Implied Volatility Measures

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Abstract

In this thesis the construction of implied volatility measures is considered. Two popular option pricing models, namely Black-Scholes model and Cox-Ross-Rubinstein binomial model, are derived, solved and their inversion is considered to obtain implied volatility estimates. In addition, current market volatility indexes used by practitioners are discussed and Chicago Board Options Exchange's (CBOE) VIX index is derived in detail.

Implied volatility measures rely heavily on the underlying assumptions of the option pricing models. In this thesis we assume the underlying asset to follow the geometric Brownian motion. The geometric Brownian motion is derived and the implications of the motion are discussed. Also, other assumptions in the pricing models are discussed.

Due to some unrealistic assumptions in the pricing models, implied volatility measures have limitations and problems. These problems are introduced and the ways to alleviate these problems are discussed.

Keywords Implied volatility, CBOE VIX index, Black-Scholes model, Cox-Ross-Rubinstein binomial model

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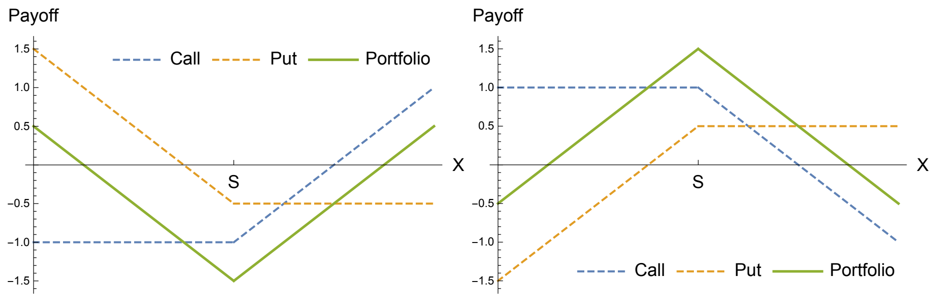
1 Introduction

In many applications of economics and finance volatility is in the central role. Volatility is the standard deviation of a time-series, such as price process, trading volume or temperature of a city, and therefore it measures how prone process is for changes, i.e. how volatile is the process we are examining. It is easy to find applications for volatility in finance such as predicting market movements or hedging portfolio, but volatility can also be used in other fields of economics, for example if we consider warehouse manager managing inventory in some industrial organization, by having idea of the volatility of inventory demand warehouse manager can retain desired inventory levels.

Let us consider one example application of volatility in detail. Suppose we are portfolio managers and we would like to cash in with our expectation of the future volatility of a stock whose price with respect to time t is denoted by $X(t)$. One way to do this is to construct a straddle portfolio as depicted in Figure 1. If we expect that the volatility is large, i.e. the stock price X is expected to deviate significantly from the strike price S, we can buy one call option and one put option with the strike price S, which yield a payoff shown by the green curve in Figure 1 (a). Then again if we expect that the volatility is small, i.e. the stock price X remains close to the strike price S, we can write one call option and one put option with the strike price S, which yield a payoff shown by the green curve in Figure 1 (b). If our expectation of the future volatility is correct and the stock price X does not experience abrupt change, we will earn positive profits on our strategies.

So how would we construct our estimate of the future volatility? One obvious way is to take historical values of our time-series, calculate the standard deviation for the historical values and use the result as an estimate for the future volatility. This estimate is called historical volatility and is one of the simplest methods to construct an estimate for the future volatility, and probably accurate in some applications such as managing inventory levels if the future volatility follows past volatility. But in the case of financial markets, literature has shown that the historical volatility is not a good estimate for the future volatility (see e.g. [1], [2]). This means that in financial markets we need to find other ways to estimate the future volatility.

To approach the issue of finding better volatility estimates for financial markets one could turn to products that are traded in financial markets and ask whether there are some products that would contain information about the future volatility. Fortunately for us, the answer for this question is yes, and these products are called options. In derivatives markets, option prices follow closely prices implied by option pricing models that take the future volatility as one of the input arguments and output the value of option at time t [3]. Because option prices are observable from the derivatives markets in real time, we can invert the option pricing models to find what is the level of volatility that would yield



(a) Long straddle

(b) Short straddle

Figure 1: Straddle strategies

the current market price of options. This estimate of volatility is called implied volatility and it measures the expected level of volatility market participants have on underlying asset.

The powerful feature of implied volatility is that it can be computed to all assets that have options written on them. This means we can calculate implied volatilities to securities (such as stocks and commodities), derivatives (such as futures) and indexes (such as S&P 500), and use these estimates to construct combined volatility estimates (such as volatility estimates for specific industry). Also, the idea of calculating implied volatilities is simple, which makes it easily applicable estimate for volatility.

Due to the beneficial properties of implied volatility, it has been widely adopted in the financial markets. Option markets around the world provide volatility indexes that are based on calculating implied volatilities, and one of the most well-known and cited such indexes is VIX index by Chicago Board of Options Exchange (CBOE) [4]. In addition to providing volatility indexes, option markets provide derivatives (futures and options) on volatility indexes, making volatility a tradable asset.

To illustrate the need for derivatives on volatility, or tradable products on volatility in general, we can return back to our example where we acted as portfolio managers trying to cash in with our expectation on the future volatility. To bet on volatility, we used straddle strategies, which would profit us in case our volatility expectation turned out to be correct and the stock price did not experience abrupt changes. But what if it was the case that in one day, let say due to an earnings release, the stock price experienced an offset in price (either positive or negative) and after the offset, stock remained to have fluctuations around that new price level as it used to have around the price level before earnings release. In this example volatility remained the same but the stock price is now away from the strike price which would affect the payoff of our straddle strategy. Therefore in a straddle, we are exposed on the price level of the underlying asset besides the volatility. In volatility products, price level effects are eliminated and we are only exposed on changes in the volatility (more applications for volatility products can be found for example from [5]).

Although implied volatility has many beneficial properties, it also has major disadvantages and shortcomings. The most imminent problem is that option pricing models that are used to calculate implied volatilities are based on assumptions that are not true in real markets. This causes the real price of options differ from theoretical values implied by option pricing models, and while price differentials are generally small, as discussed earlier, the sensitivity of implied volatility to the option price (also known as the reciprocal of vega) can be large and therefore can cause large deviations in implied volatility estimates. Because basically all option pricing models are based on some form of arbitrage argument (i.e. all riskless strategies yield risk-free return), the implied volatility is correct estimate of future volatility only if financial markets are efficient and other assumptions of option pricing models are correct (i.e. option pricing models correspond exactly to real-life markets).

In this thesis, I aim to introduce reader to implied volatility measures by introducing the two most important option pricing models, known as Black-Scholes model [6] and Cox-Ross-Rubinstein (CRR) binomial model [7], [8], alongside the assumptions of these models, how these models are inverted to find implied volatility measures, and how these models lead to volatility indexes used in real markets, such as CBOE's VIX index. Even though implied volatility indexes have proven to have significant importance in financial markets, it is important that reader understands the assumptions behind these volatility measures and the disadvantages that are implied by these assumptions.

One of the most crucial assumptions in option pricing models is the process that the underlying asset's price is assumed to obey. The most popular price process is a geometric Brownian motion which is also used in Black-Scholes model, Cox-Ross-Rubinstein binomial model, and in VIX index. Therefore we will start by introducing reader to the geometric Brownian motion and what type of characteristics it implies to the price process.

