Aalto University
School of Science
Master's Programme in Mathematics and Operations Research

Miska Rissanen

Lebesgue space norm estimates for the spherical maximal function

Master's Thesis
Espoo, February 1, 2019

Supervisor: Assistant Professor Riikka Korte
Advisor: Professor Juha Kinnunen

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ABSTRACT OF MASTER'S THESIS

Author:	Miska Rissanen
Title: Lebesgue space norm estimates for the spherical maximal function	
Date:	February 1, 2019 **Pages:** v + 49
Major:	Mathematics
Supervisor:	Assistant Professor Riikka Korte
Advisor:	Professor Juha Kinnunen

The Hardy-Littlewood maximal function, defined as the supremum of integral averages of a function over balls, is a classical and well-known tool in analysis. One essential property of the maximal function is the Hardy-Littlewood maximal inequality, which states that a weak type Lebesgue space norm estimate holds for $p = 1$, and a strong type estimate holds for all $p > 1$.
In this thesis, a more general spherical maximal operator is studied. Instead of balls, the integral average is taken over the boundary of the ball, with respect to the $n - 1$ -dimensional spherical measure. The main result of this work is a Lebesgue space norm estimate for the spherical maximal function.
We study the Fourier transform of a radially restricted spherical average operator. The dyadic Littlewood-Paley decomposition and a decay estimate for the Fourier transform of the spherical measure are used to prove $L^p \to L^q$ estimates on certain pairs (p, q). These results are then generalized to the full maximal operator, and interpolated for more general pairs (p, q).

Keywords: maximal functions, oscillatory integrals, Littlewood-Paley theory, Fourier transform, real analysis

Language: English

Aalto-yliopisto
Perustieteiden korkeakoulu
Master's Programme in Mathematics and Operations Research

DIPLOMITYÖN TIIVISTELMÄ

Tekijä:	Miska Rissanen		
Työn nimi: Lebesguen normiestimaatit pallopinnan maksimaalifunktiolle			
Päiväys:	1. helmikuuta 2019	**Sivumäärä:**	v + 49
Pääaine:	Mathematics		
Valvoja:	Apulaisprofessori Riikka Korte		
Ohjaaja:	Professori Juha Kinnunen		

Hardy-Littlewoodin maksimaalifunktio on klassinen ja hyvin tunnettu operaattori, joka on määritelty pallon yli laskettujen integraalikeskiarvojen supremumina. Yksi maksimaalifunktion tärkeä ominaisuus on Hardy-Littlewoodin maksimaalilause, jonka mukaan operaattorille pätee heikon tyypin Lebesguen normiestimaatti kun $p = 1$, ja vahvan tyypin estimaatti kaikille $p > 1$.
Tässä diplomityössä tutkitaan yleisempää pallopinnan maksimaalifunktiota. Pallojen sijaan funktion integraalikeskiarvo lasketaan pallopinnan yli $n-1$ -ulotteisen pintamitan suhteen. Työn päätulokset ovat Lebesgue avaruuden normiestimaatit pallopinnan maksimaalifunktiolle.
Tutkimme radiaalisesti rajoitettua pallopinnan integraalikeskiarvo-operaattorin Fourierin muunnosta. Dyadista Littlewood-Paleyn hajotelmaa ja pintamitan Fourierin muunnoksen vähenevää estimaattia käyttämällä todistetaan $L^p \to L^q$ -normiestimaatteja tietyille (p, q)-pareille. Nämä estimaatit voidaan yleistää täydelle maksimaalioperaattorille ja interpoloida yleisemmille (p, q)-pareille.

Asiasanat:	maksimaalifunktiot, oskilloivat integraalit, Littlewood-Paleyn teoria, Fourier-muunnos, reaalianalyysi
Kieli:	Englanti

Acknowledgements

I would like to thank my advisor Prof. Kinnunen for his guidance throughout my studies, and especially in the process of writing this thesis. His input on the topic was invaluable, and this work might have never seen the light of day without his advice on which books to open, and which ones are best left unread.

I also want to thank Prof. Korte for advising me during my first months of research, supervising this work and making it possible for me to work in the NPDE research group. To all other members of the group, I wish you the best of luck!

A special thank you goes to all my dear friends at the Guild of Physics for support in our mutual struggles in the master's thesis process. You all made my now concluding student life unforgettable.

Lastly, I must thank my partner Helinä Hakala for her loving support and encouragement. You were always there for me.

Espoo, February 1, 2019

Miska Rissanen

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Chapter 1

Introduction

The Hardy-Littlewood maximal function, defined for locally integrable functions $f : \mathbb{R}^n \to [-\infty, \infty]$, is a classical and well-known tool in analysis. It is defined as the supremum of integral averages over balls centered at x, that is

$$\mathcal{M}f(x) = \sup_{r>0} \fint_{B(x,r)} |f(y)|dy, \tag{1.1}$$

where

$$\fint_E f(y) = \frac{1}{|E|} \int_E f(y)dy.$$

In this thesis, we study the related spherical maximal function, which is defined for any function $f \in L^p(\mathbb{R}^n)$ as

$$Mf(x) = \sup_{r>0} \fint_{\partial B(x,r)} |f(y)|d\mathcal{H}^{n-1}(y). \tag{1.2}$$

Unlike in the classical maximal function, we average over the boundary of the ball with respect to the $n-1$ dimensional Hausdorff measure. The spherical maximal function is related to the classical one in fractional cases, as shown in [4]. When studying the smoothness properties of the fractional maximal function

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|dy, \tag{1.3}$$

one ends up requiring estimates for the spherical fractional maximal function

$$M_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{\partial B(x,r)} |f(y)|d\mathcal{H}^{n-1}(y). \tag{1.4}$$

The final goal would be to prove the following norm estimate for operator (1.4) used in [4]

Theorem 1.1. *Let $n \geq 2$, $p > n/(n-1)$ and*

$$0 \leq \alpha < \min\left\{\frac{n-1}{p}, n - \frac{2n}{(n-1)p}\right\}.$$

Then there exists a constant C, depending only on n, p and α, such that

$$\|M_\alpha f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

holds for $p^* = np/(n-\alpha p)$.

This is a special case of more general $L^p \to L^q$ estimates for the operator (1.2). This type of bounds are well understood, but their proofs are somewhat scattered across multiple publications. The goal of this thesis is to unify these proofs. First norm estimates for (1.2) were attained by Stein [11] in 1976 for the case $n \geq 3$ and $p = q$. The similar result for $n = 2$ are due to Bourgain [2] a decade later in 1986. Generalizations to $L^p \to L^q$ estimates were published by Schlag and Sogge in [7] and [8] in 1997. In 2002 Lee [6] finalized the results up to the endpoints.

The analysis of spherical maximal functions requires more sophisticated methods than classical maximal operators. First, covering arguments used to prove L^p boundedness of (1.1) will not work, as we are unable to countably cover sets with boundaries of balls. Rather, we will use methods in Fourier analysis and oscillatory integrals.

One essential tool is so-called Littlewood-Paley decomposition, in which functions are decomposed into components with almost disjoint Fourier supports. The components can then be studied separately and some results can be transferred back to the original function. These methods will be studied in chapter 2.2. The Fourier transform of the surface measure $\mathcal{H}^{n-1}$ restricted to the surface of the sphere $\partial B(x,r)$ will also have nice decay properties, as will be shown in chapter 2.3.

Once the prerequisites are understood, we will prove the required norm estimates for a restricted version of the spherical maximal function in chapter 3. Norm estimates will first be constructed with certain (p,q) pairs and further interpolated to other pairs. These estimates on individual points can be very technical to prove, and much of the advancements in the analysis of spherical maximal functions has been due to obtaining sharper individual norm estimates, often referred as local smoothing estimates. Some of the more advanced estimates will be cited from previous work, and can be found for example in [8], [6] and [9]. Simpler smoothing estimates nicely portray the usage of the methods from chapter 2 and will be explained in detail.

In chapter 4 the norm estimates will be extended to full spherical maximal function, again using properties of Littlewood-Paley decomposition.

1.1 Representation of the spherical maximal operator

For future analysis, let us derive other forms for (1.2)

$$\begin{aligned} Mf(x) &= \sup_{r>0} \fint_{\partial B(x,r)} |f(y)| d\mathcal{H}^{n-1}(y) \\ &= \sup_{r>0} \fint_{\partial B(0,r)} |f(x-y)| d\mathcal{H}^{n-1}(y) \\ &= \sup_{r>0} \fint_{\partial B(0,1)} |f(x-ry)| d\mathcal{H}^{n-1}(y) \\ &= \sup_{r>0} \int_{S^{n-1}} |f(x-ry)| d\sigma(y), \end{aligned} \tag{1.5}$$

where S^{n-1} is the unit sphere centered at origin, and σ is the normalized surface measure on the sphere. The measure can be attained, for example, by restricting the Hausdorff measure

$$\sigma(E) = \frac{1}{|S^{n-1}|} \mathcal{H}^{n-1}(E \cap S^{n-1}).$$

The Fourier transform, denoted $\hat{f}$, of a function f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx.$$

The inverse Fourier transform is denoted $\check{f}$. Many of the arguments in this thesis work on the Fourier side, thus another useful form for (1.2) is

$$\begin{aligned} \int_{S^{n-1}} |f(x-ry)| d\sigma(y) &= \int_{S^{n-1}} \int_{\mathbb{R}^n} e^{2\pi i (x-ry)\cdot\xi} |\hat{f}|(\xi) d\xi d\sigma(y) \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} |\hat{f}|(\xi) \int_{S^{n-1}} e^{-2\pi i y\cdot(r\xi)} d\sigma(y) d\xi \\ &= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} |\hat{f}|(\xi) \widehat{d\sigma}(r\xi) d\xi. \end{aligned} \tag{1.6}$$

The second equality follows from Fubini's Theorem and the third line is the Fourier transformation of the measure σ, which will be studied further in Section (2.3). One easily sees a third form for the spherical maximal function, arising from (1.6)

$$\begin{aligned}\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}|\hat{f}|(\xi)\widehat{d\sigma}(r\xi)d\xi &= (|\hat{f}|(\xi)\hat{d\sigma}(r\xi))\check{} \\ &= (|f(\cdot)| * \frac{1}{r^n}d\sigma(\frac{\cdot}{r}))(x) \\ &= (|f| * d\sigma_r)(x),, \end{aligned} \tag{1.7}$$

where the second equality is the time scaling property of Fourier transform.

1.2 Geometric intuitions

Any function f belonging to a L^p space has singularities only on a set E of Lebesgue measure zero, $|E| = 0$. Thus, if we require the spherical maximal function to belong into any Lebesgue space, we must also require this property. It turns out, that Theorem 1.1 implies the following

Theorem 1.2. *If a measurable set $E \subset \mathbb{R}^n$, $n \geq 2$, is of Lebesgue measure zero, then for almost every $x \in \mathbb{R}^n$, every circle centered at x intersects E on a set of Hausdorff measure zero, that is $\mathcal{H}^{n-1}(E \cap \partial B(x,r)) = 0$.*

It is easy to see that curvature plays a large role in Theorem 1.1, as well as the argument above. If one were to replace the circles with boundaries of cubes centered at x, any L^p function blowing up to infinity on one of the coordinate axes would have the maximal function be unbounded everywhere.

This also shows how any $L^p \to L^q$ estimate fails in $n = 1$, as the zero dimensional Hausdorff measure is just a counting measure and thus the single points of singularities have non-zero $\mathcal{H}^0$ measure.

1.3 Hardy-Littlewood maximal function

We now recall some facts about the classical Hardy-Littlewood maximal function (1.1). The operator is defined for any locally integrable functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ to keep the integral averages finite, but this may lead to uninteresting cases, for which the supremum diverges and $\mathcal{M}f(x) = \infty$ for every $x \in \mathbb{R}^n$. For example any non-zero polynomial satisfies local integrability, but the integral averages grow uncontrollably in r.

Let us thus limit ourselves to globally integrable $L^p(\mathbb{R}^n)$ functions for $1 \leq p \leq \infty$. The well-known Hardy-Littlewood Maximal Theorem states that $\mathcal{M}$ is an bounded operator from L^p to itself for $p > 1$ and a weak type estimate holds in L^1.

Theorem 1.3. *Let $f \in L^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$, then there exists a constant C, depending only on n and p, such that*

$$\|\mathcal{M}f\|_{L^p} \leq C \|f\|_{L^p}, \quad p > 1, \tag{1.8}$$

$$|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L^1}, \quad p = 1. \tag{1.9}$$

The proof of inequality (1.9) is based on studying the set

$$\begin{aligned} A_\lambda &= \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\} \\ &= \{x \in \mathbb{R}^n : \exists r_x > 0 \text{ s.t.} \frac{1}{\lambda} \int_{B(x,r_x)} |f(y)|dy > |B(x, r_x)|\} \end{aligned}$$

and using Vitali Covering Theorem to cover A_λ with countable union of expanded balls $\cup_{i=1}^{\infty} B(x_i, 5r_{x_i})$, for which

$$|B(x_i, r_{x_i})| < \frac{1}{\lambda} \int_{B(x_i,r_{x_i})} |f(y)|dy \leq \frac{1}{\lambda} \|f\|_{L^1}$$

and

$$\sum_{j=1}^{\infty} |B(x_i, r_{x_i})| = |\bigcup_{i=1}^{\infty} B(x_i, r_{x_i})|.$$

This implies

$$\begin{aligned} |A_\lambda| &\leq |\bigcup_{x \in A_\lambda} B(x, r_x)| \leq \sum_{j=1}^{\infty} |B(x_i, 5r_{x_i})| \\ &\leq \frac{5^n}{\lambda} \sum_{i=1}^{\infty} \int_{B(x_i,r_{x_i})} |f(y)|dy \leq \frac{5^n}{\lambda} \|f\|_{L^1}. \end{aligned}$$

The result (1.9) can then be modified to prove (1.8), but the main tool is still in covering of sets with countable union of balls.

Another way to gain (1.8) for $1 < p < \infty$ is using Marcinkiewicz Interpolation Theorem between (1.9) and the L^∞ estimate, which can be easily seen from

$$\sup_{r>0} \fint_{B(x,r)} |f(y)|dy \leq \|f\|_{L^\infty} \sup_{r>0} \fint_{B(x,r)} dy = \|f\|_{L^\infty}.$$

The Marcinkiewicz Interpolation Theorem is also a great tool for deriving estimates for the spherical maximal function and it will be studied more in chapter 2.1.

