruction fits the generic description given above. The differences lie in the details. In lieu of t and $\overline{\partial}$-equations another intermediate object and $\mathbb{R}$-linear Beltrami equations appear. While Publication II is faithful to the method in the first published proof [1], it is also possible to use scattering transforms and $\overline{\partial}$-equations in the context of the nonsmooth method [2, Chapter 18].

Although the computational burdens of these methods are heavy, they are embarrassingly parallelizable. In particular, the computations to solve the $\overline{\partial}$-equations for each $z \in \Omega$ are independent.

3.3 Comparison of the Nachman and Astala-Päivärinta methods

While Nachman's uniqueness proof assumed that $\sigma \in W^{2,p}(\Omega)$ for some $p > 1$ (essentially twice differentiable), there is no smoothness assumption in the Astala-Päivärinta proof. In applications the conductivity typically does have sharp changes, for example tissue boundaries in medical imaging, making it reasonable to expect higher quality reconstructions from an implementation based on the latter proof. However, the method based on Nachman's proof has been applied to discontinuous conductivities with success [17, 18]. Additionally, unpublished computations comparing both methods were carried out by the author of this thesis since Publication II. The results have been discouraging and no accuracy advantage in favor of the Astala-Päivärinta based method has been observed. Both methods appear to produce similar reconstructions with k-grid refinement bringing them closer in the maximum-norm. Optimistically, refinement of the new method may be possible. At least it appears necessary in order to achieve practical accuracy improvement over the former method.

4. Summary of findings

The main findings of this thesis are the following.

I A numerical solution method for the $\mathbb{R}$-linear Beltrami equation with the asymptotic condition appearing in the Astala-Päivärinta proof is presented. The numerical experiments consider the forward problem of known conductivity and computing the complex geometric optics solutions and the scattering transform. Almost two orders of magnitude speed-up over the previously published numerical method [3] is shown by the experiments. Additionally, the presented method improves the accuracy of the numerical solution by avoiding the intermediate Neumann series approximation.

II A new direct numerical solution method for EIT in dimension two is presented. The method is based on the uniqueness proof by Astala and Päivärinta [1] and assumes no smoothness of the conductivity distribution unlike the previously published $\overline{\partial}$-method based on Nachman's uniqueness proof. The solution method of Publication I is found to be applicable to the inverse problem as well.

III The numerical solution of the Beltrami equation considered in the previous two publications is shown to converge to the true solution with an estimate on the convergence rate. Numerical experiments comparing the presented discretization, the discretization used in Publication I, and the polar coordinate discretization of P. Daripa find the convergence rate of all methods comparable.

IV A polynomial approximation problem is presented to estimate the convergence rate of the $\mathbb{R}$-linear GMRES method [8] under the assumption of condiagonalizability of $M_{\#}$ in (2.1). The probability of condiagonalizability of a random matrix with standard normal distributed entries is computed precisely. A complex symmetric Lanczos-process and associated orthogonal polynomials are discovered, showing the existence of an analogue of the classical Hermitian Lanczos-process and polynomials.

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