2 Geometric Brownian motion as underlying asset's price process

Consider a deterministic system $\frac{dx(t)}{dt} = x(t) * b(x(t)) \Leftrightarrow dx(t) = x(t) * b(x(t))dt$ in integral form:

$$x(t) - x(0) = \int_0^t x(s) * b(x(s))ds. \tag{2.0.1}$$

Suppose the process has random element (i.e. noise) that is modelled as σB_t, where σ denotes volatility and process $B = (B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion (stdBM). We can write (2.0.1) as a stochastic process:

$$X_t - X_0 = \int_0^t X_s * b(X_s)\,ds + \sigma B_t, \tag{2.0.2}$$

where $X_t = X(t)$ denotes the price of the underlying asset at time t. Let us model σ as a function of the process X, i.e. $\sigma(X)$. We get from (2.0.2):

$$X_t - X_0 = \underbrace{\int_0^t X_s * b(X_s)\,ds}_{=\text{drift}} + \underbrace{\int_0^t \sigma(X_s)\,dB_s}_{=\text{noise}}. \tag{2.0.3}$$

In the differential form (2.0.3) is:

$$dX_t = X_t * b(X_t)\,dt + \sigma(X_t)\,dB_t. \tag{2.0.4}$$

By denoting $b(X_t) = \mu$ and assuming that $\sigma(X_t) = \sigma X_t$, (2.0.4) simplifies to:

$$dX_t = \mu X_t dt + \sigma X_t dB_t. \tag{2.0.5}$$

Equation (2.0.5) is the differential form of geometric Brownian motion (also known as geometric SDE) that is used to model the price of asset with respect to time t. It is composed of a drift which determines the price level of the process, and a noise which determines the level of fluctuation around the price level of the process.

To find a solution to the geometric Brownian motion, we write (2.0.5) in the integral form:

$$\int_0^t \frac{dX_s}{X_s} = \int_0^t \mu ds + \int_0^t \sigma dB_s = \mu \int_0^t ds + \sigma \int_0^t dB_s = \mu t + \sigma B_t. \tag{2.0.6}$$

Next, let us consider function $f(x) = \ln(x)$. Clearly f is twice continuously differentiable. Also the price process X is a semimartingale because it can be decomposed into a finite variation process (drift in this case) and a martingale (noise in this case, stdBM is a martingale). Thus we can use Ito's formula (see Theorem 3.22 in [9]):

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X, X\rangle_s$$

$$\Leftrightarrow \ln(X_t) - \ln(X_0) = \int_0^t \frac{1}{X_s} dX_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2} d\langle X, X\rangle_s \tag{2.0.7}$$

$$\Rightarrow d(\ln(X_t)) = \frac{dX_t}{X_t} - \frac{1}{2}\frac{1}{X_t^2} d\langle X, X\rangle_t \tag{2.0.8}$$

In (2.0.7) and (2.0.8) we need to solve the quadratic variation $\langle X, X\rangle_t$. To do this we first introduce an integral process $H \bullet M = ((H \bullet M)_t)_{t\in\mathbb{R}_+}$, where $(H \bullet M)_t$ is defined as:

$$(H \bullet M)_t = \sum_{k=1}^{\lfloor 2^n t\rfloor} H_{(k-1)2^{-n}} \left(M_{k2^{-n}} - M_{(k-1)2^{-n}}\right). \tag{2.0.9}$$

Notice that $(H \bullet M)_t \overset{n\to\infty}{\longrightarrow} \int_0^t H_s dM_s$. Quadratic variation of the integral process is introduced as Theorem 3.12 in [9] and reads:

$$\langle H \bullet M, H \bullet M\rangle_t = \int_0^t H_s^2 d\langle M, M\rangle_s. \tag{2.0.10}$$

Now $\langle X, X\rangle_t$ can be written as (denote time by T):

$$\begin{aligned}\langle X, X\rangle_t &= \langle(\mu X)\bullet T, (\mu X)\bullet T\rangle_t + \langle(\sigma X)\bullet B, (\sigma X)\bullet B\rangle_t \\ &= \int_0^t \mu^2 X_s^2 d\langle T, T\rangle_s + \int_0^t \sigma^2 X_s^2 d\langle B, B\rangle_s \\ &\overset{*}{=} \int_0^t \sigma^2 X_s^2 ds \\ &\Rightarrow d\langle X, X\rangle_t = \sigma^2 X_t^2 dt. \end{aligned} \tag{2.0.11}$$

(*) We used the fact that $\langle T, T\rangle_t = 0$ for finite variation processes, and time T is a finite variation process. Quadratic variation of the standard Brownian motion $\langle B, B\rangle_t = t$.

Therefore (2.0.7) simplifies to:

$$\ln(X_t) - \ln(X_0) = \int_0^t \frac{1}{X_s} dX_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2}\sigma^2 X_s^2 ds \tag{2.0.12}$$

$$\Leftrightarrow \ln\left(\frac{X_t}{X_0}\right) = \underbrace{\int_0^t \frac{1}{X_s} dX_s}_{=\mu t + \sigma B_t} - \frac{1}{2}\sigma^2 t = \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t. \tag{2.0.13}$$

Taking exponential of (2.0.13) gives the solution of the geometric SDE:

$$X_t = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t}. \tag{2.0.14}$$

Solution (2.0.14) can be written as:

$$\ln(X_t) = \underbrace{\ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t}_{\text{deterministic}} + \sigma \underbrace{B_t}_{\sim \mathcal{N}(0,t)} \tag{2.0.15}$$

Thus $\ln(X_t) \sim \mathcal{N}\left(\bar{\mu}, \bar{\sigma}^2\right)$, where $\bar{\mu}$ and $\bar{\sigma}^2$ are given as:

$$\bar{\mu} = \mathbb{E}\left[\ln(X_t)\right] = \ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right) t, \tag{2.0.16}$$

$$\bar{\sigma}^2 = \mathrm{Var}\left[\ln(X_t)\right] = \mathrm{Var}\left[\sigma B_t\right] = \sigma^2 t. \tag{2.0.17}$$

We say that X_t obeys a log-normal distribution because $\ln(X_t)$ (i.e. the log price) is normally distributed. The variance of log price depends on time t and volatility parameter σ which is assumed to be constant by the equation (2.0.5). The assumption that σ is constant means that in later sections when we derive implied volatility estimates for σ we should get the same estimate regardless of the strike price and the time-to-maturity of option as long as the underlying asset is the same. However, implied volatility estimates will differ depending on the time-to-maturity and the strike price of option [10], referred as the term structure of volatility and the volatility smile, respectively. These issues will be discussed in detail in later sections.

Now, as we have introduced the process the price of underlying asset obeys we can move on to derive the Black-Scholes and CRR-binomial tree option pricing models. In next two sections we will derive these models and show how they can be used to calculate implied volatility estimates for underlying assets.

3 Option pricing models

In this section we aim to derive the two most common option pricing models, namely Black-Scholes model and CRR-binomial model. We will do derivations in detail to provide the reader with thorough understanding of these models before moving on to the implied volatility measures. Good understanding of these option pricing models is a prerequisite to fully understand the implied volatility measures.

To begin, let us introduce assumptions that we will use in the derivation of Black-Scholes and CRR-binomial models:

(i) Short-term interest rate is known and constant.

(ii) Underlying asset's price process $X = (X_t)_{t \in \mathbb{R}_+}$ is a geometric Brownian motion.

(iii) Underlying asset pays no dividend or other distribution.

(iv) Option $V(X,t)$ is an European option, i.e. executable only on the maturity date.

(v) There are no transaction costs.

(vi) We can buy and sell any fraction of the underlying asset.

(vii) Markets are efficient, i.e. riskless profit opportunities yield the risk-free interest rate.

Assumptions (iii) and (iv) can easily be relaxed, i.e. we could allow the underlying asset to pay distributions and we could consider American options instead of European options, but for the sake of simplicity we will stick to these two assumptions. The idea of analysis does not differ whether we allow asset to pay distributions or whether we consider American options instead of European options.