It is important to note, that Theorem 1.3 produces only $L^p \to L^p$ estimates. To gain similar $L^p \to L^q$ estimates, one need to add a fractional term to $\mathcal{M}$ and study the fractional maximal function (1.3) and a related operator, called Riesz potential

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy$$

for $0 < \alpha < n$. We easily see that

$$\frac{r^\alpha}{|S^n| r^n} \int_{B(x,r)} |f(y)| dy \leq \frac{1}{|S^n|} \int_{B(x,r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \leq C I_\alpha |f(x)|,$$

and taking supremum in r gives a pointwise bound

$$\mathcal{M}_\alpha f(x) \leq I_\alpha |f(x)|.$$

For the Riesz potential, $L^p \to L^q$ estimates are known as the Hardy-Littlewood-Sobolev Lemma

Lemma 1.1. *For $\alpha > 0$, $p > 1$, and $\alpha < p/n$, there exists C, depending only on n,p and α, such that*

$$\|I_\alpha f\|_{L^{p^*}(\mathbb{R}^n)} \leq c \|f\|_{L^p(\mathbb{R}^n)}, \quad p^* = \frac{pn}{n - \alpha p},$$

holds for every $f \in L^p(\mathbb{R}^n)$.

With this lemma and the above pointwise bound, we can construct $L^p \to L^q$ bounded maximal operators by choosing a proper fractional exponent α

Theorem 1.4. *Let $1 < p \leq q$, and choose $\alpha = n(1/p - 1/q)$, then for any $f \in L^p(\mathbb{R}^n)$ we have*

$$\|\mathcal{M}_\alpha f\|_{L^q(\mathbb{R}^n)} \leq c \|f\|_{L^p(\mathbb{R}^n)}.$$

Here $q = p^*$ from Lemma 1.1, but we interpret the result so that a proper fractional constant is chosen based on the required p and q.

The Hardy-Littlewood maximal functions have many essential applications in analysis. For example the Lebesgue Differentation Theorem is a consequence of the weak L^1-boundedness of the maximal function.

Theorem 1.5. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(x) - f(y)| dy = 0$$

holds for almost every $x \in \mathbb{R}^n$.

Another application is in the theory of Sobolev spaces, where Sobolev functions can be characterized by pointwise boundedness by the maximal function. This result actually generalizes to analysis on metric spaces, where maximal functions are also an useful tool. See, for example [5, Chapter 5].

Theorem 1.6. *If $u \in W^{1,p}(\mathbb{R}^n)$, $1 < p \leq \infty$ a Sobolev function, then there exists $c(n)$, so that*

$$|u(x) - u(y)| \leq c|x - y|(\mathcal{M}|Du|(x) + \mathcal{M}|Du|(y)),$$

for all x, y outside some set N of Lebesgue measure zero.

Similar argument is also true the other way, so that if there exists a function $g \in L^p(\mathbb{R}^n)$, such that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)),$$

then $u \in W^{1,p}(\mathbb{R}^n)$.

Note that both of these applications describe local behavior of functions. In this sense $\mathcal{M}$ is a local operator, as the supremum of integral averages is never attained in $r \to \infty$ for L^p functions.

Chapter 2

Prerequisites

In this chapter we go over some techniques and results, fundamental to the proof Lebesgue space norm estimates for the spherical maximal function. Interpolation theorems allow us to interpolate endpoint estimates to wide range of $L^p \to L^q$ estimates, Littlewood-Paley theory localizes problems on the Fourier side and results about the Fourier transform of the spherical measure σ shows us important decay properties of $\widehat{d\sigma}$ in (1.6).

2.1 Interpolation of operators

2.1.1 Operators on Lebesgue and Lorentz spaces

As the main claim in Theorem 1.1 is that for any L^p function the spherical maximal function belongs to some Lebesgue space, in this section we will define these spaces, as well as Lorentz spaces which are generalizations of typical L^p spaces.

For $\mathbb{R}^n$, the Lebesgue space of power p, or L^p space, is the space of measurable functions with Lebesgue integrable pth powers. A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$ with norm $\|f\|_{L^p(\mathbb{R}^n)}$ if

$$\|f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f|^p dx \right)^{\frac{1}{p}} < \infty.$$

The limit space $L^\infty(\mathbb{R}^n)$ is the space of essentially bounded functions

$$\|f\|_{L^\infty(\mathbb{R}^n)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.$$

Going forward, we will leave out the underlying space in the norm, unless significant.

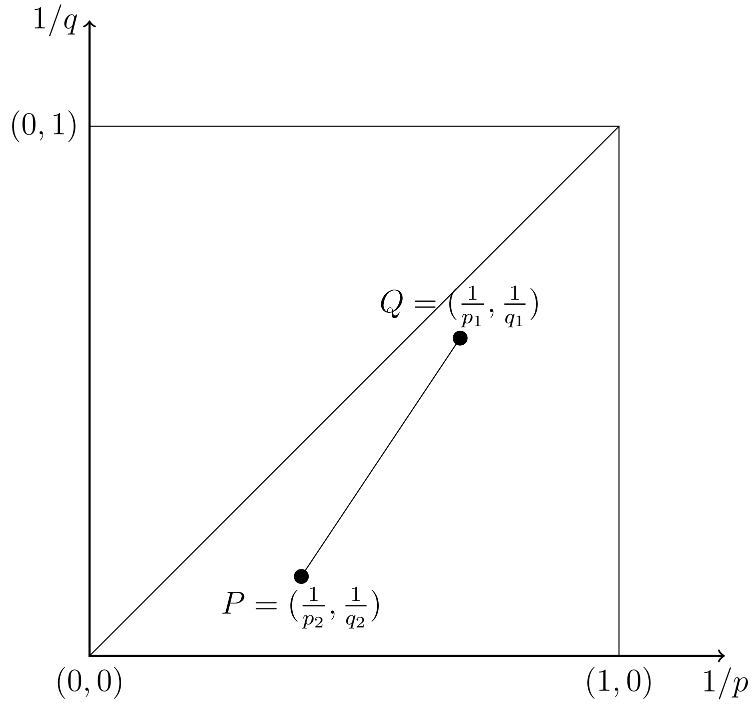

Figure 2.1: The $(1/p, 1/q)$ -coordinate system in which we study the bounds for the spherical maximal function. Given two points P and Q, in which $L^{p_i} \to L^{q_i}$ bounds exist, we are interested in the line segment between these points. The diagonal line represents the easier $L^p \to L^p$ bounds, whereas more general techniques are needed for other points. Also, for most theorems, $p_i \leq q_i$ is required, restricting us to the lower triangle.

Given an operator T acting on two Lebesgue spaces L^{p_1}, L^{p_2} so that

$$f \in L^{p_i} \Rightarrow Tf \in L^{q_i}, \quad i = 1, 2,$$

interpolation theory shows that T can have this type of boundedness in L^p spaces with $p_1 < p < p_2$. This concept is easily understood in a $(1/p, 1/q)$ -coordinate system, as shown in Figure 2.1.1. In this setting, the question is just proving linear interpolation between two points in the rectangle $[0, 1] \times [0, 1]$.

When working with operators on function spaces, the mapped functions might not belong to L^p but in some more general space. These spaces $L^{p,r}$ are called Lorentz spaces, and they include the classical weak L^p space $L^{p,\infty}$.

Definition 2.1. *For $1 \leq p < \infty$, $0 < r \leq \infty$, the Lorenz space $L^{p,r}$ with* *respect to measure μ consists of all measurable f, for which*

$$\|f\|_{L^{p,r}} = \begin{cases} p^{\frac{1}{r}} \left(\int_0^\infty \lambda^r \mu\{x \in \mathbb{R}^n : |f(x)| \geq \lambda\}^{\frac{r}{p}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{\lambda > 0} \lambda \mu\{x \in \mathbb{R}^n : |f(x)| > \lambda\}^{\frac{1}{p}}, & r = \infty, \end{cases}$$

is finite.

Note that $\|\cdot\|_{L^{p,r}}$ fails the triangle inequality and is instead a quasinorm, so that for every f, g there exists a $K > 0$ for which

$$\|f + g\|_{L^{p,r}} \leq K(\|f\|_{L^{p,r}} + \|g\|_{L^{p,r}})$$

holds. Next we define the type of Lorentz norm estimates needed for interpolation theorems.

Definition 2.2. *An operator T is said to be of weak type (p, q) if it is bounded from $L^{p,1}$ into $L^{q,\infty}$, that is, if there exists a constant M, such that*

$$\|Tf\|_{L^{q,\infty}} \leq M \|f\|_{L^{p,1}}$$

for all $f \in L^{p,1}$. Alternatively, T is of strong type (p, q) if

$$\|Tf\|_{L^q} \leq \|f\|_{L^p}$$

holds for all $f \in L^p$.

2.1.2 Interpolation theorems

The classical Marcinkiewicz Interpolation Theorem states that if a sublinear operator T is of weak type (p_i, p_i), $i = 1, 2$ and $1 \leq p_1 < p_2 \leq \infty$, then T maps L^p to L^p strongly for $p_1 < p < p_2$. A well-known example is the Hardy-Littlewood maximal function, for which $L^p \to L^p$ boundedness is achieved for any $p > 1$ by interpolating between a weak $(1, 1)$ bound and a strong (∞, ∞) bound.

As we are searching for estimates where q might differ from p, we need a more general version of the Marcinkiewicz Interpolation Theorem. If $P = (1/p_1, 1/q_1)$ and $Q = (1/p_2, 1/q_2)$ lie inside the cube $[0, 1] \times [0, 1] \in \mathbb{R}^2$ and T is of weak type at these points, we want strong bounds on the open line between P and Q, namely when p and q are as

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \tag{2.1}$$

and $0 < \theta < 1$. If we would additionally have a third bound at some point $W = (1/p_3, 1/q_3) \in [0,1] \times [0,1]$, repeated interpolation between the line PQ and point W achieves strong bounds in the interior of the triangle PQW. With some restrictions, the interpolation can be repeated for additional points to the interior of a convex polygon in the cube $[0,1] \times [0,1]$. We must require $p_i \leq q_i$, which restricts us to the lower triangle of the cube, and we are unable to interpolate parallel to the coordinate axes. Let us now state the general Marcinkiewicz Interpolation Theorem.

Theorem 2.1. *Let $1 \leq p_1 < p_2 < \infty$ and $1 \leq q_1, q_2 \leq \infty$ with $q_1 \neq q_2$ and $p_i \leq q_i$. Then for any sublinear operator, for which*

$$\|Tf\|_{L^{q_i,\infty}} \leq M_i \|f\|_{L^{p_i,1}}, \quad i = 1, 2,$$

holds, there exists a constant $C = C(\theta, p_1, p_2, q_1, q_2)$, such that

$$\|Tf\|_{L^q} \leq C M_1^{\theta} M_2^{1-\theta} \|f\|_{L^p} \tag{2.2}$$

holds for $0 < \theta < 1$ and p and q are defined as in (2.1).

The proof is standard and can be founds on [1, Chapter 4.4]. Recall also the complex method of interpolation, the Riesz-Thorin Interpolation Theorem

Theorem 2.2. *Suppose $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Then for any linear operator T, for which*

$$\|Tf\|_{L^{q_i}} \leq M_i \|f\|_{L^{p_i}}, \quad i = 1, 2,$$

holds, T is of strong type (p, q), so that

$$\|Tf\|_{L^q} \leq M_1^{\theta} M_2^{1-\theta} \|f\|_{L^p}$$

holds for $0 < \theta < 1$ and p and q are defined as in (2.1)

Altough this method of interpolation would allow us to work on estimates, where $q < p$ and gives sharper bounds, the complex method requires the operator T to be linear and endpoint estimates to be strong, so it is not well suited for our study.

Next we prove a interpolation result for sums of operators from [6]. It is somewhat different from other interpolation theorems, as we are requiring strong bounds on the endpoints, and are left with weak bounds on the interpolation segment. However, the gain is boundedness for a sum of operators.

Lemma 2.1. *Let $\epsilon_1, \epsilon_2 > 0$. If a sequence $\{T_j\}$, $j = 0, 1, 2, ...$ of sublinear operators is bounded with*

$$\|T_j f\|_{L^{q_1}} \leq M_1 2^{\epsilon_1 j} \|f\|_{L^{p_1}}, \quad \|T_j f\|_{L^{q_2}} \leq M_2 2^{-\epsilon_2 j} \|f\|_{L^{p_2}}, \tag{2.3}$$

for some $1 \leq p_1, p_2 < \infty$, $1 \leq q_1, q_2 < \infty$. Then the sum operator $T = \sum T_j$ is bounded as

$$\|Tf\|_{L^{q,\infty}} \leq C M_1^{\theta} M_2^{1-\theta} \|f\|_{L^{p,1}},$$

where $\theta = \epsilon_2/(\epsilon_1 + \epsilon_2)$, and p and q are as in (2.1).

Proof. Set $T_N = \sum_0^N T_j$ and $T^N = \sum_{N+1}^{\infty} T_j$, for some $N \in \mathbb{Z}$. For these sums we can find bounds

$$\begin{aligned}\|T_N f\|_{L^{q_1}} &\leq \sum_{j=0}^{N} \|T_j f\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \sum_{j=0}^{N} 2^{\epsilon_1 j} \\ &\leq C(\epsilon_1) M_1 2^{N\epsilon_1} \|f\|_{L^{p_1}}\end{aligned}$$

as we have $2^{\epsilon N}/(1 - 2^{-\epsilon}) \geq \sum_{j=0}^{N} 2^{\epsilon j}$, for any $\epsilon > 0$. We also have a bound

$$\begin{aligned}\left\|T^N f\right\|_{L^{q_2}} \leq \sum_{j=N+1}^{\infty} \|T_j f\|_{L^{q_2}} &\leq M_2 \|f\|_{L^{p_2}} \sum_{j=N+1}^{\infty} 2^{-\epsilon_2 j} \\ &= M_2 \|f\|_{L^{p_2}} \left(\sum_{j=0}^{\infty} 2^{-\epsilon_2 j} - \sum_{j=0}^{N} 2^{-\epsilon_2 j} \right) \\ &= \frac{2^{-\epsilon_2 N}}{1 - 2^{-\epsilon_2}} M_2 \|f\|_{L^{p_2}} \\ &= C(\epsilon_2) M_2 2^{-\epsilon_2 N} \|f\|_{L^{p_2}}.\end{aligned}$$

Let E be a measurable set. By the estimates above estimates and Chebyshev's inequality, we have for some $\lambda > 0$

$$\begin{aligned}|\{x : |T\chi_E(x)| > \lambda\}| &\leq |\{x : |T_N \chi_E(x)| > \tfrac{1}{2}\lambda\}| + |\{x : |T^N \chi_E(x)| > \tfrac{1}{2}\lambda\}| \\ &\leq \left(\frac{1}{\frac{1}{2}\lambda}\right)^{q_1} \int_{\mathbb{R}^n} |T_N \chi_E|^{q_1} dx + \left(\frac{1}{\frac{1}{2}\lambda}\right)^{q_2} \int_{\mathbb{R}^n} |T^N \chi_E|^{q_2} dx \\ &\leq C(\lambda^{-q_1} \|T\chi_E\|_{L^{q_1}}^{q_1} + \lambda^{-q_2} \|T\chi_E\|_{L^{q_2}}^{q_2}) \\ &\leq C(M_1^{q_1} 2^{N\epsilon_1 q_1} |E|^{q_1/p_1} \lambda^{-q_1} + M_2^{q_2} 2^{-N\epsilon_2 q_2} |E|^{q_2/p_2} \lambda^{-q_2}).\end{aligned}$$

In the sum above, the first term is strictly increasing and other one strictly decreasing with respect to N. We can remove the dependency on N by minimizing the sum with respect to N, and after some lengthy computation, we get

$$|\{x : |T\chi_E(x)| > \lambda\}| \leq C(M_1^\theta M_2^{1-\theta}|E|^{1/p}\lambda^{-1})^q,$$

which is exactly the required weak estimate

$$\lambda|\{x : |T\chi_E(x)| > \lambda\}|^{1/q} \leq CM_1^\theta M_2^{1-\theta}|E|^{1/p}$$

for indicator functions. By sublinearity, this extends to simple functions and further to L^p functions with approximation by simple functions. □

Thus, if we are able to find strong bounds for each component of a sum of operators on some endpoints, so that the norm converges to zero at the other endpoint, we gain weak boundedness for the summed operator, on some point between. Note that if (2.3) holds with $\epsilon_1 = 0$ on one end point, we can insert any value to the power of two and so modify θ to gain the result on the whole open line between the endpoints.

2.2 Littlewood-Paley theory

Next we study the basics of a tool in harmonic analysis, called Littlewood-Paley theory. The theory is bases on decomposing a function f on the Fourier side to nearly disjoint pieces, essentially supported on an annulus around radius 2^j. As these pieces are compactly supported and nearly disjoint, they can be more easily analyzed, and some nice properties can then be transferred back to the original function f.

First recall the Plancherel's identity

$$\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2}.$$

Immediate way of disjointing the Fourier side of a function, would be to use indicator functions of disjoint sets. Say $f \in L^2$ and construct f_j, $j \in \mathbb{Z}$, so that

$$\hat{f}_j(\xi) = \hat{f}(\xi)\chi_{\{2^j \leq |\xi| < 2^{j+1}\}}(\xi).$$

As the decomposing sets are disjoint, Plancherel's identity gives

$$\|f\|_{L^2} = \left\|\hat{f}\right\|_{L^2} = \left\|\sum_{j\in\mathbb{Z}} \hat{f}_j\right\|_{L^2} = \sum_{j\in\mathbb{Z}} \left\|\hat{f}_j\right\|_{L^2} = \sum_{j\in\mathbb{Z}} \|f_j\|_{L^2}.$$

This is of course a great way of decomposing functions in L^4, but unfortunately this type of rough decomposition fails miserably in L^p, for any $p \neq 2$. Even single components f_j will not be bounded, and for example estimation by Young's convolution inequality fails when $n \geq 2$. Say $\varphi = \chi_{B(0,1)}$ to get

$$\|f_j\|_{L^p} = \|f * \check{\varphi}\|_{L^p} \leq \|f\|_{L^p} \|\check{\varphi}\|_{L^1} .$$

It is known that the Fourier transform of unit ball is not integrable, so f_j cant be bound in L^p. To overcome this, in Littlewood-Paley theory, we use smooth bump functions instead of strict indicator functions. The appropriate space is the space of Schwartz functions

Definition 2.3. *The Schwartz space $\mathcal{S}$ consists of rapidly decaying smooth functions, so that*

$$\mathcal{S} = \{f \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha D^k f(x)| < \infty\}$$

for all multi-indices $\alpha, k \in \mathbb{N}^n$.

One important property of $\mathcal{S}$ is that Fourier transform is a automorphism on this space, so that $\hat{f} \in \mathcal{S}$ for all Schwartz functions f.

To construct the Littlewood-Paley decomposition, we first define compactly supported smooth bump functions around radius 2^j that localize $\hat{f}$. One easily sees that these are Schwartz functions. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial, decreasing function, such that $\phi(|\xi|) = 1$ when $|\xi| \leq 1$ and $\phi(|\xi|) = 0$ when $|\xi| \geq 2$. Now we set

$$\varphi(\xi) = \phi(\xi) - \phi(2\xi),$$

and notice that φ is a bump function supported on the open annulus $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$ and $\varphi(\xi) = 1$ around $|\xi| = 1$. By scaling the function φ we achieve a partition of unity

$$\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \backslash \{0\},$$

where $\varphi_j(\xi) = \varphi(\xi/2^j)$ is supported on $\{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. The only problematic point is $\xi = 0$, as all φ_j are supported away from origin. Fortunately, for our application we are mostly interested in the higher frequencies $j \geq 0$. We can let $\varphi_0(\xi) = \phi(\xi)$ and have a partition of unity of the whole space

$$\sum_{j \geq 0} \varphi_j(\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.4}$$

This can be seen, as for any $\xi \in \mathbb{R}^n$, only at most two terms in the sum are non-zero. For $|\xi| \le 1$, only φ_0 is supported and it has value 1. If $1 < |\xi| < 2$, we get

$$\varphi_0(\xi) + \varphi_0(2^{-1}\xi) - \varphi_0(\xi) = \varphi_0(2^{-1}\xi) = 1,$$

and for general annulus $\{\xi \in \mathbb{R}^n : 2^j < |\xi| \le 2^{j+1}\}$ we have

$$\begin{aligned}\varphi_j(\xi) + \varphi_{j+1}(\xi) &= \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi) + \varphi_0(2^{-j-1}\xi) - \varphi_0(2^{-j}\xi)\\ &= -\varphi_0(2^{-j+1}\xi) + \varphi_0(2^{-j-1}\xi)\\ &= -\varphi_0(2^1) + \varphi_0(2^{-1})\\ &= 1.\end{aligned}$$

Now we can define operators P_j and S_j by

$$\widehat{P_j f}(\xi) = \varphi_j(\xi)\hat{f}(\xi), \tag{2.5}$$

$$\widehat{S_j f}(\xi) = \phi(\xi/2^j)\hat{f}(\xi). \tag{2.6}$$

The operator P_j smoothly localizes f on the Fourier side to the annulus $|\xi| \approx 2^j$, and S_j to a ball of radius 2^j. To see how these operators modify the functions, we observe

$$S_j f(x) = (f(\cdot) * (2^{nj}\check{\phi}(2^j\cdot))(x) = \int_{\mathbb{R}^n} f(x - 2^{-j}y)\check{\phi}(y)dy,$$

where $\check{\phi}$ is a Schwartz function with total mass $\int \check{\phi} = \phi(0) = 1$. As $\check{\phi}$ decays quickly, the convolution is roughly an averaging operator of f around x on scale 2^{-j}. Thus on smaller than 2^{-j} scales around x, $S_j f$ stays nearly constant.

As $S_{j+1} f = 1$ on $|\xi| \le 2^{j+1}$, the following equality holds for $P_j f$

$$P_j f = S_{j+1} P_j f = \int_{\mathbb{R}^n} P_j f(x - 2^{-(j+1)}y)\check{\phi}(y)dy.$$

So $P_j f$ stays constant on scales smaller than $2^{-(j+1)}$ around x. On the other hand $S_{j-1} f$ is supported on $|\xi| \le 2^{j-1}$ which is outside of supp φ_j and we get

$$S_{j-1} P_j f = \int_{\mathbb{R}^n} P_j f(x - 2^{-(j-1)}y)\check{\phi}(y)dy = 0.$$

This roughly asserts that $P_j f$ has mean zero on scales larger than 2^{-j+1}. Of course $P_j f$ also has zero mean over the whole space

$$\int_{\mathbb{R}^n} P_j f(x)dx = \widehat{P_j f}(0) = 0,$$

so some sort of oscillatory nature is expected for Pjf.

With decomposition $f = \sum_{j\in\mathbb{Z}} P_j f$, we are interested in how regularity properties pass from f to the Littlewood-Paley decomposition and back. Bounding P_j and S_j in L^p is easy by Young's convolution inequality

$$\begin{aligned}\|P_j\|_{L^p} &= \|(f * \check{\varphi}_j)(x)\|_{L^p} \\ &\leq \|f\|_{L^p} \left\| \frac{1}{2^{jn}} \check{\varphi}\left(\frac{\cdot}{2^j}\right) \right\|_{L^1} \\ &\leq \|f\|_{L^p} \|\check{\varphi}\|_{L^1} \\ &\leq C \|f\|_{L^p} .\end{aligned}$$

Here $\|\check{\varphi}\|_{L^1}$ can be explicitly calculated and while $\check{\varphi}(x)$ has zero mean over $\mathbb{R}^n$, the norm is non-zero due to absolute values being taken. Similar calculation works for $\|S_j f\|_{L^p}$. Now triangle inequality gives us the cheap Littlewood-Paley inequality

$$\sup_j \|f_j\|_{L^p} \leq \|f\|_{L^p} \leq \left\| \sum_j f_j \right\|_{L^p} \leq \sum_j \|f_j\|_{L^p} . \tag{2.7}$$

To transfer regularity properties back from individual components, we would like to have some sort of reverse result of form

$$\sum_j \|f_j\|_{L^p} \approx \|f\|_{L^p} ,$$

but this is asking too much. Instead we have somewhat similar result for the so called Littlewood-Paley square function Sf

$$Sf(x) := \left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}} . \tag{2.8}$$

The L^2 case is again informative due to Plancherel's identity. For every $\xi \in \mathbb{R}^n$, there at most two non-zero components in the decomposition

$$1^2 = (\varphi_j(\xi) + \varphi_{j+1}(\xi))^2 = \varphi_j(\xi)^2 + \varphi_{j+1}(\xi)^2 + 2\varphi_j(\xi)\varphi_{j+1}(\xi),$$

so that

$$\varphi_j(\xi)^2 + \varphi_{j+1}(\xi)^2 \approx 1$$

and we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2}^2 = \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} \varphi_j(\xi)^2 \hat{f}(\xi)^2 d\xi$$
$$\approx \int_{\mathbb{R}^n} \hat{f}(\xi)^2 d\xi$$
$$= \|f\|_{L^2}^2 .$$

Also

$$\left\| \left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^2}^2 = \left(\sum_{j\in\mathbb{Z}} \|P_j f\|_{L^2}^2 \right)^{\frac{1}{2}} .$$

Somewhat similar argument folds for $p \neq 2$ as the Littlewood-Paley Theorem:

Theorem 2.3. *With Sf as in* (2.8)*, for any $1 < p < \infty$, there exists C, depending only on n and p, such that*

$$\|Sf\|_{L^p} \leq C \|f\|_{L^p}$$

holds any $f \in L^p$

A converse result also holds, so that

$$\|Sf\|_{L^p} \approx \|f\|_{L^p}$$

for any $1 < p < \infty$. This can be seen as almost orthogonality of Littlewood-Paley components in L^p, as combined with (2.7) we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \approx \|f\|_{L^p} = \left\| \sum_{j\in\mathbb{Z}} P_j f(x) \right\|_{L^p} ,$$

so that

$$\left(\sum_{j\in\mathbb{Z}} |P_j f(x)|^2 \right)^{\frac{1}{2}} \approx \sum_{j\in\mathbb{Z}} P_j f(x)$$

in L^p.

The compact Fourier supports of Littlewood-Paley components provides many beneficial properties. One is the Bernstein's inequality, of which we will need a modification of. The inequality allows L^p to L^q bound for a function if the Fourier transform of the function is supported on a ball around origin.

Lemma 2.2. *For* $1 \leq p \leq q \leq \infty$ *and any function* $f \in L^p$*, for which* $\operatorname{supp} \hat{f} \subset B(0,r)$*, there exists a constant* C*, depending only on* r *and* n*, so that*

$$\|f\|_{L^q} \leq C \|f\|_{L^p}, \tag{2.9}$$

holds.

Proof. Let γ be a $C_0^\infty(B(0,2r))$ function with $\gamma(\xi) = 1$ on $B(0,r)$ and $\gamma(\xi) \leq 1$. Now $\hat{f}(\xi) = \gamma(\xi)\hat{f}(\xi)$, and by taking the inverse Fourier transform and applying Young's inequality we have

$$\begin{aligned} \|f\|_{L^q} &\leq \|\check{\gamma} * f\|_{L^q} \\ &\leq \|\check{\gamma}\|_{L^k} \|f\|_{L^p} \\ &\leq C_r \|f\|_{L^p}, \end{aligned}$$

where the norm of $\check{\gamma}$ is estimated by

$$\|\check{\gamma}\|_{L^k} \leq \|\check{\gamma}\|_{L^1} + \|\check{\gamma}\|_{L^\infty} \leq C_r. \tag{2.10}$$

The first inequality is due to convexity of the mapping $1/p \to \|\gamma\|_{L^p}$ for $1 \leq p \leq \infty$, $\gamma \in L^1 \cap L^\infty$. The second inequality follows from $\gamma(0) = 1$ and $\sup \varphi(\xi) = 1$ supported on the ball $B(0,2r)$. □

2.3 Fourier transform of the spherical measure

To take advantage of the form (1.6) of the spherical maximal function, we need some information on the Fourier transform of the spherical measure $\widehat{d\sigma}$. In this section we derive an useful decay estimate using the method of stationary phase for oscillatory integrals.

For a finite measure μ on $\mathbb{R}^n$, the Fourier transform $\widehat{d\mu}$ is defined as

$$\widehat{d\mu}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\mu(x)$$

For the spherical measure this becomes

$$\begin{aligned} \widehat{d\sigma}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} d\sigma(x) \\ &= \int_{S^{n-1}} e^{-2\pi i\lambda(x\cdot e_n)} d\sigma(x), \quad \lambda = |\xi|, \end{aligned} \tag{2.11}$$

due to the rotational symmetry of $d\sigma$. Here $e_n = (0, 0, ..., 1)$ is the nth unit vector. This integral can be explicitly solved to give

$$\widehat{d\sigma}(\xi) = \frac{2\pi}{|\xi|^{(n-2)/2}} J_{(n-2)/2}(2\pi|\xi|),$$

as is done in [3, Appendix B]. Here J_ν is the Bessel function of order ν. We will construct another representation using techniques from theory of oscillatory integrals.