3.1 Black-Scholes model

To derive Black-Scholes model, let us begin by considering portfolio consisting of one stock (or could be any other asset that has options written on it) with the price $X(t)$ at time t, and call options with value (or payoff) $V(X,t)$ to underlying stock. We want to delta-hedge this portfolio, meaning that the value of the portfolio remains unchanged when small change in the stock price X happens. If the change in stock price, denoted by ΔX, is small we can approximate the change in the value of option ΔV by the first-order Taylor expansion:

$$dV = V(X + dX, t) - V(X,t) = \frac{\partial V}{\partial X} dX. \tag{3.1.1}$$

Because the value of the call option increases as the stock price increases, i.e. $\frac{\partial V}{\partial X} > 0$, a long position in stock requires a short position in call option to obtain a delta-hedged portfolio. The value of the portfolio, denoted by $W(X, V(X,t))$ is:

$$W(X, V(X,t)) = X - KV(X,t), \tag{3.1.2}$$

where $K \in \mathbb{R}$ is the number of call options in the delta-hedged portfolio. To have a delta-hedged position, small change in the price of the stock should not change the value of the portfolio. Thus the number of call options in the portfolio is:

$$\begin{aligned}
&W(X + \Delta X, V(X + \Delta X, t)) - W(X, V(X,t)) \\
&= X + \Delta X - KV(X + \Delta X, t) - X + KV(X,t) \\
&= \Delta X - K(V(X + \Delta X, t) + V(X,t)) = 0 \\
&\Rightarrow K(V(X + \Delta X, t) - V(X,t)) = \Delta X \\
&\Leftrightarrow K\frac{\partial V}{\partial X}\Delta X = \Delta X \\
&\Rightarrow K = 1 \Big/ \left(\frac{\partial V}{\partial X}\right).
\end{aligned} \tag{3.1.3}$$

Thus the value of the portfolio (also called the equity) at time t is:

$$W(X, V(X,t)) = X - \frac{1}{\partial V/\partial X}V(X,t). \tag{3.1.4}$$

The change of the portfolio value with respect to small time step dt is:

$$dW = dX - \frac{1}{\partial V/\partial X}dV, \tag{3.1.5}$$

where the change in call options value can be written using Ito's formula (see Theorem 3.22 in [9]):

$$\begin{aligned}
&V(X + dX, t + dt) - V(X,t) \\
&= \int_{\tau}^{\tau+d\tau} \frac{\partial V}{\partial X}dX_s + \int_{\tau}^{\tau+d\tau} \frac{\partial V}{\partial t}ds + \frac{1}{2}\int_{\tau}^{\tau+d\tau} \frac{\partial^2 V}{\partial X^2}d\langle X, X\rangle_s + \\
&\int_{\tau}^{\tau+d\tau} \frac{\partial^2 V}{\partial X \partial t}d\langle X, t\rangle_s + \frac{1}{2}\int_{\tau}^{\tau+d\tau} \frac{\partial^2 V}{\partial t^2}d\langle t, t\rangle_s \\
&\overset{*}{=} \int_{\tau}^{\tau+d\tau} \frac{\partial V}{\partial X}dX_s + \int_{\tau}^{\tau+d\tau} \frac{\partial V}{\partial t}ds + \frac{1}{2}\int_{\tau}^{\tau+d\tau} \frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2 ds \\
&= \int_{\tau}^{\tau+d\tau} \frac{\partial V}{\partial X}dX_s + \int_{\tau}^{\tau+d\tau} \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right) ds \\
&\Rightarrow dV = \frac{\partial V}{\partial X}dX_t + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right) dt.
\end{aligned} \tag{3.1.6}$$

(*) We assume the stock price obeys geometric Brownian motion. Thus $d\langle X, X\rangle_t = \sigma^2 X_t^2 dt$. *Also the quadratic variation of finite variation processes is zero.*

By substituting (3.1.6) to (3.1.5) we get the change of the portfolio value with respect to time:

$$\begin{aligned} dW &= dX - \frac{1}{\partial V/\partial X}\left(\frac{\partial V}{\partial X} dX_t + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right) dt\right) \\ &= -\frac{dt}{\partial V/\partial X}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right). \end{aligned} \tag{3.1.7}$$

Because the value of the delta-hedged portfolio is independent of the stock price, it has no risk. Therefore the return dW must equal the risk-free interest rate r_f (this is the arbitrage argument of Black-Scholes model):

$$\begin{aligned} dW &= -\frac{dt}{\partial V/\partial X}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right) \\ &= W * r_f dt = \left(X_t - \frac{1}{\partial V/\partial X} V(X,t)\right) * r_f dt \\ &\Rightarrow -\frac{1}{\partial V/\partial X}\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial X^2}\sigma^2 X_t^2\right) = r_f X_t - r_f \frac{1}{\partial V/\partial X} V(X,t) \\ &\Rightarrow \frac{\partial V}{\partial t} = r_f V(X,t) - r_f X_t \frac{\partial V}{\partial X} - \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X^2}. \end{aligned} \tag{3.1.8}$$

The stochastic differential equation (3.1.8) is called the Black-Scholes equation for the value of a call option. By denoting the maturity date (or expiration date) of the option by T and the execution price (or strike price) by S, we can write boundary conditions for (3.1.8):

$$\begin{cases} V(0,t) = 0, \forall t \in [0,T] & \text{(3.1.9a)} \\ V(X_t, t) \overset{X_t \to \infty}{\longrightarrow} X_t & \text{(3.1.9b)} \\ V(X_T, T) = \max\{X_T - S, 0\} & \text{(3.1.9c)} \end{cases}$$

To solve (3.1.8) with the boundary conditions (3.1.9a)-(3.1.9c) we will transform (3.1.8) to an ordinary diffusion equation with a change of variables and solve this diffusion equation with its Green's function (this is done in Appendix A). This procedure yields solution:

$$V(X,t) = X_t \Psi(d_+(\tau, X)) - S e^{-r_f \tau} \Psi(d_-(\tau, X)), \tag{3.1.10}$$

where $\Psi(\bullet)$ is the cumulative distribution function (CDF) of the standard normal distribution and $d_{\pm}(\tau, x) = \frac{\ln(X_t/S)+\left(r_f \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$, $\tau = T - t$. To price put options we use the put-call parity which makes (3.1.10) applicable also in the pricing of put options.

3.2 Cox-Ross-Rubinstein binomial model

Consider a stock with the price $X_k, k \in \mathbb{Z}_+$ at state k. In state k stock can either go up or down. The most simple such system has $k \in \{n, n+1\}$ and is called a two state system.

Let us consider a portfolio that buys H stocks (H also called hedging ratio) and sells one call option with the strike price S. Option is maturing at the ending state of the system (in two state system when $k = n+1$). By denoting the value of the option by V, the initial investment in the state $k = n$ and the payoff in maturity (state $k = n+1$):

$$k = n : W_n = HX_n - V_n \tag{3.2.1}$$

$$\begin{aligned} k = n+1 : W^{(\mathrm{up})}_{n+1} &= HX^{(\mathrm{up})}_{n+1} - \max\left\{X^{(\mathrm{up})}_{n+1} - S, 0\right\} \text{ OR} \\ W^{(\mathrm{down})}_{n+1} &= HX^{(\mathrm{down})}_{n+1} - \max\left\{X^{(\mathrm{down})}_{n+1} - S, 0\right\}. \end{aligned} \tag{3.2.2}$$

Now let us find H such that the payoff for the portfolio in the maturity is the same regardless of the price of the stock in the maturity:

$$\begin{aligned} W^{(\mathrm{up})}_{n+1} = W^{(\mathrm{down})}_{n+1} &\Rightarrow HX^{(\mathrm{up})}_{n+1} - \max\left\{X^{(\mathrm{up})}_{n+1} - S, 0\right\} \\ &= HX^{(\mathrm{down})}_{n+1} - \max\left\{X^{(\mathrm{down})}_{n+1} - S, 0\right\} \\ \Rightarrow H &= \frac{\max\left\{X^{(\mathrm{up})}_{n+1} - S, 0\right\} - \max\left\{X^{(\mathrm{down})}_{n+1} - S, 0\right\}}{X^{(\mathrm{up})}_{n+1} - X^{(\mathrm{down})}_{n+1}}. \end{aligned} \tag{3.2.3}$$

With the hedge ratio given by (3.2.3) portfolio's payoff in the maturity does not depend on the price of the stock, therefore we have a certain (i.e. riskless) payoff. For a riskless payoff investors require the risk-free interest rate r_f (this is the arbitrage argument of binomial models). By continuous interest compounding we have:

$$W_n = W_{n+1} e^{-r_f T}, \tag{3.2.4}$$

where T denotes the time length of moving from state n to $n+1$ (e.g. hour, day, week, month, etc.). By substituting (3.2.1) and (3.2.2) to (3.2.4) we obtain the price of the call option in state n:

$$\begin{aligned} HX_n - V_n &= \left(HX_{n+1}^{(\text{up})} - \max\left\{X_{n+1}^{(\text{up})} - S, 0\right\}\right) e^{-r_f T} \\ \Rightarrow V_n &= HX_n - \left(HX_{n+1}^{(\text{up})} - \max\left\{X_{n+1}^{(\text{up})} - S, 0\right\}\right) e^{-r_f T}. \end{aligned} \tag{3.2.5}$$

For multiple state systems, we use the two-state model starting from the last two states (i.e. maturity state and state before maturity) and backpropagate the result all the way to the first node.