Integrals similar to (2.11), that generalize to form

$$I(\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\phi(x)} a(x) d\mu(x), \tag{2.12}$$

are called oscillatory integrals due to the oscillatory nature of $e^{i\lambda\phi(x)}$. Here a represents the amplitude of the oscillation and ϕ is a real valued function controlling the oscillation and thus called the phase function. The value λ can be thought of as the frequency of the oscillation, and in general we are interested in the order of magnitude of $|I(\lambda)|$ as $\lambda \to \infty$. One way to control the integral is to localize to regions where the phase function has or does not have a critical point. In these regions one can use the following principles of nonstationary and stationary phase respectively. Further analysis of oscillatory integrals can be found for example in [12, Chapter 8].

Proposition 2.1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function so that $\nabla\phi(p) \neq 0$ for some p. If $a \in C_0^\infty(\mathbb{R}^n)$ is supported in a small neighborhood of p, then*

$$\left| \frac{d^k I(\lambda)}{d\lambda^k} \right| \leq C_{k,N} \lambda^{-N}$$

for all $N \geq 0$.

In other words, $|I|$ decays rapidly near nonstationary points.

Proposition 2.2. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function so that $\nabla\phi(p) = 0$ and the Hessian $H_\phi(p)$ invertible for some p. If $a \in C_0^\infty(\mathbb{R}^n)$ is supported in a small neighborhood of p, then*

$$\left| \frac{d^k}{d\lambda^k} (e^{\pi i\lambda\phi(p)} I(\lambda)) \right| \leq C_k \lambda^{-((n-1)/2+k)}.$$

With these propositions in mind, let us construct a smooth partition of unity $\{q_j\}_{j=1}^k$ over S^{n-1} so that q_1 is supported near e_n and q_2 near $-e_n$. The

Fourier transform (2.11) takes the form

$$\widehat{d\sigma}(\xi) = \int_{S^{n-1}} e^{-2\pi i\lambda(x\cdot e_n)} q_1(x)d\sigma(x) + \int_{S^{n-1}} e^{2\pi i\lambda(x\cdot e_n)} q_2(x)d\sigma(x) \tag{2.13}$$

$$+ \sum_{j=3}^{k} \int_{S^{n-1}} e^{-2\pi i\lambda(x\cdot e_n)} q_j(x)d\sigma(x). \tag{2.14}$$

The segments away form $\pm e_n$ can be written as

$$\int_{\mathbb{R}^{n-1}} e^{-2\pi i\lambda(\phi_j(x)\cdot e_n)} a_j(x)dx,$$

where $a_j : \mathbb{R}^{n-1} \to [0,1]$ is a bump function and $\phi_j : \mathbb{R}^{n-1} \to S^{n-1} \subset \mathbb{R}^n$ maps the bumps smoothly to the support of q_j. Thus we have an oscillatory integral of the form (2.12) with $\phi = (\phi_j \cdot e_n)$ and

$$\nabla(\phi_j(x)\cdot e_n) = \nabla\phi_{j,n}(x) = \left(\frac{\partial}{\partial x_1}\phi_{j,n}(x), \frac{\partial}{\partial x_2}\phi_{j,n}(x), ..., \frac{\partial}{\partial x_{n-1}}\phi_{j,n}(x)\right),$$

where $\phi_{j,n}$ is the nth component of the map. The gradient can be observed to be zero only at when $\phi_j(x) = \pm e_n$ but as we are away from these poles we have $\nabla\phi \neq 0$, and use the principle of nonstationary phase to bound the integrals with some rapidly decreasing function. This suggests that the main contribution comes from the two segments q_1, q_2 near e_n. For these segments, we use mappings $\phi : \mathbb{R}^{n-1} \to S^{n-1}$, $\phi(x) = (x, \pm\sqrt{1-|x|^2})$ near the origin of $\mathbb{R}^{n-1}$, so that

$$\int_{S^{n-1}} e^{-2\pi i\lambda(x\cdot e_n)} q_1(x)d\sigma(x) = \int_{\mathbb{R}^{n-1}} e^{-2\pi i\lambda\sqrt{1-|x|^2}} \frac{a_1(x)}{\sqrt{1-|x|^2}}dx.$$

The Hessian of the phase function $\phi(x) = \sqrt{1-|x|^2}$ turns out to be $-I_{n-1}$ at the origin, and thus is invertible. Similar estimate holds for the component near $-e_n$. Now the principle of stationary phase gives us the estimate for the whole integral (2.11).

$$\begin{aligned}
\widehat{d\sigma}(\xi) &= \int_{\mathbb{R}^{n-1}} e^{-2\pi i|\xi|\sqrt{1-|x|^2}} \frac{a_1(x)}{\sqrt{1-|x|^2}}dx + \int_{\mathbb{R}^{n-1}} e^{2\pi i|\xi|\sqrt{1-|x|^2}} \frac{a_2(x)}{\sqrt{1-|x|^2}}dx \\
&\quad + \sum_{j=3}^{k} \int_{\mathbb{R}^{n-1}} e^{-2\pi i|\xi|(\phi_j(x)\cdot e_n)} a_j(x)dx \\
&= e^{2\pi i|\xi|} e^{\pi i|\xi|(-2)} \int_{\mathbb{R}^{n-1}} e^{\pi i|\xi|(-2\sqrt{1-|x|^2})} \frac{a_1(x)}{\sqrt{1-|x|^2}}dx \\
&\quad + e^{-2\pi i|\xi|} e^{\pi i|\xi|(2)} \int_{\mathbb{R}^{n-1}} e^{\pi i|\xi|(2\sqrt{1-|x|^2})} \frac{a_2(x)}{\sqrt{1-|x|^2}}dx \\
&\quad + \sum_{j=3}^{k} \int_{\mathbb{R}^{n-1}} e^{-2\pi i|\xi|(\phi_j(x)\cdot e_n)} a_j(x)dx \\
&= e^{2\pi i|\xi|}\omega_+(|\xi|) + e^{-2\pi i|\xi|}\omega_-(|\xi|) + y(|\xi|),
\end{aligned}$$

where

$$\left| \frac{d^k}{d|\xi|^k} \omega_\pm(|\xi|) \right| \le C_k |\xi|^{-((n-1)/2+k)} \tag{2.15}$$

and

$$\left| \frac{d^k}{d|\xi|^k} y(|\xi|) \right| \le C_{k,N} |\xi|^{-N}$$

for any $N > 0$. As y decays rapidly, it can be absorbed to the other terms to have form

$$\widehat{d\sigma}(\xi) = e^{2\pi i|\xi|}\omega_+(|\xi|) + e^{-2\pi i|\xi|}\omega_-(|\xi|). \tag{2.16}$$

This controls the Fourier transform when $|\xi|$ is large. For small $|\xi|$, we use the fact

$$|\widehat{d\sigma}(\xi)| \le \int_{S^{n-1}} |e^{-2\pi i x\cdot\xi}| d\sigma(x) = 1$$

to see that $\widehat{d\sigma}(\xi)$ is bounded near 0. Thus $\omega_\pm$ must have upper bound by

$$|\frac{d^k}{d|\xi|^k} \omega_\pm(|\xi|)| \le C_k \frac{1}{(1+|\xi|)^{(n-1)/2+k}}. \tag{2.17}$$

The form (2.16) with decay (2.15) is not unique to the hypersphere. One can deduce similar estimates with the same rate of decay for other hypersurfaces, assuming they satisfying certain smoothness properties. For example the Gaussian curvature needs to be nonzero everywhere, to avoid problems similar to the one in section 1.2.

To combine some results of this chapter, we now prove the following result from [3, Chapter 5.5], related to Littlewood-Paley bump functions and spherical measure.

Lemma 2.3. *Let φ_j be a bump function as in* (2.4)*, and $d\sigma$ as in* (2.16)*. Then for any $M > n$ there exists $C_M < \infty$, such that*

$$|(\check{\varphi}_j * d\sigma)(x)| \leq \frac{C_M 2^j}{(1+|x|)^M} \tag{2.18}$$

holds.

Proof. As $\check{\varphi}_j$ is a Schwartz function, for any N, there exists C_N, such that

$$|(\check{\varphi}_j * d\sigma)(x)| \leq \int_{S^{n-1}} \frac{C_N 2^{jn}}{(1+2^j|x-y|)^N} d\sigma(y).$$

Now the integral can be estimated by studying the following dyadic partition of space:

$$\begin{cases} S_{-1} = S^{n-1} \cap \{y \in \mathbb{R}^n : 2^j|x-y| \leq 1\}, \\ S_r = S^{n-1} \cap \{y \in \mathbb{R}^n : 2^r \leq 2^j|x-y| \leq 2^{r+1}\}, \quad r \geq 0. \end{cases}$$

For $r \leq j$, the sets have $\mathcal{H}^{n-1}$ measure at most $c_n 2^{(r+1-j)(n-1)}$, and require $|x|$ to be small so that $x \in B(0,3)$. For large $r > j$, the measure of the sets is bounded by $\mathcal{H}^{n-1}(S^{n-1}) = c_n$ and requires $|x| \leq 2^{r+1-j} + 1$. Combining these estimates gives

$$\begin{aligned} \int_{S^{n-1}} \frac{C_N 2^{jn}}{(1+2^j|x-y|)^N} d\sigma(y) &\leq C_N 2^{nj} \Bigg(\sum_{r=-1}^{j} \frac{\chi_{B(0,3)}(x)\mathcal{H}^{n-1}(S_r)}{2^{rN}} \\ &\qquad + \sum_{r \geq j+1} \frac{\chi_{B(0,2^{r+1-j}+1)}(x)\mathcal{H}^{n-1}(S_r)}{2^{rN}} \Bigg) \\ &\leq C_N \Bigg(\frac{C_M}{(1+|x|)^M} \sum_{r=-1}^{j} \frac{2^{(r+1-j)(n-1)+nj}}{2^{rN}} \\ &\qquad + \frac{1}{(1+|x|)^M} \sum_{r \geq j+1} \frac{2^{jn}(1+2^{r+2-j})^M c_n}{2^{rN}} \Bigg) \\ &\leq \frac{C_{M,N,n} 2^j}{(1+|x|)^M}, \end{aligned}$$

if $N > M > n$. Here we used the estimate

$$\chi_{B(0,R)}(x) \leq \frac{(1+R)^M}{(1+|x|)^M}$$

for any $M \geq 1$. By choosing for example $N = M + 1$ we have the required estimate. $\square$

Chapter 3

Main theorem

We are now ready to state the main result of the thesis. The cases $n = 2$ and $n \geq 3$ are handled separately as the former is more complicated. This is due to the dimension in the exponent of the estimate (2.17) for $\widehat{d\sigma}$ results in some sums not converging in $n = 2$. Most of the analysis will be carried out in a localized case $1 < r < 2$, improving to the global bound $r > 0$ afterwards in chapter 4. Let us denote

$$\overline{M}f(x) = \sup_{1<r<2} \int_{S^{n-1}} |f(x - ry)| d\sigma(y). \tag{3.1}$$

We will also first prove the result for nonnegative Schwartz functions, extending to L^p functions in chapter 4.

3.1 Case $n \geq 3$

For the case $n \geq 3$, the main result to be proved is the following theorem.

Theorem 3.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$ be nonnegative Schwartz function. Set points*

- $P_1 = (0, 0)$,
- $P_2 = (\frac{n-1}{n}, \frac{n-1}{n})$,
- $P_3 = (\frac{n-1}{n}, \frac{1}{n})$,
- $P_4 = (\frac{n^2-n}{n^2+1}, \frac{n-1}{n^2+1})$

and $\mathcal{Q}$ as the closed polygon spanned by the points. Then if $(1/p, 1/q) \in \mathcal{Q} \setminus ([P_2P_3] \cup \{P_4\})$, there is a constant C depending only on n, p and q, such that for operator (3.1)

$$\left\|\overline{M}f\right\|_{L^q} \leq C \left\|f\right\|_{L^p}. \tag{3.2}$$

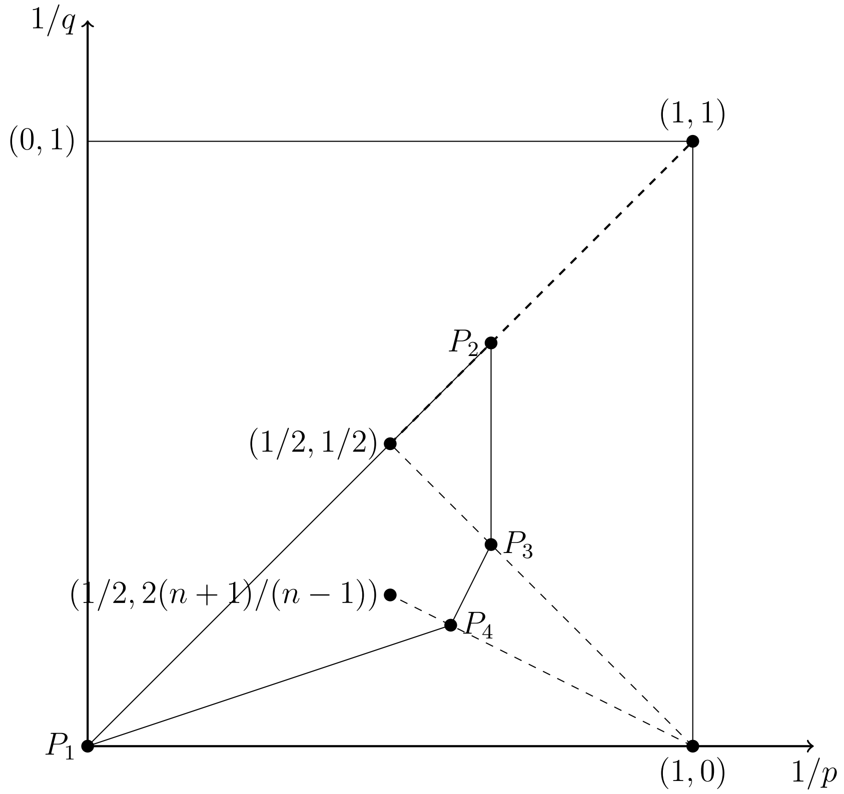

Figure 3.1: The polygon from Theorem 3.1 and interpolation segments in the case $n = 3$. Using Lemma 2.1 on the dashed line segments gives weak bounds on the points P_1, P_2, P_3, P_4. Marcinkiewicz Interpolation Theorem then gives the full theorem when $(1/p, 1/q)$ is inside the closed polygon spanned by the points P_i except the points P_2, P_3, P_4.

The strategy for the proof is to prove estimates in some $(1/p, 1/q)$ points for single Littlewood-Paley components and use Lemma 2.1 to gain weak bounds on points P_2, P_3, P_4. The strong $L^\infty \to L^\infty$ bound on pointP_1 is trivial for Schwartz functions. Then repeated use of Marcinkiewicz Interpolation Theorem proves the estimate (3.2) inside the closed polygon. The case $n = 3$ is illustrated in Figure 3.1.