Alternative approach, which yields exactly the same result as the method above, to value options in 2-state system is to consider the probability for the stock price moving up, denoted by p_u. Let us determine p_u such that the expected payoff from the stock is the same now and in the next state, i.e.:

$$X_n = \mathbb{E}\left[X_{n+1}\right] \Leftrightarrow X_n = p_u X_{n+1}^{(\text{up})} + (1 - p_u) X_{n+1}^{(\text{down})}. \tag{3.2.6}$$

A risk neutral person is indifferent of selling the stock now or selling it in the next state if the portfolio yields a risk-free return:

$$X_n = \mathbb{E}\left[X_{n+1}\right] e^{-r_f T} \Leftrightarrow X_n = \left(p_u X_{n+1}^{(\text{up})} + (1 - p_u) X_{n+1}^{(\text{down})}\right) e^{-r_f T}. \tag{3.2.7}$$

This gives p_u:

$$p_u = \frac{X_n e^{r_f T} - X_{n+1}^{(\text{down})}}{X_{n+1}^{(\text{up})} - X_{n+1}^{(\text{down})}}. \tag{3.2.8}$$

Using p_u we can write the value of the option V:

$$\begin{aligned} V_n &= \mathbb{E}\left[V_{n+1}\right] e^{-r_f T} = \left(p_u V_{n+1}^{(\text{up})} + (1 - p_u) V_{n+1}^{(\text{down})}\right) e^{-r_f T} \\ &= \left(p_u \max\left\{X_{n+1}^{(\text{up})} - S, 0\right\} + (1 - p_u) \max\left\{X_{n+1}^{(\text{down})} - S, 0\right\}\right) e^{-r_f T}. \end{aligned} \tag{3.2.9}$$

The problem with binomial option pricing is that we need to predetermine the price of the stock in each state of the system $k \in \mathbb{Z}_+$. One way to overcome this issue is to use a stochastic price movement, which is the idea in CRR-binomial model.

Let us denote the logarithm of price (i.e. log-price) of the stock at state n

by $\ln(X_n)$. Now in CRR-model the log-price can either increase by $\ln(u)$ or decrease by $\ln(d)$. This implies possible price movements:

$$\ln\left(X_{n+1}^{(\text{up})}\right) = \ln(X_n) + \ln(u) \Rightarrow X_{n+1}^{(\text{up})} = X_n * u \tag{3.2.10}$$
$$\ln\left(X_{n+1}^{(\text{down})}\right) = \ln(X_n) - \ln(d) \Rightarrow X_{n+1}^{(\text{down})} = X_n * d. \tag{3.2.11}$$

Next step is to determine u and d. By substituting (3.2.10) and (3.2.11) to (3.2.8) we get the probability of up movement:

$$p_u = \frac{e^{r_f T} - d}{u - d}. \tag{3.2.12}$$

In a risk neutral world, we also require that the volatility of portfolios are the same, not only the expected returns. By assuming a small time step between states n and $n+1$ (i.e. T is small), we can approximate:

$$\begin{aligned}
\text{Var}(X_{n+1}) &\overset{*}{=} X_n^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right) \\
&\overset{**}{\approx} X_n^2(1 + 2\mu T)\sigma^2 T \\
&= X_n^2\left(\sigma^2 T + 2\mu\sigma^2 \underbrace{T^2}_{\approx 0}\right) \\
&\approx X_n^2 \sigma^2 T
\end{aligned} \tag{3.2.13}$$

(*) By assumption (ii), $\ln(X_{n+1}) \sim \mathcal{N}\left(\bar{\mu}, \bar{\sigma}^2\right)$, *where* $\bar{\mu} = \ln(X_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t$ *and* $\bar{\sigma}^2 = \sigma^2 t$. *Thus* $Var(X_{n+1}) = X_n^2 e^{2\mu T}\left(e^{\sigma^2 T} - 1\right)$.
(**) For small x, $e^x \approx 1 + x$.

By using the approximation of $\text{Var}(X_{n+1})$, we get:

$$\begin{aligned}
&\underbrace{\text{Var}(X_{n+1}))}_{\approx X_n^2\sigma^2 T} = \mathbb{E}\left[X_{n+1}^2\right] - \mathbb{E}\left[X_{n+1}\right]^2 \\
&= p_u u^2 X_n^2 + (1 - p_u)\, d^2 X_n^2 - \left(p_u u X_n + (1 - p_u)\, d X_n\right)^2 \\
&\Leftrightarrow X_n^2 \sigma^2 T = p_u u^2 X_n^2 + (1 - p_u)\, d^2 X_n^2 - p_u^2 u^2 X_n^2 - \\
&2p_u(1 - p_u)\, u d X_n^2 - (1 - p_u)^2 d^2 X_n^2 \\
&\Rightarrow \sigma^2 T = p_u u^2 + (1 - p_u)\, d^2 - p_u^2 u^2 - 2p_u(1 - p_u)\, u d - (1 - p_u)^2 d^2 \\
&= \left(p_u - p_u^2\right)(u - d)^2.
\end{aligned} \tag{3.2.14}$$

By substituting (3.2.12) to (3.2.14), we get:

$$\begin{aligned}
\sigma^2 T &= \frac{(u-d)\left(e^{r_f T}-d\right)-\left(e^{r_f T}-d\right)^2}{(u-d)^2}(u-d)^2 \\
&= (u-d)\left(e^{r_f T}-d\right)-\left(e^{r_f T}-d\right)^2 \\
&= ue^{r_f T} - ud - de^{r_f T} + d^2 - e^{2r_f T} + 2e^{r_f T}d - d^2 \\
&= (u+d)e^{r_f T} - ud - e^{2r_f T}. \qquad (3.2.15)
\end{aligned}$$

CRR-method assumes that $d = \frac{1}{u}$. By substituting this into (3.2.15) we get:

$$\begin{aligned}
&\sigma^2 T = \left(u + \frac{1}{u}\right) e^{r_f T} - 1 - e^{2r_f T} \\
&\Rightarrow u + \frac{1}{u} = e^{-r_f T}\left(\sigma^2 T + 1\right) + e^{r_f T} \approx (1 - r_f T)\left(\sigma^2 T + 1\right) + (1 + r_f T) \\
&= \sigma^2 T + 1 - \sigma^2 r_f \underbrace{T^2}_{\approx 0} - r_f T + 1 + r_f T = 2 + \sigma^2 T \\
&\Rightarrow u^2 - \left(2 + \sigma^2 T\right) u + 1 = 0 \\
&\Rightarrow u = \frac{2 + \sigma^2 T \pm \sqrt{\left(2 + \sigma^2 T\right)^2 - 4}}{2} = \frac{2 + \sigma^2 T \pm \sqrt{4 + 4\sigma^2 T + \sigma^4 T^2 - 4}}{2} \\
&= \frac{2 + \sigma^2 T \pm 2\sqrt{\sigma^2 T}}{2} \\
&= 1 + \underbrace{\frac{\sigma^2 T}{2}}_{\ll \sigma\sqrt{T}} \pm \sigma\sqrt{T} \\
&\overset{u \geq 1}{=} 1 + \sigma\sqrt{T} \approx e^{\sigma\sqrt{T}}. \qquad (3.2.16)
\end{aligned}$$

We obtained $u = e^{\sigma\sqrt{T}}$ and respectively $d = \frac{1}{u} = e^{-\sigma\sqrt{T}}$ in Cox-Ross-Rubinstein method. It is important to keep in mind that in CRR-method we assume a small time step between states, i.e. T is small. It is worth noting that consecutive up and down movements will cancel each others, i.e. $d * u = 1$.