We start by constructing a Littlewood-Paley decomposition of the operator $\overline{M}$. Let f_j be defined by $\hat{f}_j = \varphi_j \hat{f}$, where $\varphi_j(\xi)$ is a localizing bump function similar to (2.4). We will denote $\overline{M}_j f = \overline{M} f_j$ and have

$$\begin{aligned}\overline{M}f &= \sup_{1<r<2} \int_{S^{n-1}} |\sum_{j=0}^{\infty} f_j(x-ry)|d\sigma(x) \\ &\leq \sup_{1<r<2} \int_{S^{n-1}} |f_0(x-ry)|\sigma(x) + \sum_{j=1}^{\infty} \sup_{1<r<2} \int_{S^{n-1}} |f_j(x-ry)|\sigma(x) \\ &\leq \overline{M}_0 f + \sum_{j=1}^{\infty} (\overline{M}_j f)\end{aligned}$$

due to the sublinearity of the maximal operator. The low frequencies in $\overline{M}_0 f$ are easy to bound with Benrstein's inequality and a well-known lemma in real analysis [3, Chapter 2.1.2]

Lemma 3.1. *Let $K \in L^1(\mathbb{R}^n)$ be a nonnegative, decreasing and radial function. Then for $K_\epsilon = 1/\epsilon^n K(x/\epsilon)$,*

$$\sup_{\epsilon>0} |(f * K_\epsilon)(x)| \leq \|K\|_{L^1} \mathcal{M}f(x)$$

holds for every $x \in \mathbb{R}^n$.

Let $\tilde{\varphi} \in C_0^\infty(\mathbb{R}^n)$ be bump function such that $\tilde{\varphi} = 1$ on the support of φ_0. Thus $\tilde{\varphi}\varphi_0 = \varphi_0$ and we can write the first Littlewood-Paley component $\overline{M}_0 f$ in the form (1.5) as

$$\begin{aligned}\overline{M}_0 f &= \sup_{1<r<2} \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \hat{f}(\xi)\tilde{\varphi}(\xi)\varphi_0(\xi)\widehat{d\sigma}(r\xi)d\xi \\ &= \sup_{1<r<2} \left[(\hat{f}\tilde{\varphi})\check{} * \tilde{d\sigma}_r \right](x).\end{aligned}$$

Here $\tilde{d\sigma}_r(x) = \frac{1}{r^n}\tilde{d\sigma}(\frac{x}{r})$ is the inverse Fourier transform of $\varphi_0(\xi)\widehat{d\sigma}(r\xi)$. As φ_0 is a radial Schwartz function, $\check{\varphi}_0$ is also a radial Schwartz function and thus has an decreasing majorant in $C_N(1+|x|)^{-N}$ for any $N \geq 0$. Now $\tilde{d\sigma}$ can be estimated from above as

$$\begin{aligned}|\tilde{d\sigma}(x)| &= |(\varphi_0(\xi)\widehat{d\sigma}(r\xi))\check{}(x)| = |(\check{\varphi}_0 * d\sigma_r)(x)| \\ &= \left| \int_{S^{n-1}} \check{\varphi}_0(x-ry)d\sigma(y) \right| \\ &\leq \frac{C_N}{(1+|x|)^N} \int_{S^{n-1}} d\sigma(y) \\ &= \frac{C_N}{(1+|x|)^N} \\ &= g(x).\end{aligned}$$

Naturally we can choose N depending on the dimension n, so that $g \in L^1(\mathbb{R}^n)$ and Lemma 3.1 gives

$$\overline{M}_0 f(x) \leq \sup_{r>0} \left[(\hat{f}\tilde{\varphi})\check{} * g_r \right](x) \leq \|g\|_{L^1} \mathcal{M}(\hat{f}\tilde{\varphi})\check{}(x).$$

Taking L^q, $q > 1$ norm on both sides and using the Hardy-Littlewood maximal inequality, we have

$$\left\|\overline{M}_0 f\right\|_{L^q} \leq C \left\|(\hat{f}\tilde{\varphi})\check{}\right\|_{L^q}.$$

As $(\hat{f}\tilde{\varphi})\check{}$ has its Fourier transform supported on ball around zero, we can use Bernstein's inequality 2.2 for any $1 \leq p \leq q$, $1 < q$ and Young's inequality for convolutions to gain the desired $L^p \to L^q$ bound

$$\begin{aligned}\left\|\overline{M}_0 f\right\|_{L^q} &\leq C \left\|(\hat{f}\tilde{\varphi})\check{}\right\|_{L^q} \\ &= C \left\|f * (\tilde{\varphi})\check{}\right\|_{L^p} \\ &\leq C \left\|(\tilde{\varphi})\check{}\right\|_{L^1} \|f\|_{L^p} \\ &\leq C \|f\|_{L^p}.\end{aligned}$$

For the high frequencies $j \geq 1$, we require the following estimates

$$\left\|\overline{M}_j f\right\|_{L^1} \leq C 2^j \|f\|_{L^1}, \tag{3.3}$$

$$\left\|\overline{M}_j f\right\|_{L^\infty} \leq C 2^j \|f\|_{L^1}, \tag{3.4}$$

$$\left\|\overline{M}_j f\right\|_{L^2} \leq C 2^{-j\frac{n-2}{2}} \|f\|_{L^2}, \tag{3.5}$$

$$\left\|\overline{M}_j f\right\|_{L^{2(n+1)/(n-1)}} \leq C 2^{-j\frac{n^2-2n-1}{2n+2}} \|f\|_{L^2}. \tag{3.6}$$

As the first two estimates are divergent in j, and the other two are convergent, Lemma 2.1 between (3.3) and (3.5), (3.4) and (3.5), (3.4) and (3.6) produces bounds of weak type respectively on P_2, P_3, P_4.

Let us now prove the above bounds for Littlewood-Paley components when $j \geq 1$. We will use the decay property of $\widehat{d\sigma}$ and study integrals of the form

$$A_j f(x,r) = \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + r|\xi|)} \frac{\hat{f}(\xi)\varphi_j(\xi)}{(1+|\xi|)^{(n-1)/2}} d\xi, \tag{3.7}$$

so that

$$\overline{M}_j f(x) \leq \sup_{1<r<2} A_j f(x,r).$$

For many of the estimates, we use the following embedding type lemma from [6]. Other simpler inequalities can be used for some of the estimates but the following works for multiple cases.

Lemma 3.2. *For a smooth function u defined on $\mathbb{R}^n \times I$, where I is a bounded interval, and $1 < p \leq \infty$*

$$\left\| \sup_{r\in I} |u(x,r)| \right\|_{L^p(\mathbb{R}^n)} \leq C \left(\|u\|_{L^p(\mathbb{R}^n\times I)} + \|u\|^{(p-1)/p}_{L^p(\mathbb{R}^n\times I)} \|\partial_r u\|^{1/p}_{L^p(\mathbb{R}^n\times I)} \right). \quad (3.8)$$

Proof. We begin by fixing x and take u as a smooth function $u : I \to \mathbb{R}$. To keep the right-hand side of (3.8) finite, we must assume $u(r)^p \in W^{1,1}(I)$, $p > 1$. Here $W^{1,1}(I)$ is a one dimensional Sobolev space and thus has an inclusion $W^{1,1}(I) \subset L^\infty(I)$ due to Sobolev functions being absolutely continuous on lines. We have the inequality

$$\begin{aligned}\sup_{r\in I} |u(r)^p| &\leq C(\|u^p\|_{L^1(I)} + \|\partial_r u^p\|_{L^1(I)}) \\ &\leq \int_I |u^p| dr + \int_I p|u^{p-1}||\partial_r u| dr \\ &\leq \int_I |u^p| dr + p \left[\int_I |u^{p-1}|^{p/(p-1)} dr \right]^{(p-1)/p} \left[\int_I |\partial_r u|^p dr \right]^{1/p}\end{aligned}$$

by applying Hölder's inequality on the second term. Now letting x vary and integrating over $\mathbb{R}^n$ we get

$$\begin{aligned}\int_{\mathbb{R}^n} \sup_{r\in I} |u(x,r)^p| dx \leq C\Bigg(& \|u\|^p_{L^p(\mathbb{R}^n\times I)} \\ & + p \int_{\mathbb{R}^n} \left[\int_I |u|^p dr \right]^{\frac{p-1}{p}} \left[\int_I |\partial_r u|^p dr \right]^{\frac{1}{p}} dx \Bigg) \\ \leq C & \left(\|u\|^p_{L^p(\mathbb{R}^n\times I)} + p \|u\|^{\frac{p-1}{p}}_{L^p(\mathbb{R}^n\times I)} \|\partial_r u\|^{\frac{1}{p}}_{L^p(\mathbb{R}^n\times I)} \right).\end{aligned}$$

Here we used Hölder's inequality again on the second term. Now we can take the pth root and use the inequality $p^{1/p} \leq 2$, $p > 0$ to gain (3.8) for $1 < p < \infty$. As the constant does not depend on p this can be extended to $p = \infty$ for the full result. □

The $L^2 \to L^2$ estimate now follows easily from the previous lemma and (3.7), as $A_j f$ is smooth due to f being a Schwartz function

$$\begin{aligned}\|\overline{M}_j f\|_{L^2(\mathbb{R}^n)} \leq C\Big(& \|A_j f\|_{L^2(\mathbb{R}^n\times[1,2])} \\ & + \|A_j f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n\times[1,2])} \|\partial_t A_j f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n\times[1,2])} \Big).\end{aligned} \quad (3.9)$$

For the right hand side we can calculate

$$|A_j f(x,r)| = \left| \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + r|\xi|)} \frac{\hat{f}(\xi)\varphi_j(\xi)}{(1+|\xi|)^{(n-1)/2}} d\xi \right|$$
$$\leq C 2^{-j(\frac{n-1}{2})} \left| \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{2\pi i r|\xi|} \hat{f}(\xi)\varphi_j(\xi) d\xi \right|$$

and

$$\left| \frac{\partial}{\partial r} A_j f(x,r) \right| = \left| \int_{\mathbb{R}^n} e^{2\pi i(x\cdot\xi + r|\xi|)} 2\pi i|\xi| \frac{\hat{f}(\xi)\varphi_j(\xi)}{(1+|\xi|)^{(n-1)/2}} d\xi \right|$$
$$\leq C 2^{-j(\frac{n-1}{2}-1)} \left| \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{2\pi i r|\xi|} \hat{f}(\xi)\varphi_j(\xi) d\xi \right|,$$

as φ_j localizes ξ to the annulus of radius 2^j. Combining these estimates to (3.9) we achieve

$$\left\| \overline{M}_j f \right\|_{L^2\mathbb{R}^n} \leq C \Big(2^{-j(\frac{n-1}{2})} \left\| U f_j \right\|_{L^2(\mathbb{R}^n \times [1,2])}$$
$$+ 2^{-j(\frac{n-1}{2})\frac{1}{2}} 2^{-j(\frac{n-1}{2}-1)\frac{1}{2}} \left\| U f_j \right\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n \times [1,2])} \left\| U f_j \right\|^{\frac{1}{2}}_{L^2(\mathbb{R}^n \times [1,2])} \Big)$$
$$= C \left(2^{-j(\frac{n-1}{2})} + 2^{-j(\frac{n-2}{2})} \right) \left\| U f_j \right\|_{L^2(\mathbb{R}^n \times [1,2])}$$
$$\leq C 2^{-j(\frac{n-2}{2})} \left\| U f_j \right\|_{L^2(\mathbb{R}^n \times [1,2])},$$

where

$$U f_j(x,r) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} e^{2\pi i r|\xi|} \hat{f}_j(\xi) d\xi \tag{3.10}$$

is a type of Fourier integral operator. By Plancherel's identity, instead of $Uf_j(x,r)$, we can study the L^2 norm of $\hat{f}_j(\xi)e^{2\pi i r|\xi|}$. Using Euler's formula on the exponential, we now easily easily gain

$$\left\| U f_j(x,r) \right\|_{L^2(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\hat{f}_j(\xi) e^{2\pi i r|\xi|}|^2 d\xi \right)^{\frac{1}{2}} \tag{3.11}$$
$$= \left(\int_{\mathbb{R}^n} \sqrt{\hat{f}_j(\xi)^2 \cos^2(2\pi r|\xi|) + \hat{f}_j(\xi)^2 \sin^2(2\pi r|\xi|)}^{\,2} d\xi \right)^{\frac{1}{2}}$$
$$= \left\| \hat{f}_j \right\|_{L^2(\mathbb{R}^n)}$$
$$= \left\| f_j \right\|_{L^2(\mathbb{R}^n)}.$$

We see that the operator U maps L^2 to itself and we get rid of the dependency on r. Thus the estimate converts to

$$\begin{aligned}\left\|\overline{M}_j f\right\|_{L^2(\mathbb{R}^n)} &\leq C2^{-j(\frac{n-2}{2})} \left\|U f_j\right\|_{L^2(\mathbb{R}^n\times[1,2])} \\ &= C2^{-j(\frac{n-2}{2})} \left(\int_1^2 \left\|f_j\right\|^2_{L^2(\mathbb{R}^n)} dr\right)^{\frac{1}{2}} \\ &= C2^{-j(\frac{n-2}{2})} \left\|f_j\right\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

As the L^p norm of individual Littlewood-Paley components in bounded by the norm of the full function (2.7), we have

$$\left\|\overline{M}_j f\right\|_{L^2(\mathbb{R}^n)} \leq C2^{-j(\frac{n-2}{2})} \left\|f\right\|_{L^2(\mathbb{R}^n)}.$$

Now one can use this estimate to actually gain the $L^2 \to L^2$ result for full f, not just the Littlewood-Paley components. By Minkowski inequality

$$\begin{aligned}\left\|\overline{M} f\right\|_{L^2} = \left\|\overline{M}\left(\sum_{j=0}^{\infty} f_j\right)\right\|_{L^2} &\leq \sum_{j=0}^{\infty} \left\|\overline{M}_j f\right\|_{L^2} \\ &\leq C\left\|\overline{M}_0 f\right\|_{L^2} + \sum_{j=1}^{\infty} \left\|\overline{M}_j f\right\|_{L^2} \\ &\leq C\left(\left\|f\right\|_{L^2} + \sum_{j=1}^{\infty} 2^{-j(\frac{n-2}{2})} \left\|f\right\|_{L^2}\right) \\ &= C\left\|f\right\|_{L^2}(1 + \sum_{j=1}^{\infty} 2^{-j(\frac{n-2}{2})}) \\ &\leq C(n)\left\|f\right\|_{L^2}\end{aligned}$$

for $n \geq 3$. This L^2 estimate also nicely portrays, why $n = 2$ need separate treatment. For $n \geq 3$ the series $\sum_{j\geq 1} \left\|\overline{M}_j f\right\|_{L^2}$ converges by the above estimate, but in $n = 2$ we lose the information on j and can not produce such a result.

The bounds (3.3) and (3.4) can be easily derived from Lemma 2.3, as this gives a pointwise bound for the L^∞ estimate as well as a bound for the integral via Young's convolution inequality. Let us choose $M = n + 1$ in

(2.18) so the constant C_M depends only on n. The L^∞ bound follows from

$$\begin{aligned}
|\overline{M}_j f(x)| &= |\sup_{1<r<2}(f * (\check{\varphi}_j * d\sigma_r))(x)| \\
&= \left|\sup_{1<r<2}\int_{\mathbb{R}^n} f(x-y)(\check{\varphi}_j * d\sigma_r)(x)dy\right| \\
&\leq \sup_{1<r<2}\int_{\mathbb{R}^n} |f(x-y)|\frac{C_M r^{-n}2^j}{(1+r^{-1}|x|)^M}dy \\
&\leq C2^j\int_{\mathbb{R}^n}|f(x-y)|dy \\
&= C2^j \|f\|_{L^1}.
\end{aligned}$$

Taking essential supremum in x of this pointwise inequality gives the result

$$\left\|\overline{M}_j f\right\|_{L^\infty} \leq C2^j \|f\|_{L^1}.$$

For the L^1 case we calculate

$$\begin{aligned}
\left\|\overline{M}_j f\right\|_{L^1} &= \left\|\sup_{1<r<2}(f * (\check{\varphi}_j * d\sigma_r))\right\|_{L^1} \\
&\leq \|f\|_{L^1}\left\|\sup_{1<r<2}\check{\varphi}_j * d\sigma_r\right\|_{L^1} \\
&\leq C2^j \|f\|_{L^1}
\end{aligned}$$

by again using the estimate (2.18) and

$$\begin{aligned}
\left\|\sup_{1<r<2}\check{\varphi}_j * d\sigma_r\right\|_{L^1} &\leq \int_{\mathbb{R}^n}\sup_{1<r<2}\frac{C_M r^{-n}2^j}{(1+r^{-1}|x|)^M}dx \\
&\leq \int_{\mathbb{R}^n}\frac{C_M 2^j}{(1+|x|)^M}dx \\
&\leq C(n)2^j.
\end{aligned}$$

The final estimate (3.6) is a consequence of Strichartz's estimates in [13].

Now that the estimates (3.3) - (3.6) are understood, we can finally give a proof for the $n \geq 3$ case.

Proof of Theorem 3.1. Let f be a nonnegative Schwartz function. Thus $f(x) \leq \|f\|_{L^\infty} < \infty$, for every $x \in \mathbb{R}^n$ and

$$\overline{M}f(x) \leq \int_{S^{n-1}} \|f\|_{L^\infty} d\sigma(y) = \|f\|_{L^\infty}.$$

Taking supremum in x then gives the L^∞ result

$$\left\|\overline{M}f\right\|_{L^\infty} \leq \|f\|_{L^\infty}.$$

Next we use the interpolation Lemma 2.1 on the bounds (3.3) and (3.5) for the spherical maximal functions over the Littlewood-Paley components f_j. The exponents are $\epsilon_1 = 1$ and $\epsilon_2 = (n-2)/2$, so the weak bound is found with θ as

$$\theta = \frac{\epsilon_2}{\epsilon_1 + \epsilon_2} = \frac{n-2}{2}\frac{2}{n} = \frac{n-2}{n}.$$

Thus p and q are as

$$\frac{1}{q} = \frac{1}{p} = \theta\frac{1}{1} + (1-\theta)\frac{1}{2} = \frac{n-1}{n},$$

giving the weak estimate on point P_2

$$\left\|\overline{M}f\right\|_{L^{(n-1)/n,\infty}} \leq C\,\|f\|_{L^{(n-1)/n,1}}.$$

Interpolating this weak type estimate with the strong type $L^\infty \to L^\infty$ estimate with the Marcinkiewicz Interpolation Theorem 2.1 gives the full $L^p \to L^p$ result

$$\left\|\overline{M}f\right\|_{L^p} \leq C\,\|f\|_{L^p}, \quad \frac{n-1}{n} < p \leq \infty. \tag{3.12}$$

Similarly, Lemma 2.1 gives weak type bounds on P_3 and P_4 by interpolating from (3.4) to (3.5) and (3.4) to (3.6) respectively. Marcinkiewicz interpolation between P_3 and P_4 gives strong bounds on the open line segment (P_3, P_4) which can be interpolated with the $L^p \to L^p$ estimates for the result inside polygon $\mathcal{Q}$. Interpolation between P_4 and P_1 finalizes the proof. Note that the Marcinkiewicz Interpolation Theorem can not produce estimates on the line segment $[P_2, P_3]$ as it is vertical in the $(1/p, 1/q)$ coordinates. □

3.2 Case $n = 2$

The planar case $n = 2$ is more complicated, and requires separate treatment. First of all, the L^2 estimate (3.5) fails to converge and we cannot interpolate between this and the $L^1 \to L^\infty$ estimate to gain a weak bound on any point in the lower triangle of the $(1/p, 1/q)$ unit square. One can not even deduce the $L^p \to L^p$ estimates due to this lack of convergence in L^2. After seeing the result (3.12) for $n \geq 3$, one could still believe similar estimate would exist for $n = 2$ with $(n-1)/2 = 1/2$ being the weak endpoint result. This is true, as the following theorem.

Theorem 3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a non negative Schwartz function. Set points*

- $O = (0, 0)$,
- $P = (1/2, 1/2)$,
- $Q = (2/5, 1/5)$

and $\mathcal{T}$ as the closed triangle spanned by the points. Then if $(1/p, 1/q) \in \mathcal{T} \setminus \{P, Q\}$ there is a constant C depending only on n, p, and q, such that for operator 3.1

$$\left\|\overline{M}\right\|_{L^q} \leq C \left\|f\right\|_{L^p}.$$

Again we produce some estimates for $\overline{M}_j f$ and use Lemma 2.1 to gain weak estimates for the full operator $\overline{M}f$, which can be interpolated by Marcinkiewicz Interpolation Theorem. Figure 3.2 illustrates the different points and lines used in the proof.

Let us first prove the $L^p \to L^p$ estimates in the half open diagonal $[O, P)$. The $L^\infty \to L^\infty$ estimate is again trivial for the smooth Schwartz functions. As the L^2 estimate (3.5) for $n = 2$ is

$$\left\|\overline{M}_j f\right\|_{L^2} \leq C \left\|f\right\|_{L^2}, \tag{3.13}$$

we need to find some other point with negative exponent of 2^j to use Lemma 2.1. One such estimate is

$$\left\|\overline{M}_j f\right\|_{L^4} \leq C 2^{-j(\frac{1}{8}-\epsilon)} \left\|f\right\|_{L^4}, \quad \text{for all } \epsilon > 0, \tag{3.14}$$

from [9, Chapter 2.4]. For small enough $0 < \epsilon < \frac{1}{8}$, the exponent is negative, and for any $2 \leq p < 4$ we get a weak type estimate

$$\left\|\overline{M}_j f\right\|_{L^{p,\infty}} \leq C \left\|f\right\|_{L^{p,1}}.$$

Interpolating this with the $L^\infty \to L^\infty$ estimate gives the full diagonal result

$$\left\|\overline{M}_j f\right\|_{L^p} \leq C \left\|f\right\|_{L^p}, \quad p > 2. \tag{3.15}$$

To extend to $L^p \to L^q$ bounds, we need some estimates to interpolate with. One could use Lemma 2.1 on (3.17) and (3.4) to gain weak estimates on $1/p = 3/9$, $q = 2/9$, but we can actually improve from this. In [6], the following estimate is derived for operators of type (3.10) in $n = 2$.

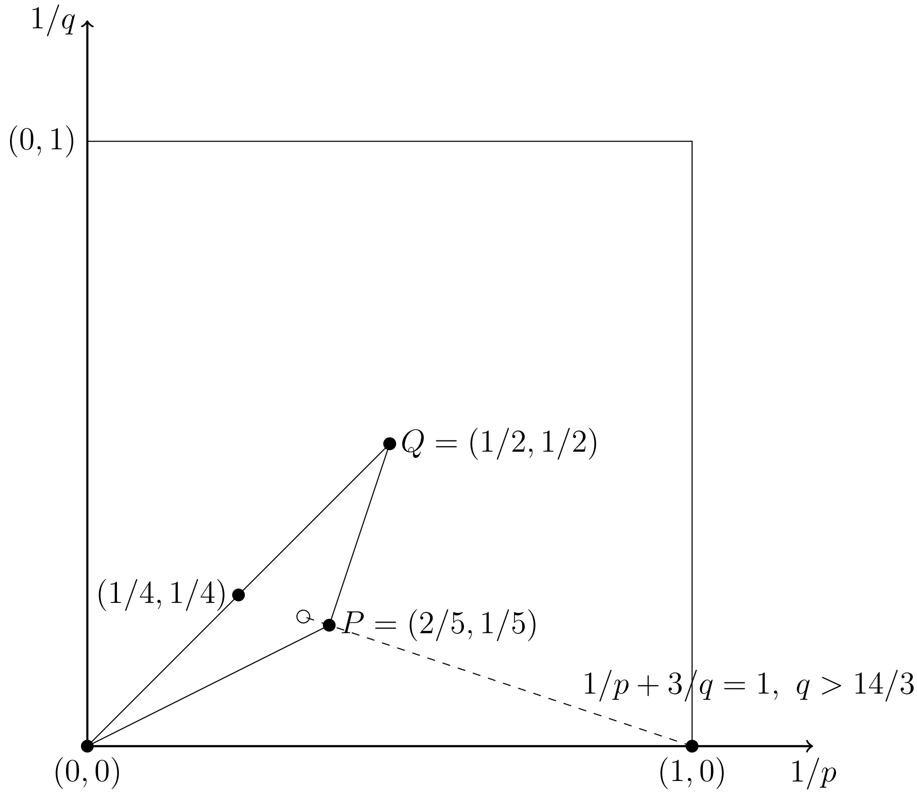

Figure 3.2: Case $n = 2$. Theorem 3.2 holds in the closed triangle O, P, Q, minus the points P, Q. For the $L^p \to L^p$ estimate, the bound (3.14) is derived at $p = q = 4$. Proposition 3.16 holds on the dashed line, and can be used to prove weak estimate on the half open line $[P, Q)$.

Proposition 3.1. *If* supp $\hat{f} \subset \{\xi \in \mathbb{R}^2 : |\xi| \sim N\}$, *then for* $1/p + 3/q = 1$, $14/3 < q \leq \infty$,

$$\left(\int_{\mathbb{R}^2}\int_1^2 |Uf(x,r)|^q drdx\right)^{\frac{1}{q}} \leq CN^{3/2-6/q} \|f\|_{L^p(\mathbb{R}^2)}, \tag{3.16}$$

where Uf *is the Fourier integral operator* (3.10).

The proof of this proposition is very technical and will not be discussed here. We now prove Theorem 3.2.

Proof of Theorem 3.2. Let $f \in \mathcal{S}$ and f_j, $j \geq 1$ be a Littlewood-Paley component of f. Choose $I = [1,2]$ in Lemma 3.2.

$$\left\|\overline{M}_j f\right\|_{L^q(\mathbb{R}^2)} \leq C\Big(\|A_j f\|_{L^q(\mathbb{R}^2\times[1,2])} + \|A_j f\|_{L^q(\mathbb{R}^2\times[1,2])}^{\frac{1-q}{q}} \|\partial_t A_j f\|_{L^q(\mathbb{R}^2\times[1,2])}^{\frac{1}{q}} \Big),$$

where $A_j f(x,r)$ is as in (3.7). With similar estimations as in the L^2 case of $n \geq 3$, we get

$$\|A_j f\|_{L^q(\mathbb{R}^2\times[1,2])} \leq C 2^{-\frac{j}{2}} \|U f_j\|_{L^q(\mathbb{R}^2\times[1,2])}$$

and

$$\|\partial_t A_j f\|_{L^q(\mathbb{R}^2\times[1,2])} \leq C 2^{\frac{j}{2}} \|U f_j\|_{L^q(\mathbb{R}^2\times[1,2])}.$$

So for any $q > 1$,

$$\begin{aligned}\left\|\overline{M}_j f\right\|_{L^q(\mathbb{R}^2)} &\leq C\Big(2^{-\frac{j}{2}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])} \\ &\qquad + 2^{-\frac{j}{2}\frac{1-q}{q}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])}^{\frac{1-q}{q}} 2^{\frac{j}{2}\frac{1}{q}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])}^{\frac{1}{q}} \Big) \\ &\leq C\Big(2^{-\frac{j}{2}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])} + 2^{-\frac{j}{2}+\frac{j}{q}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])}\Big) \\ &\leq C 2^{-\frac{j}{2}+\frac{j}{q}} \|U_j f\|_{L^q(\mathbb{R}^2\times[1,2])}\end{aligned}$$

holds. Now we restrict q as in Proposition 3.16 and see

$$\begin{aligned}\left\|\overline{M}_j f\right\|_{L^q(\mathbb{R}^2)} &\leq C 2^{-\frac{j}{2}+\frac{j}{q}} 2^{\frac{3j}{2}-\frac{6j}{q}} \|f_j\|_{L^p(\mathbb{R}^2)} \\ &\leq C 2^{j(1-\frac{5}{q})} \|f\|_{L^p(\mathbb{R}^2)}, \end{aligned} \tag{3.17}$$

when $1/q < 3/14$ and $1/p + 3/q = 1$. The exponent is negative when $1/q > 1/5$, $p < 2/5$, so Lemma 2.1 with the L^2 bound (3.13) produces the result in the interior of $\mathcal{T}$, as we get weak type estimates as close to the line (P,Q) as we want, and can interpolate witht he $L^\infty \to L^\infty$ result. We can extend the result to the closure of $\mathcal{T}$ by first interpolating (3.17) and (3.13). Choose p, q so that (3.17) holds, then Marcinkiewicz interpolation gives

$$\left\|\overline{M}_j f\right\|_{L^{\tilde{q}}} \leq C 1^{1-\theta} 2^{j(1-\frac{5}{q})\theta} \|f\|_{L^{\tilde{p}}},$$

where $\tilde{p}, \tilde{q}$ is contained in the closed triangle spanned by points $(5/14, 3/14), (1/2, 1/2), (1, 0)$, but not on the closed line segment $[(5/14, 3/14), (1/2, 1/2)]$. Using the property $1/p + 3/q = 1$, one can compute

$$\left(1 - \frac{5}{q}\right)\theta = \frac{1}{2}\left(\frac{3}{\tilde{p}} - \frac{1}{\tilde{q}} - 1\right).$$

On the left of line $[P, Q)$, the exponent is negative, and on the right positive. Thus we can find points so that interpolating with Lemma 2.1 gives

$$\left\|\overline{M}_j f\right\|_{L^{q,\infty}} \leq C \left\|f\right\|_{L^{p,1}}$$

on $[P, Q)$. These can be interpolated with each other to have strong type bounds on (P, Q), and interpolating the weak type bound

$$\left\|\overline{M}_j f\right\|_{L^{5,\infty}} \leq C \left\|f\right\|_{L^{5/2,1}} \tag{3.18}$$

on point P with the $L^\infty \to L^\infty$ result gives strong bounds also on $[O, P)$. Thus strong type bounds have been established on $\mathcal{T} \setminus \{P, Q\}$ and in addition weak type bound (3.18) holds on point P. □

Chapter 4

Finalizing the proof

With the Theorems 3.1 and 3.2 in place for the local spherical maximal operator (3.1) and Schwartz functions, we now want to extend to similar results when r is allowed to vary over $r > 0$ and f is a general L^p function.