Now, as we have derived the two most common option pricing models along with their assumptions, we are ready to construct implied volatility measures. In the next section we will show how implied volatility measures are computed from option pricing models and how these relate to volatility indexes used in markets such as CBOE's VIX index. Also, we will discuss in detail the problems and shortcomings of implied volatility measures, which are important to keep in mind when using implied volatility measures in practise.

4 Implied volatility

As we saw in the sections 3.1 and 3.2, where we derived Black-Scholes and CRR-binomial option pricing models, volatility σ of the underlying asset was one of the inputs that we used in the option pricing models to determine the value of a option at time t. In both of these models, volatility σ was introduced in the noise term of the geometric Brownian motion, which resulted in the stock price to be log-normally distributed. This assumption yielded closed-form solution for option price in Black-Scholes model, and deterministic up ($u = e^{\sigma\sqrt{T}}$) and down ($d = e^{-\sigma\sqrt{T}}$) movements in CRR-binomial tree.

One challenge in option pricing models is that volatility of underlying asset is not directly observable from the market, as is the case for the other parameters of these two models, i.e. the price of the stock X, the risk-free interest rate r_f, the time-to-maturity T and the option price V or P (call and put respectively). Therefore, because we have only one unknown variable in our model, namely volatility, option pricing models such as Black-Scholes and CRR-binomial tree can be used to find the volatility. This is done by inverting the option pricing model to find the volatility of the underlying asset such that the option pricing model gives the correct price of the option. The volatility resulting from the option pricing model inversion is called implied volatility.

To be more precise, let us denote the option pricing model by function $f : (\sigma, \Omega) \to V$, where Ω denotes the set of input parameters of the option pricing model besides volatility σ. Thus the option pricing model can be seen as a function f that gets inputs, in which volatility σ is one of the arguments, and outputs the price of the option V. Because we find values of V and Ω from the markets we can find the implied volatility σ by the inverse function $f^{-1} : (V, \Omega) \to \sigma$. This is how we theoretically find the implied volatility by using option pricing models such as Black-Scholes and CRR-binomial tree.

Although it is theoretically simple to find implied volatilities, practically it is more challenging, because in general case there is no closed-form expression for the inverse function f^{-1} (in certain cases it is possible to find a closed form solution for f^{-1} by using approximations, see e.g. [11]). If we do not have closed form expression for the inverse function, we can find the implied volatility numerically by solving a root finding equation:

$$f(\sigma, \Omega) - V = 0. \tag{4.0.1}$$

Equation (4.0.1) can be solved with any root finding method, such as Newton's method, Quasi-Newton's methods (e.g. secant method), Bisection or some other fixed point iteration schemes. It is important to notice that some root finding algorithms work better than the others depending on the option pricing model we use to find the implied volatility. For example, when using Black-Scholes model it is convenient to use Newton's method because there is closed form expression for $\frac{\partial f}{\partial \sigma}$, but in the case of CRR-binomial model computation of $\frac{\partial f}{\partial \sigma}$ is costly because closed from expression does not exist and finding the derivative requires lengthy chain rule expressions (similar phenomena as in neural networks). If we want to find implied volatilities in real-time it is important to choose the root finding algorithm appropriately.

As we now know how to calculate implied volatilities by inverting option pricing models or solving equations of form (4.0.1) by root finding algorithms, next step is to use implied volatilities to construct implied volatility measures and indexes. However, before moving to implied volatility measures and indexes, we need to discuss the term structure of implied volatility and implied volatility smile. These two phenomena are intrinsic part of implied volatilities and are essential to understand and keep in mind when constructing implied volatility measures and indexes.

4.1 Term structure of implied volatility and implied volatility smile

In Black-Scholes and CRR-binomial tree models we assumed that the price of the underlying asset X_t follows the geometric Brownian motion:

$$dX_t = X_t \left(\mu dt + \sigma dB_t \right), \tag{4.1.1}$$

where $dX_t = X_t - X_0$ denoted the price change from initial time t_0 to time t, μ was the drift parameter, σ was the volatility parameter and B_t was the standard Brownian motion. It is important to notice that in these two option pricing models we assumed σ to be constant over time and over option strike prices. This means that when calculating implied volatilities with these option pricing models, we should get the same implied volatility regardless of the time-to-maturity T or the strike price S of the option.

However, the empirical evidence shows that options with different time-to-maturities and strike prices have different implied volatilities (see [10]). In [10] authors had three key findings:

1. Short maturity out-of-money (i.e. for call option strike price much higher than the current stock price, for puts vice versa) calls are overpriced in the market compared to the prices predicted by Black-Scholes.
2. The price bias between the market price and the price predicted by Black-Scholes is statistically significant but the bias reverses after long periods of time.
3. None of the option pricing models up that date were able to explain that bias reversal in 2.

Because the option price increases when the volatility increases, i.e. $\frac{\partial V}{\partial \sigma} > 0$, overpriced products have higher implied volatilities than underpriced products. Thus by [10] we get the following conclusions:

(a) Short maturity out-of-money calls have higher implied volatilities than longer maturity counterparts.

(b) At-the-money calls on the first period provided higher implied volatilities on longer time-to-maturity options. On the second period the shorter time-to-maturity provided higher implied volatilities (trend reversal). The first period was 23.8.1976-21.10.1977 while the second period was 24.10.1977-31.8.1978.

(c) In the first period lower strike price corresponded to higher implied volatility. On the second period the higher striking price corresponded to higher implied volatility.

Conclusions (a) and (b) tell that implied volatility varies over time. This is referred to as the term structure of volatility. Then again, conclusion (c) tells that the implied volatility depends also on the strike price. This is referred to as the volatility smile.

The empirical evidence by [10] shows that option prices predicted by Black-Scholes (and other option pricing models) deviate from the true market values both over time and over strike price. This means that the implied volatility changes over time and over strike price which is against the assumptions of the price process we used in the derivation of Black-Scholes model and CRR-binomial model. This can be due to two reasons: first, market imperfections (such as transaction costs) may systematically prevent option prices to take their true theoretical values or secondly, the price of the underlying asset do not follow the geometric Brownian motion with a constant volatility (i.e. our model assumptions are not correct).

To explain the term structure and smile of implied volatility, more recent option pricing models have assumed the volatility to vary over time. Widely used assumption in these models is to assume the volatility to follow a certain random process. This class of option pricing models is known as stochastic volatility models. One popular stochastic volatility model is Heston's model [12] in which the underlying asset's volatility is assumed to follow Cox-Ingersoll-Ross process [13]. In this thesis we will not discuss stochastic volatility models.

In the construction of implied volatility measures we can address the term structure and smile of implied volatility by calculating implied volatilities across different maturities and strike prices, and use a weighting scheme to calculate the final estimate of the implied volatility. Many weighting schemes are discussed in the literature, and we will shortly mention some of them in the next section.

4.2 Implied volatility measures and indexes

As was noted in the Section 4.1, options with different time-to-maturities and strike prices have different implied volatilities. Therefore it is not straightforward to construct measures of implied volatilities because depending on the option type, we get different estimates for implied volatilities.