4.1 Global maximal function

Most of the applications for maximal operators are related to the local behavior of functions, such as Lebesgue Differentiation Theorem and Sobolev space applications. In addition, for L^p, $p < \infty$ functions, most of the mass is concentrated on some bounded domain in $\mathbb{R}^n$. Thus, for any L^p function, we can scale down and study the spherical maximal function defined by

$$M_\alpha f(x) = \sup_{0<r<1} r^\alpha |f * d\sigma_r(x)|, \tag{4.1}$$

where we allow r to approach zero and α is a fractional constant depending on p and q.

When $p = q$, we get $\alpha = 0$ and have the regular spherical maximal function, but for the off-diagonal estimates, the fractional constant is needed for achieving $L^p \to L^q$ bounds for the Hardy-Littlewood maximal function.

The following lemma from [7] allows us to extend to the global case whenever the local spherical maximal function converges for Littlewood-Paley components.

Lemma 4.1. *Let $1 < p \leq q \leq \infty$ and $q \geq 2$. Suppose that for some $\beta < 0$*

$$\left\| \overline{M} f_j \right\|_{L^q} \leq C 2^{j\beta} \left\| f_j \right\|_{L^p} \tag{4.2}$$

holds for all $j = 1, 2, ...,$ $f_j \in \mathcal{S}$ *and* supp $\hat{f}_j \subset \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. *Then with* $\alpha = n(1/p - 1/q)$,

$$\|M_\alpha f\|_{L^q} \leq C \|f\|_{L^p}$$

holds for all $f \in \mathcal{S}$.

Bounds of type (4.2) can be derived in the regions defined by Theorems 3.1 and 3.2 by interpolating bounds of type (3.3) - (3.6).

For the proof of Lemma 4.1, first try to set $\alpha = 0$, for $p < q$ and let $f = \sum_0^\infty f_j$ be the Littlewood-Paley decomposition of $f \in \mathcal{S}$. Now interpret the supremum over $0 < r < 1$ as

$$\begin{aligned} Mf &= \sup_{k \geq 0} \sup_{r \sim 2^{-k}} |f * d\sigma_r| \\ &= \sup_{k \geq 0} \sup_{r \sim 2^{-k}} \left| \sum_{j=0}^{\infty} f_j * d\sigma_r \right|. \end{aligned}$$

Next, divide the decomposition into low and high frequencies around some integer k

$$Mf \leq \sup_{k \geq 0} \sup_{r \sim 2^{-k}} \left| \sum_{j=0}^{k} f_j * d\sigma_r \right| + \sup_{k \geq 0} \sup_{r \sim 2^{-k}} \left| \sum_{j=k+1}^{\infty} f_j * d\sigma_r \right|. \qquad (4.3)$$

Let us now study the left and right term independently.

For the low frequency part, the sum $\sum_{j=0}^{k} f_j$ is a function whose Fourier transform is supported on a ball of radius 2^k. By recalling the operator (2.6) from chapter 2.2 we see that this is the same as operating f by S_k. The operator S_k has the property that $S_k f$ is nearly constant on scales smaller thatn 2^{-k} and as we are taking a supremum with $r \sim 2^{-k}$, the spherical average is roughly the same as the average over $B(x, 2^{-k})$. Thus the left part of (4.3) should be dominated by the Hardy-Littlewood maximal function. This can also be calculated as below.

$$\begin{aligned} \left| \left(\sum_{j=0}^{k} f_j * d\sigma_r\right)(x) \right| &= \left| \fint_{\partial B(x,r)} (f * \phi_{2^{-k}})(y) d\mathcal{H}^{n-1}(y) \right| \\ &\leq \fint_{\partial B(x,r)} \left| \int_{\mathbb{R}^n} f(z) \phi_{2^{-k}}(y - z) dz \right| d\mathcal{H}^{n-1}(y), \end{aligned}$$

where

$$\phi_{2^{-k}}(x) = 2^{nk} \phi(2^k x) = (\hat{\phi}(\cdot/2^k)\check{)}(x)$$

is a Schwartz function as in chapter 2.2, and has a radial, decreasing, positive and integrable majorant $|\phi(x)| \leq \Psi_N(x)$, so that

$$\phi_{2^{-k}}(x) \leq \Psi_{N,2^{-k}}(x) = 2^{nk}\Psi_N(2^k x) = \frac{C_N 2^{kn}}{(1+2^k|x|)^N}.$$

Now the convolution $(f * \phi_{2^{-k}})(y)$ can be majorized by $\mathcal{M}f(x)$ by

$$\begin{aligned} |(f * \phi_{2^{-k}})(y)| &\leq \int_{\mathbb{R}^n} |f(z)||\Psi_{N,2^{-k}}(y-z)|dz \\ &= \int_{B(y,2^{-k})} |f(z)|\Psi_{N,2^{-k}}(y-z)dz \\ &\quad + \int_{\mathbb{R}^n \setminus B(y,2^{-k})} |f(z)|\Psi_{N,2^{-k}}(y-z)dz, \end{aligned}$$

where

$$\begin{aligned} \int_{B(y,2^{-k})} |f(z)|\Psi_{N,2^{-k}}(y-z)dz &\leq C2^k \int_{B(x,C2^{-k})} |f(z)|\Psi_n(2^k(y-z))dz \\ &\leq C \fint_{B(x,C2^{-k})} |f(z)|C_n dz \\ &\leq C(n)\mathcal{M}f(x). \end{aligned}$$

We can switch from integrating the ball $B(y,2^{-k})$ to $B(x,C2^{-k})$ as $|x-y| \sim 2^{-k}$. For the other part, denote $\psi_N(|y|) = \Psi_N(y)$ and $\Phi_y : t \to \int_{B(y,t)} |f(z)|dz$ so that $\Phi_y'(t) = \int_{\partial B(y,t)} |f(z)|d\mathcal{H}^{n-1}(z)$. Integration by parts then gives

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B(y,2^{-k})} |f(z)|\Psi_{N,2^{-k}}(y-z)dz &= \int_{2^{-k}}^{\infty} \Phi_y'(t)\psi_{N,2^{-k}}(t)dt \\ &= -\Phi_y(2^{-k})\psi_{N,2^{-k}}(2^{-k}) \\ &\quad + \int_{2^{-k}}^{\infty} \Phi_y(t)(-\psi_{N,2^{-k}}'(t))dt \\ &= -2^{kn}\psi_N(2^k 2^{-k}) \int_{B(y,2^{-k})} |f(z)|dz \\ &\quad + \int_{2^{-k}}^{\infty} \frac{1}{t^n} \int_{B(y,t)} |f(z)|dz(-\psi_{N,2^{-k}}'(t)t^n)dt \\ &\leq C\mathcal{M}f(x)(\int_0^{\infty} -\psi_{N,2^{-k}}'(t)t^n dt - \psi_N(1)) \\ &= C\mathcal{M}f(x)(\int_0^{\infty} \psi_{N,2^{-k}}(t)nt^{n-1}dt - \psi_N(1)) \\ &\leq C(n)\mathcal{M}f(x), \end{aligned}$$

by setting $N = n + 1$ and again moving to integrals over $B(x, C2^{-k})$ and $B(x, Ct)$. Now the left integral in (4.3) is majorized as

$$\begin{aligned}\sup_{k\geq 0}\sup_{r\sim 2^{-k}}\left|(\sum_{j=0}^{k} f_j * d\sigma_r)(x)\right| &\leq \sup_{k\geq 0}\sup_{r\sim 2^{-k}}\left|\fint_{\partial B(x,r)} C\mathcal{M}f(x)d\mathcal{H}^{n-1}(y)\right| \\ &= C\mathcal{M}f(x).\end{aligned}$$

We could now take L^q norm on both sides and use Hardy-Littlewood Theorem to bound the expression by $||f||_{L^q}$, but as we are looking for a L^p bound, we need to add a proper fractional term r^α, where $\alpha = n(1/p - 1/q)$, to the expression and use the results from chapter 1.3 to get

$$\left\|\sup_{k\geq 0}\sup_{r\sim 2^{-k}} r^\alpha \left|\sum_{j=0}^{k} f_j * d\sigma_r\right|\right\|_{L^q} \leq \|C\mathcal{M}_\alpha f\|_{L^q} \leq C\,\|f\|_{L^p}\,, \tag{4.4}$$

which is the desired result for the low-frequency part of (4.3).

The high-frequency part of 4.3 seems more complex than the lower part. What we essentially have is a function consisting of the oscillations in f with "frequency" higher than 2^k. Thus it is hard to say anything about averages on the scale $r \sim 2^{-k}$ as the function can contain substantial oscillations on much higher frequencies. What we can work with, are the local estimates (4.2), so let us try to modify the expression in the right integral of (4.3) into a local maximal argument. For fixed $k > 0$, we have

$$\begin{aligned}\sup_{r\sim 2^{-k}}\left|(\sum_{j>k} f_j * d\sigma_r)(x)\right| &\leq C\sum_{j>k}\sup_{r\sim 2^{-k}}\int_{\partial B(0,1)}|f(x-ry)|d\sigma(y) \\ &\leq C\sum_{j>k}\sup_{1<t<2}\int_{\partial B(0,1)}|f(x-\frac{t}{2^k}y)|d\sigma(y) \\ &= C\sum_{j>k}\sup_{1<t<2}2^{kn}\int_{\partial B(0,1)}\left|\frac{1}{2^{kn}}f(\frac{1}{2^k}(2^k x - ty)\right|d\sigma(y) \\ &= C\sum_{j>k}\sup_{1<t<2}2^{kn}((f_j)_{2^k} * d\sigma_t)(2^k x) \\ &= C\sum_{j>k}(\overline{M}(f_j)_{2^k})_{2^{-k}}(x),\end{aligned}$$

where $g_{2^k}(x) = 2^{-nk}g(x/2^k)$, so we have a sum of scaled local maximal functions. Next we majorize the supremum over k with a sum, and take L^q

norms, so that

$$\begin{aligned}
\left\| \sup_{k>0} \sup_{r\sim 2^{-k}} r^{\alpha} \left| \sum_{j=k+1}^{\infty} f_j * d\sigma_r \right| \right\|_{L^q} &\leq \left\| \left(\sum_{k\geq 0} \left(\sum_{j>k} 2^{-k\alpha} (\overline{M}(f_j)_{2^k})_{2^{-k}} \right)^q \right)^{\frac{1}{q}} \right\|_{L^q} \\
&= \left(\int_{\mathbb{R}^n} \sum_{k\geq 0} \left(\sum_{j>k} 2^{-k\alpha} (\overline{M}(f_j)_{2^k})_{2^{-k}} \right)^q dx \right)^{\frac{1}{q}} \\
&= \left(\sum_{k\geq 0} \int_{\mathbb{R}^n} \left(\sum_{j>k} 2^{-k\alpha} (\overline{M}(f_j)_{2^k})_{2^{-k}} \right)^q dx \right)^{\frac{1}{q}} \\
&= \left(\sum_{k\geq 0} \left\| \sum_{j>k} 2^{-k\alpha} (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q}^q \right)^{\frac{1}{q}} \\
&\leq \left(\sum_{k\geq 0} \left(\sum_{j>k} 2^{-k\alpha} \left\| (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q} \right)^q \right)^{\frac{1}{q}}.
\end{aligned}$$

To make sense of the norm of the scaled maximal function, we can calculate

$$\begin{aligned}
2^{-k\alpha} \left\| (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q} &= 2^{-k\alpha} \left(\int_{\mathbb{R}^n} 2^{knq} \overline{M}(f_j)_{2^k} (2^k x)^q dx \right)^{\frac{1}{q}} \\
&= 2^{-k\alpha} \left(\int_{\mathbb{R}^n} 2^{knq-kn} \overline{M}(f_j)_{2^k} (y)^q dy \right)^{\frac{1}{q}} \\
&= 2^{-k\alpha+kn-kn\frac{1}{q}} \left\| \overline{M}(f_j)_{2^k} \right\|_{L^q}.
\end{aligned}$$

The scaled function $(f_j)_{2^k}$ can be expressed as

$$(f_j)_{2^k} = \frac{1}{2^{kn}} f_j(2^{-k}x) = \frac{1}{2^{kn}} \int_{\mathbb{R}^n} e^{2\pi i \frac{x}{2^k}\cdot\xi} \hat{f}(\xi) \varphi_j(\xi) d\xi,$$

and by change of variables $\gamma = \frac{\xi}{2^k}$, we get

$$(f_j)_{2^k} = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\gamma} \hat{f}(2^k\gamma) \varphi(\frac{2^k}{2^j}\gamma) d\gamma,$$

which has support on $|\gamma| \sim 2^{j-k}$, as supp $\varphi \subset \{\gamma \in \mathbb{R}^n : |\gamma| \sim 1\}$. Thus the assumption (4.2) gives

$$\begin{aligned}
\left\| \overline{M}(f_j)_{2^k} \right\|_{L^q} &\leq 2^{(j-k)\beta} \left\| (f_j)_{2^k} \right\|_{L^p} \\
&= 2^{(j-k)\beta} \left(\int_{\mathbb{R}^n} \frac{1}{2^{knp}} f_j(2^{-k}x)^p dx \right)^{\frac{1}{p}} \\
&= 2^{(j-k)\beta - kn + kn\frac{1}{p}} \left\| f_j \right\|_{L^p}.
\end{aligned}$$