To alleviate this problem, one way to construct implied volatility measures is to find implied volatilities for options with different maturities and strike prices and then use weighted sum as the estimate of the underlying asset's volatility. There has been many proposals in the literature of how weighting should be done. In [14] some of the most popular weighting schemes are listed and here we repeat them below:

$$\begin{cases} \hat{\sigma} = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N}\sigma & \text{(4.2.1a)} \\ \hat{\sigma} = \dfrac{1}{\sum_{i=1}^{N} w_i}\sqrt{\displaystyle\sum_{i=1}^{N} w_i^2\sigma_i^2} & \text{(4.2.1b)} \\ \hat{\sigma} = \dfrac{\sum_{i=1}^{N}\sigma_i \frac{\partial V_i}{\partial \sigma_i}\frac{\sigma_i}{V_i}}{\sum_{i=1}^{N}\frac{\partial V_i}{\partial \sigma_i}\frac{\sigma_i}{V_i}} & \text{(4.2.1c)} \\ \arg\min\limits_{\hat{\sigma}}\left\{\displaystyle\sum_{i=1}^{N} w_i\left[V_i - f\left(\hat{\sigma}\right)\right]\right\} & \text{(4.2.1d)} \end{cases}$$

Equation (4.2.1a) finds $\hat{\sigma}$ as an arithmetic mean of implied volatilities of options with different time-to-maturities T and strike prices S. Although this approach is simple, it does not give very accurate prediction of the implied volatility because option pricing models predict the value of options better for some combinations of T and S than others. Therefore it would make sense to put more weight on options which can be accurately priced by option pricing models. Also some options are more sensitive to volatility than others. This means that we are prone to high estimation error when estimating volatility by volatility insensitive options, and thus it would make sense to put higher weight on volatility sensitive options than insensitive ones.

Equation (4.2.1b) is suggested in [15]. The weights are chosen to be $w_i = \frac{\partial V_i}{\partial \sigma_i}$ (also called vega). Equation (4.2.1c) is suggested in [16] and for weights they use option price elasticies to volatility. Equation (4.2.1d) is suggested in [2] in which author does not propose any superior way of choosing weights w_i.

In market implied volatility indexes, weighting schemes of type (4.2.1a)-(4.2.1d) are used. The most followed and cited such indexes are CBOE's VIX and RVX indexes which measure the implied volatility of S&P500 and Russell 2000 indexes respectively, and HSI Volatility Index which measures the implied volatility of Hong Kong's Hang Seng Index. All these indexes are calculated in the same fashion, by considering wide range of options with different maturities and strike prices and weighting the resulting implied volatilities to obtain the final value of volatility index. In the next section we will discuss CBOE's VIX index in detail. We will discuss what is the goal of VIX index, what type of options it uses to calculate the implied volatility and how VIX index is calculated. In [4] the general form of VIX index formula is given, and we will also derive this formula.

4.3 CBOE's VIX index

As discussed in [4], VIX index is calculated by using standard and weekly SPX (S&P 500 Index options) call and put options. Standard SPX options have monthly expiration. The SPX options that are used to calculate VIX expire on Fridays (for standard SPX options 3$^{\text{rd}}$ Friday of each month and weekly SPX options expire all other Fridays). The goal of VIX index is to measure the 30-day expected volatility in S&P 500 index. CBOE uses so called near- and next-term put and call SPX options with 23 to 37 days to expiration and with wide range of strike prices (notice the term structure of volatility and volatility smile is addressed). Options expiring in 30 calendar days or less are considered near-term. Standard and Weekly SPX options are combined in VIX index to match the 30 day target time-frame as closely as possible. The composition of options and weights that are used to calculate VIX changes continuously.

In VIX calculation, time-to-maturity T of options is denoted in minutes (precision used by practitioners):

$$T = \frac{M_{\text{current day}} + M_{\text{settlement day}} + M_{\text{other days}}}{\text{Minutes in a year}}, \tag{4.3.1}$$

where M denotes minutes remaining until midnight on the specific date. The risk-free interest rates r_f in VIX calculation are based on the U.S. Treasury yield curve rates. The yield curve is obtained by cubic interpolation between treasury yield points to derive yields for relevant SPX options.

In [4] the general form of VIX index formula is given as:

$$\sigma^2 = \frac{2}{T} \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right) - \frac{1}{T}\left(\frac{F}{S_0} - 1\right)^2, \tag{4.3.2}$$

where VIX-index $= \sigma * 100$, T is time-to-maturity, F is the forward index level derived from the index option prices, S_0 is the first strike price below the forward index level F, S_i is the strike price of the i'th out-of-money option, ΔS_i is the difference between the i'th consecutive strike prices, r_f is the risk-free interest rate and $Q\left(S_i\right)$ is the midpoint of bid-ask spread for each option with the strike price S_i.

To derive (4.3.2) we assume that the log-price of the underlying asset is normally distributed (as has been the convention in this thesis). We start by the identity (2.0.8) that we obtained in the Section 2 for the geometric Brownian motion:

$$d\left(\ln\left(X_t\right)\right) = \frac{dX_t}{X_t} - \frac{1}{2}\frac{1}{X_t^2}d\langle X, X\rangle_t \overset{d\langle X,X\rangle_t = \sigma^2 X_t^2 dt}{=} \frac{dX_t}{X_t} - \frac{1}{2}\sigma^2 dt. \tag{4.3.3}$$

By integrating (4.3.3) from the initial time up to maturity we get:

$$\ln\left(\frac{X_T}{X_0}\right) = \underbrace{\int_0^T \frac{1}{X_s}\,dX_s}_{=\mu T + \sigma B_T} - \frac{1}{2}\sigma^2 T = \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T. \tag{4.3.4}$$

Taking the expected value of (4.3.4) we get:

$$\begin{aligned} \mathbb{E}\left[\ln\left(\frac{X_T}{X_0}\right)\right] &= \mu T - \frac{1}{2}\mathbb{E}\left[\sigma^2\right]T \\ \Rightarrow \mathbb{E}\left[\sigma^2\right] &= \frac{2}{T}\left(\mu T - \mathbb{E}\left[\ln\left(\frac{X_T}{X_0}\right)\right]\right). \end{aligned} \tag{4.3.5}$$

Next by remembering that $\ln\left(X_t\right) \sim \mathcal{N}\left(\bar{\mu}, \bar{\sigma}^2\right)$ with $\bar{\mu} = \ln\left(X_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)t$ and $\bar{\sigma}^2 = \sigma^2 t$, we can use the properties of a log-normally distributed random variable:

$$\begin{aligned} F = \mathbb{E}\left[X_T\right] &= e^{\bar{\mu} + \frac{\bar{\sigma}^2}{2}} = X_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \frac{1}{2}\sigma^2 T} = X_0 e^{\mu T} \\ \Rightarrow \mu T &= \ln\left(\frac{F}{X_0}\right). \end{aligned} \tag{4.3.6}$$

By substituting (4.3.6) to the expression of $\mathbb{E}\left[\sigma^2\right]$, i.e. the equation (4.3.5), we get:

$$\mathbb{E}\left[\sigma^2\right] = \frac{2}{T}\left(\ln\left(\frac{F}{X_0}\right) - \mathbb{E}\left[\ln\left(\frac{X_T}{X_0}\right)\right]\right). \tag{4.3.7}$$

Next we will rewrite the expected value $\mathbb{E}\left[\ln\left(\frac{X_T}{X_0}\right)\right]$. By introducing a deterministic variable $\alpha \geq 0$, we can write:

$$
\begin{aligned}
\mathbb{E}\left[\ln\left(\frac{X_T}{X_0}\right)\right] &= \mathbb{E}\left[\ln\left(\frac{X_T}{\alpha}\right)\right] + \ln\left(\frac{\alpha}{X_0}\right) \\
&\overset{*}{=} \frac{\overbrace{\mathbb{E}\left[X_T\right]}^{=F}}{\alpha} - \Bigg(1 + \mathbb{E}\left[\int_0^\alpha \frac{1}{S^2}\max\left\{S - X_T, 0\right\} dS\right] + \\
&\mathbb{E}\left[\int_\alpha^\infty \frac{1}{S^2}\max\left\{X_T - S, 0\right\} dS\right]\Bigg) + \ln\left(\frac{\alpha}{X_0}\right) \\
&= \frac{F}{\alpha} - \Bigg(1 + \int_0^\alpha \frac{1}{S^2}\mathbb{E}\left[\max\left\{S - X_T, 0\right\}\right] dS + \\
&\int_\alpha^\infty \frac{1}{S^2}\mathbb{E}\left[\max\left\{X_T - S, 0\right\}\right] dS\Bigg) + \ln\left(\frac{\alpha}{X_0}\right) \\
&\overset{**}{=} \frac{F}{\alpha} - \left(1 + \int_0^\alpha \frac{1}{S^2}e^{r_f T}P(t,S)dS + \int_\alpha^\infty \frac{1}{S^2}e^{r_f T}V(t,S)dS\right) + \ln\left(\frac{\alpha}{X_0}\right) \\
&\overset{***}{\approx} \frac{F}{\alpha} - \left(1 + \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2}e^{r_f T}Q\left(t, S_k\right)\right) + \ln\left(\frac{\alpha}{X_0}\right) \qquad (4.3.8)
\end{aligned}
$$

(*) $\int_0^\alpha \frac{1}{S^2}\max\left\{S - X_T, 0\right\} dS = \begin{cases} 0, & \alpha \le X_T \\ \int_{X_T}^\alpha \frac{1}{S^2}\left(S - X_T\right) dS = \frac{X_T}{\alpha} + \ln\left(\frac{\alpha}{X_T}\right) - 1, & \alpha > X_T \end{cases}$,

also $\int_\alpha^\infty \frac{1}{S^2}\max\left\{X_T - S, 0\right\} dS = \begin{cases} 0, & \alpha \ge X_T \\ \int_\alpha^{X_T} \frac{1}{S^2}\left(X_T - S\right) dS = \frac{X_T}{\alpha} + \ln\left(\frac{\alpha}{X_T}\right) - 1, & \alpha < X_T \end{cases}$.

$\Rightarrow \int_0^\alpha \frac{1}{S^2}\max\left\{S - X_T, 0\right\} dS + \int_\alpha^\infty \frac{1}{S^2}\max\left\{X_T - S, 0\right\} dS = \frac{X_T}{\alpha} + \ln\left(\frac{\alpha}{X_T}\right) - 1$

$\Rightarrow \ln\left(\frac{X_T}{\alpha}\right) = \frac{X_T}{\alpha} - \left(1 + \int_0^\alpha \frac{1}{S^2}\max\left\{S - X_T, 0\right\} dS + \int_\alpha^\infty \frac{1}{S^2}\max\left\{X_T - S, 0\right\} dS\right)$.

(**) By the rational pricing assumption the current price of call and put options (V and P respectively) equals the discounted expected payoff in the maturity, i.e. $\mathbb{E}\left[\max\left\{X_T - S, 0\right\}\right] = e^{r_f T}V(t,S)$ *and* $\mathbb{E}\left[\max\left\{S - X_T, 0\right\}\right] = e^{r_f T}P(t,S)$.

(***) Introduce a function $Q(t,S) = \begin{cases} V(t,S), & \alpha < X_T \\ P(t,S), & \alpha \ge X_T \end{cases}$. *Then we can write* $\int_0^\alpha \frac{1}{S^2}e^{r_f T}P(t,S)dS + \int_\alpha^\infty \frac{1}{S^2}e^{r_f T}V(t,S)dS = \int_0^\infty \frac{1}{S^2}e^{r_f T}Q(t,S)dS$. *Denote the ascending sequence of strike prices by* $\left\{S_k\right\}_{k\in\{1,\ldots,N\}}$. *Then we can approximate the integral by*
$\int_0^\infty \frac{1}{S^2}e^{r_f T}Q(t,S)dS = \sum_{k=1}^N \frac{1}{S_k^2}e^{r_f T}Q\left(t,S_k\right)\underbrace{\left(S_k - S_{k-1}\right)}_{=\Delta S_k} = \sum_{k=1}^N \frac{\Delta S_k}{S_k^2}e^{r_f T}Q\left(t,S_k\right)$.

Thus the expression of $\mathbb{E}\left[\sigma^2\right]$ simplifies to:

$$\begin{aligned}
\mathbb{E}\left[\sigma^2\right] &= \frac{2}{T}\left(\ln\left(\frac{F}{X_0}\right) - \ln\left(\frac{\alpha}{X_0}\right) - \frac{F}{\alpha} + \left(1 + \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right)\right)\right) \\
&= \frac{2}{T}\left(\ln\left(\frac{F}{\alpha}\right) - \frac{F}{\alpha} + \left(1 + \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right)\right)\right) \\
&\overset{****}{\approx} \frac{2}{T}\left(\frac{F}{S_0} - 1 - \frac{\left(\frac{F}{S_0} - 1\right)^2}{2} - \frac{F}{S_0} + \left(1 + \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right)\right)\right) \\
&= \frac{2}{T}\left(\sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right) - \frac{\left(\frac{F}{S_0} - 1\right)^2}{2}\right) \\
&= \frac{2}{T} \sum_{k=1}^{N} \frac{\Delta S_k}{S_k^2} e^{r_f T} Q\left(t, S_k\right) - \frac{1}{T}\left(\frac{F}{S_0} - 1\right)^2 . \qquad (4.3.9)
\end{aligned}$$

(****) Taylor expansion:

$\ln\left(\frac{F}{\alpha}\right) = \ln\left(1 + \left(\frac{F}{\alpha} - 1\right)\right) = \frac{F}{\alpha} - 1 - \frac{\left(\frac{F}{\alpha}-1\right)^2}{2} + \underbrace{\frac{\left(\frac{F}{\alpha} - 1\right)^3}{3} - \ldots}_{\approx 0} \approx \frac{F}{\alpha} - 1 - \frac{\left(\frac{F}{\alpha}-1\right)^2}{2}.$

We have $\frac{F}{\alpha} \approx 1$ if we choose $\alpha = S_0$ to be the first strike price below the forward index level F (as was the definition in VIX formula).

In (4.3.9) we obtained the general form of VIX formula which was given in (4.3.2).

5 Conclusion

In this thesis we introduced the reader to implied volatility measures by discussing how implied volatilities are calculated from option pricing models, and how implied volatility measures are used in real-life markets. We derived the most common option pricing models (i.e. Black-Scholes model and CRR-binomial model) and showed how these models are inverted to find implied volatilities. We discussed the intrinsic problems implied volatilities have (i.e. the term structure of implied volatility and implied volatility smile) and how we could alleviate these problems by considering wide range of options with different maturities and strike prices and constructing implied volatility measures by using a weighted sum of implied volatilities instead of the values implied by single options.

As we saw in the derivation of the option pricing models, the assumptions we make in the derivation of the model affect greatly how accurately the option pricing model values options when compared to the option prices obtained in the markets. The general setting of the option pricing models used to calculate implied volatilities is that all model parameters excluding volatility are directly observable from the markets, therefore the accuracy of the pricing model is crucial because it directly affects how accurate the implied volatility estimates are.

As the assumptions of our pricing models affected the implied volatility estimates, also the market conditions had a significant effect. The option pricing models usually occupy some form of arbitrage argument which requires that risk-free profit opportunities yield the risk-free return, and the expected profit is always an increasing function of the investment riskiness in general. If this type of market efficiency does not exist in the markets the implied volatility estimates are not correct even if other pricing model assumptions correspond to the real-life scenario. Therefore it was highlighted in the thesis that our option pricing models, and thus implied volatility estimates, become inaccurate if either model assumptions are not correct or if markets are not efficient which imply that the real option prices will not converge toward their theoretical values.