So the simplified norm of the scaled maximal function is

$$\begin{aligned} 2^{-k\alpha} \left\| (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q} &= 2^{-k\alpha + kn - kn\frac{1}{q} - kn + kn\frac{1}{p}} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p} \\ &= 2^{k(n(1/p-1/q)-\alpha)} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p}, \end{aligned}$$

where $\alpha = n(1/p - 1/q)$, giving

$$2^{-k\alpha} \left\| (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q} = 2^{(j-k)\beta} \left\| f_j \right\|_{L^p}.$$

Thus the norm estimate for the high frequency part becomes

$$\begin{aligned} \left(\sum_{k \geq 0} \left(\sum_{j > k} 2^{-k\alpha} \left\| (\overline{M}(f_j)_{2^k})_{2^{-k}} \right\|_{L^q} \right)^q \right)^{\frac{1}{q}} &= \left(\sum_{k \geq 0} \left(\sum_{j > k} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p} \right)^q \right)^{\frac{1}{q}} \\ &\leq \left(\sum_{k \in \mathbb{Z}} \left(\sum_{j > k} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p} \right)^q \right)^{\frac{1}{q}}. \end{aligned}$$

This can be seen as a l^q sequence norm of a discrete convolution

$$\begin{aligned} \sum_{j > k} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p} &= \sum_{j \in \mathbb{Z}} 2^{-(k-j)\beta} \chi_{\{-(k-j) > 0\}} \left\| f_j \right\|_{L^p} \\ &= (\left\| f_j \right\|_{L^p} * 2^{-j\beta} \chi_{\{-j>0\}})(k). \end{aligned}$$

Next we use Young's inequality for convolutions to see

$$\begin{aligned} \left(\sum_{k \in \mathbb{Z}} \left(\sum_{j > k} 2^{(j-k)\beta} \left\| f_j \right\|_{L^p} \right)^q \right)^{\frac{1}{q}} &= \left\| \left\| f_j \right\|_{L^p} * 2^{-j\beta} \chi_{\{-j>0\}} \right\|_{l^q} \\ &\leq \left\| \left\| f_j \right\|_{L^p} \right\|_{l^q} \left\| 2^{-j\beta} \chi_{\{-j>0\}} \right\|_{l^1} \\ &= \sum_{j > 0} 2^{j\beta} \left(\sum_{j \in \mathbb{Z}} \left\| f_j \right\|_{L^p}^q \right)^{\frac{1}{q}} \\ &\leq C(\beta) \left(\sum_{j \in \mathbb{Z}} \left\| f_j \right\|_{L^p}^q \right)^{\frac{1}{q}}, \end{aligned}$$

as $\beta < 0$. Now we use the sequence space inclusion $l^p \subset l^q$, for $p \leq q$ and choose $s = \max\{2, p\}$, so that $s \leq q$, to get

$$C(\beta) \left(\sum_{j \in \mathbb{Z}} \left\| f_j \right\|_{L^p}^q \right)^{\frac{1}{q}} \leq C(\beta) \left(\sum_{j \in \mathbb{Z}} \left\| f_j \right\|_{L^p}^s \right)^{\frac{1}{s}}.$$

Recall the Minkowski's integral inequality

Lemma 4.2. *For measurable $F : X \times Y \to \mathbb{R}$ and $1 \le p < \infty$*

$$\left(\int_Y \left(\int_X |F(x,y)|dx\right)^p dy\right)^{\frac{1}{p}} \le \int_X \left(\int_Y |F(x,y)|^p dy\right)^{\frac{1}{p}} dx$$

holds.

Interpreting the sum over $j \in \mathbb{Z}$ as an integral, we have $1 \le s/p < \infty$ and use the above inequality

$$\begin{aligned}\left(\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^s\right)^{\frac{1}{s}} &= \left(\sum_{j\in\mathbb{Z}} \left(\int_{\mathbb{R}^n} |f_j|^p dx\right)^{\frac{s}{p}}\right)^{\frac{p}{s}\frac{1}{s}\frac{s}{p}} \\ &\le \left(\int_{\mathbb{R}^n} \left(\sum_{j\in\mathbb{Z}} |f_j|^{\frac{s}{p}p}\right)^{\frac{p}{s}} dx\right)^{\frac{1}{p}} \\ &= \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^s\right)^{\frac{1}{s}} \right\|_{L^p}.\end{aligned}$$

With $2 \le s$, we can use the sequence space inclusion again to pass to the square functions

$$\left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^s\right)^{\frac{1}{s}} \right\|_{L^p} \le \left\| \left(\sum_{j\in\mathbb{Z}} f_j^2\right)^{\frac{1}{2}} \right\|_{L^p}.$$

Now we can finally apply the Littlewood-Paley Theorem 2.3 to achieve

$$\left\| \sup_{k\ge 0} \sup_{r\sim 2^{-k}} r^\alpha \left| \sum_{j=k+1}^{\infty} f_j * d\sigma_r \right| \right\|_{L^q} \le C \left\| \left(\sum_{j\in\mathbb{Z}} f_j^2\right)^{\frac{1}{2}} \right\|_{L^p} \le C \|f\|_{L^p}. \tag{4.5}$$

With these results, we can prove Lemma 4.1.

Proof of Lemma 4.1. Let $f = \sum_{j\ge 0} f_j$ be the Littlewood-Paley decomposition of $f \in \mathcal{S}$. For the fractional spherical maximal function (4.1), we have

$$\begin{aligned}M_\alpha f(x) &= \sup_{k\ge 0} \sup_{r\sim 2^{-k}} \left| (\sum_{j\ge 0} f_j * d\sigma_r)(x) \right| \\ &\le \sup_{k\ge 0} \sup_{r\sim 2^{-k}} \left| (\sum_{j=0}^{k} f_j * d\sigma_r)(x) \right| + \sup_{k\ge 0} \sup_{r\sim 2^{-k}} \left| (\sum_{j=k+1}^{\infty} f_j * d\sigma_r)(x) \right|.\end{aligned}$$

Taking L^q norm on both sides gives

$$\|M_\alpha f\|_{L^q} \leq \left\| \sup_{k\geq 0} \sup_{r\sim 2^{-k}} \left| \sum_{j=0}^{k} f_j * d\sigma_r \right| \right\|_{L^q} + \left\| \sup_{k\geq 0} \sup_{r\sim 2^{-k}} \left| \sum_{j=k+1}^{\infty} f_j * d\sigma_r \right| \right\|_{L^q}.$$

The results (4.4) and (4.5) above state, that there exists C_1, C_2 depending only on n, p and q, such that

$$\left\| \sup_{k\geq 0} \sup_{r\sim 2^{-k}} \left| \sum_{j=0}^{k} f_j * d\sigma_r \right| \right\|_{L^q} \leq C_1 \|f\|_{L^p}$$

and

$$\left\| \sup_{k\geq 0} \sup_{r\sim 2^{-k}} \left| \sum_{j=k+1}^{\infty} f_j * d\sigma_r \right| \right\|_{L^q} \leq C_2 \|f\|_{L^p}$$

hold. Combining these gives the required norm estimate for the fractional maximal function

$$\|M_\alpha f\|_{L^q} \leq (C_1 + C_2) \|f\|_{L^p} \leq C \|f\|_{L^p}.$$

□

Note that Lemma 4.1 only covers $q \geq 2$, in order to pass to the square functions. For $n \geq 3$, this leaves out the diagonal $(n-1)/n < p = q < 2$ and other estimates, but for these parts the estimates can be proved independently for the full maximal function. The L^2 estimate (3.5) can be proved for the full maximal function by slight modification of the arguments, as is done in [3, Chapter 5.5]. The $L^1 \to L^1$ (3.3) can be proved by Lemma 3.1, but the estimate is only of weak type, as Hardy-Littlewood Theorem 1.3 holds only weakly in L^1.

Marcinkiewicz interpolation between these bounds gives estimates for the full operator with $(n-1)/n < p = q < 2$ and further interpolation between these estimates and point P_3 in Theorem 3.1 proves the result for any $q < 2$ points for witch the localized result holds.

4.2 Generalization to L^p functions

This far, most of the analysis has been done on Schwartz functions to avoid any problems arising from Fourier transforms and Littlewood-Paley decompositions. We now extend these results to general L^p functions using basic approximation with simple functions. We will also consider measurability of the spherical maximal operator. The approach presented here is similar to [10].

This measurability of an spherical maximal operator M should not be thought of as trivial. For example, consider a non-measurable set on $[0,1]$ mapped to some curve $\gamma : [0,1] \to \mathbb{R}^2$. This set, as a set of measure zero, is of course measurable with respect to the two dimensional Lebesgue measure, but for the spherical maximal function, restring to spheres and one dimensional Hausdorff measures, the integral over $\partial B(x,r) \cap \gamma$ might not be defined for some x and r. For this, the following result is derived from the $L^p \to L^q$-norm estimates.

Lemma 4.3. *Let $E \subset \mathbb{R}^n$, $n \geq 2$ be a set of measure zero. Then for almost every $x \in \mathbb{R}^n$*

$$\mathcal{H}^{n-1}(E \cap \partial B(x,r)) = 0$$

for all $r > 0$.

Let us first assume that $L^p \to L^q$ estimates hold for some spherical maximal function M and all $f \in \mathcal{S}$, for some (p,q) pair. To generalize the arguments to indicator functions, we will approximate open sets with increasing sequences of Schwartz functions and the following lemma.

Lemma 4.4. *Let f and f_i be non-negative Borel measurable functions and assume $f_i(x)$ increases to $f(x)$ for every $x \in \mathbb{R}^n$, then*

$$Mf(x) = \lim_{i\to\infty} Mf_i(x). \tag{4.6}$$

Proof. As f_i is increasing, clearly

$$Mf(x) \geq Mf_i(x), \quad \text{for all } i \in \mathbb{N},$$

and for fixed $r > 0$

$$\begin{aligned}\lim_{i\to\infty} \fint_{\partial B(x,r)} |f_i(y)| d\mathcal{H}^{n-1}(y) &= \fint_{\partial B(x,r)} \lim_{i\to\infty} |f_i(y)| d\mathcal{H}^{n-1}(y) \\ &= \fint_{\partial B(x,r)} |f(y)| d\mathcal{H}^{n-1}(y)\end{aligned}$$

by Monotone Convergence Theorem. Combining these gives

$$\lim_{i\to\infty} Mf_i(x) \geq \fint_{\partial B(x,r)} |f(y)| d\mathcal{H}^{n-1}(y)$$

for every $x \in \mathbb{R}^n$ and $r > 0$. Taking supremum in r on the right hand side results in

$$\lim_{i\to\infty} Mf_i(x) \geq Mf(x) \geq Mf_i(x), \text{for all } i \in \mathbb{N},$$

implying the result. □

Now setting f to be the indicator function of an open set E and f_i to be a sequence of increasing bump functions converging to f pointwise, proves Mf to be measurable as a limit of measurable functions. Applying the $L^p \to L^q$ estimates to equality (4.6) gives the norm estimates for indicator functions of open sets.

Now we can prove Lemma 4.3. Let E be any set of measure zero in $\mathbb{R}^n$ and $\tilde{E}$ a Borel set of measure zero, so that $E \subset \tilde{E}$. Now let $\{O_i\}_{i=1}^{\infty}$ be a sequence of open sets, such that

$$\begin{aligned} |O_i| &\to 0, \\ \tilde{E} &\subset O_i, \\ O_{i+1} &\subset O_i, \end{aligned}$$

for all i. Then

$$M\chi_{\tilde{E}} \leq \lim_{i\to\infty} M\chi_{O_i}$$

and

$$\|M\chi_{\tilde{E}}\|_{L^p} \leq \left\|\lim_{i\to\infty} M\chi_{O_i}\right\|_{L^p} \leq \lim_{i\to\infty} \|M\chi_{O_i}\|_{L^p} = C \lim_{i\to\infty} \|\chi_{O_i}\|_{L^p} = 0.$$

This implies $M\chi_{\tilde{E}}(x) = 0$ almost everywhere, so that

$$\mathcal{H}^{n-1}(E \cap \partial B(x,r)) \leq \mathcal{H}^{n-1}(\tilde{E} \cap \partial B(x,r)) = 0$$

for every $r > 0$ at almost every x.

Every Lebesgue measurable set E can be expressed as an union of a Borel set and a set of measure zero $E = E_B \cup E_N$. Intersecting a Borel set E_B with a sphere $\partial B(x,r)$ is of course measurable with respect to $\mathcal{H}^{n-1}$, so for almost every $x \in \mathbb{R}^n$

$$\begin{aligned} \mathcal{H}^{n-1}(E_B \cap \partial B(x,r)) &\leq \mathcal{H}^{n-1}(E \cap \partial B(x,r)) \\ &\leq \mathcal{H}^{n-1}(E_B \cap \partial B(x,r)) + \mathcal{H}^{n-1}(E_N \cap \partial B(x,r)) \\ &\leq \mathcal{H}^{n-1}(E_B \cap \partial B(x,r)) \end{aligned}$$

holds for any $r > 0$, proving that $E \cap \partial B(x, r)$ is measurable.

Let $f = \sum_{i=0}^{k} a_i \chi_{E_i}$ be a simple function, where $a_i \in \mathbb{R}$, $E_i \subset \mathbb{R}^n$ and Lebesgue measurable. Now f restricted to $\partial B(x, r)$ is $\mathcal{H}^{n-1}$ measurable and Mf is well defined for almost every $x \in \mathbb{R}^n$. For every i, let $\tilde{E}_{i_j}$ be a converging sequence of open sets, approximating the Borel part of E_i.

$$\begin{aligned} \|Mf\|_{L^q} \leq \sum_{i=0}^{k} a_i \left\| \lim_{j \to \infty} M\chi_{\tilde{E}_{i_j}} \right\|_{L^q} &\leq \sum_{i=0}^{k} a_i \lim_{j \to \infty} \left\| M\chi_{\tilde{E}_{i_j}} \right\|_{L^q} \\ &\leq C \sum_{i=0}^{k} a_i \lim_{j \to \infty} \left\| \chi_{\tilde{E}_{i_j}} \right\|_{L^p} \\ &= C \sum_{i=0}^{k} a_i \|\chi_{E_i}\|_{L^p} \\ &= C \|f\|_{L^p} . \end{aligned}$$

Finally any non-negative $f \in L^p$ function can be approximated by a increasing sequence of simple functions almost everywhere, and similar result to Lemma 4.4 can be established, showing Mf is well defined and measurable almost everywhere, and

$$\|Mf\|_{L^q} \leq C \|f\|_{L^p} .$$

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