6 References

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7 Appendix A (Solving Black-Scholes equation)

In the Section 3.1 we obtained the Black-Scholes stochastic differential equation (SDE) with the boundary conditions:

$$\begin{cases} \dfrac{\partial V}{\partial t} = r_f V(X,t) - r_f X_t \dfrac{\partial V}{\partial X} - \dfrac{1}{2}\sigma^2 X_t^2 \dfrac{\partial^2 V}{\partial X^2} & (7.0.1a) \\ \text{With the boundary conditions:} \\ V(0,t) = 0, \forall t \in [0,T] & (7.0.1b) \\ V(X_t,t) \overset{X_t \to \infty}{\longrightarrow} X_t & (7.0.1c) \\ V(X_T,T) = \max\{X_T - S, 0\} & (7.0.1d) \end{cases}$$

To solve (7.0.1a) with the boundary conditions (7.0.1b)-(7.0.1d), let us begin by rewriting (7.0.1a):

$$\begin{aligned} \frac{\partial V}{\partial t} &= r_f V(X,t) - r_f X_t \frac{\partial V}{\partial X} - \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X^2} && \Big| \text{Multiply by } e^{-r_f t} \\ &\overset{*}{\Rightarrow} \frac{\partial\left(e^{-r_f t}V\right)}{\partial t} + r_f e^{-r_f t} V = r_f e^{-r_f t} V - \\ r_f X_t &\frac{\partial\left(e^{-r_f t}V\right)}{\partial X} - \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2\left(e^{-r_f t}V\right)}{\partial X^2} && | u = e^{-r_f t} V \\ &\Rightarrow \frac{\partial u}{\partial t} = -r_f X_t \frac{\partial u}{\partial X} - \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 u}{\partial X^2} \\ &\Rightarrow \frac{\partial u}{\partial t} + r_f X_t \frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 u}{\partial X^2} = 0. && (7.0.2) \end{aligned}$$

We will do a change of variables $X = e^y$ and $t = T - s$. This yields (**):

$$\begin{aligned} &-\frac{\partial u}{\partial s} + r_f \frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2 \left(-\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \\ &\Leftrightarrow -\frac{\partial u}{\partial s} + \left(r_f - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2 \frac{\partial^2 u}{\partial y^2} = 0. && (7.0.3) \end{aligned}$$

(*) $\frac{\partial\left(e^{-r_f t}V\right)}{\partial t} + r_f e^{-r_f t} V = -r_f e^{-r_f t} V + e^{-r_f t}\frac{\partial V}{\partial t} + r_f e^{-r_f t} V = e^{-r_f t}\frac{\partial V}{\partial t}$

(**) $X\frac{\partial u}{\partial X} = e^y \frac{\partial u}{\partial e^y}\frac{\partial y}{\partial y} = e^y \frac{\partial u}{\partial y}\Big/\left(\frac{\partial e^y}{\partial y}\right) = \frac{\partial u}{\partial y}$. *Therefore* $\frac{\partial u}{\partial X} = \frac{1}{X}\frac{\partial u}{\partial y}$ *and* $\frac{\partial^2 u}{\partial X^2} = \frac{\partial}{\partial X}\left(\frac{1}{X}\frac{\partial u}{\partial y}\right) = -\frac{1}{X^2}\frac{\partial u}{\partial y} + \frac{1}{X}\frac{\partial}{\partial X}\left(\frac{\partial u}{\partial y}\right) = -\frac{1}{X^2}\frac{\partial u}{\partial y} + \frac{1}{X}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial X}\right) = -\frac{1}{X^2}\frac{\partial u}{\partial y} + \frac{1}{X^2}\frac{\partial^2 u}{\partial y^2} \Rightarrow X^2\frac{\partial^2 u}{\partial X^2} = -\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2}$. *Also* $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial (T-s)}\frac{\partial s}{\partial s} = \frac{\partial u}{\partial s}\Big/\left(\frac{\partial (T-s)}{\partial s}\right) = -\frac{\partial u}{\partial s}$.

Next, we do a substitution $z = y + \left(r_f - \frac{1}{2}\sigma^2\right)\tau$ and $\tau = s$. This yields partial derivatives:

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial}{\partial z}, \frac{\partial}{\partial s} = \frac{\partial}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial s} = \left(r_f - \frac{1}{2}\sigma^2\right)\frac{\partial}{\partial z} + \frac{\partial}{\partial \tau}. \quad (7.0.4)$$

Using (7.0.4) we can simplify (7.0.3) to:

$$\begin{aligned}
&-\frac{\partial u}{\partial s} + \left(r_f - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} = 0 \\
&\Rightarrow -\left(r_f - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial z} - \frac{\partial u}{\partial \tau} + \left(r_f - \frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial z} + \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2} = 0 \\
&\Leftrightarrow \frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2}, \quad (7.0.5)
\end{aligned}$$

where $u(z, \tau)$, i.e. u is a function of z and τ. This is the ordinary diffusion equation. The option value at the maturity (7.0.1d) becomes an initial condition (remember $\tau = s = T - t$):

$$u(z, 0) = e^{-r_f T}\max\{X_T - S, 0\}. \quad (7.0.6)$$

The value of call option is:

$$\begin{aligned}
V = e^{r_f t}u(z,\tau) &= e^{r_f t}u\left(y + \left(r_f - \frac{1}{2}\sigma^2\right)\tau, T - t\right) \\
&= e^{r_f t}u\left(\ln(X) + \left(r_f - \frac{1}{2}\sigma^2\right)(T - t), T - t\right). \quad (7.0.7)
\end{aligned}$$

The only thing we remain to do is to solve the diffusion equation $\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2}$ with the initial condition $u(z, 0) = e^{-r_f T}\max\{X_T - S, 0\}$. This diffusion equation can be reduced to the standard heat equation by a change of variables $\tau' = \frac{1}{2}\sigma^2\tau$. This gives us the standard diffusion equation:

$$\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{\partial u}{\partial \tau'}\frac{\partial \tau'}{\partial t} = \frac{1}{2}\sigma^2\frac{\partial u}{\partial \tau'} \\
&\Rightarrow \frac{1}{2}\sigma^2\frac{\partial u}{\partial \tau'} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial z^2} \\
&\Rightarrow \frac{\partial u}{\partial \tau'} = \frac{\partial^2 u}{\partial z^2}. \quad (7.0.8)
\end{aligned}$$

This type of diffusion equation is called a Cauchy problem (i.e. we are dealing in the upper half-plane of (z, τ')-plane). Green's function for the heat equation, denoted by $\Phi(z, \tau')$, is given by:

$$\Phi(z,\tau') = \frac{1}{\sqrt{4\pi\tau'}} e^{-|x|^2/(4\tau')} = \frac{1}{\sqrt{2\pi\sigma^2\tau}} e^{-|x|^2/\left(2\sigma^2\tau\right)}, \quad z \in \mathbb{R}, \tau' > 0. \quad (7.0.9)$$

Green's function for the Cauchy problem can be derived from the Fourier theorem (first convert the diffusion equation to the Fourier side, find the solution there and convert back to the original domain, i.e. the Fourier inversion).

The solution for the Cauchy problem is:

$$\begin{aligned} u(z,\tau) = (\Phi \otimes g)(z) &= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} e^{-|z-\zeta|^2/\left(2\sigma^2\tau\right)} g(\zeta) d\zeta \\ &= \frac{e^{-r_f T}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} e^{-|z-\zeta|^2/\left(2\sigma^2\tau\right)} \max\left\{e^{\zeta} - S, 0\right\} d\zeta, \end{aligned} \quad (7.0.10)$$

where $\otimes$ denotes the convolution operator. By substituting (7.0.10) to the formula of the call option value expressed in (7.0.7), we get:

$$\begin{aligned} V(X,t) = e^{r_f t} u(z,\tau) &= e^{r_f t} u\left(\ln(X) + \left(r_f - \frac{1}{2}\sigma^2\right)\tau, \tau\right) \\ &= \frac{e^{-r_f(T-t)}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} e^{-\left|\ln(X)+\left(r_f-\frac{1}{2}\sigma^2\right)\tau-\zeta\right|^2/\left(2\sigma^2\tau\right)} \max\left\{e^{\zeta} - S, 0\right\} d\zeta \\ &= \frac{e^{-r_f\tau}}{\sqrt{2\pi\sigma^2\tau}} \int_{-\infty}^{\infty} e^{-\left|\ln(X)+\left(r_f-\frac{1}{2}\sigma^2\right)\tau-\zeta\right|^2/\left(2\sigma^2\tau\right)} \max\left\{e^{\zeta} - S, 0\right\} d\zeta. \end{aligned} \quad (7.0.11)$$

Simplification of (7.0.11) gives:

$$V(X,t) = X_t \Psi\left(d_+(\tau, X)\right) - S e^{-r_f\tau} \Psi\left(d_-(\tau, X)\right), \quad (7.0.12)$$

where $\Psi(\bullet)$ is the CDF of the standard normal distribution and $d_\pm(\tau, x) = \frac{\ln(X_t/S)+\left(r_f \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$, $\tau = T - t$.