Publication 7

Dlala, E., Belahcen, A., Arkkio, A., “A Fast fixed-point method for solving magnetic field problems in media of hysteresis”, *IEEE Transactions on Magnetics*, to be published, in press, January 2008.

A Fast Fixed-Point Method for Solving Magnetic Field Problems in Media of Hysteresis

Emad Dlala, Anouar Belahcen, and Antero Arkkio

***Abstract*— The paper proposes a new fixed-point method for solving time-stepping hysteretic field problems. The method is aimed to speed up the convergence of the fixed-point solution and enhance the applicability of the fixed-point iteration. The method makes use of the differential reluctivity and produces a locally convergent solution. A 1D finite-element procedure is performed to test the method by computing the magnetic field in a ferromagnetic laminated steel. The efficiency, stability, and applicability of the method are assessed in which the method is proven simple and remarkably fast.**

***Index Terms*— Contraction mapping, finite-elements, fixed-point, magnetic field, nonlinear hysteresis, time-stepping.**

I. Introduction

THE FIXED-POINT iteration has become a technique of growing importance in the area of electromagnetics, especially when medium of hysteresis is considered. The advantages of this technique lie mainly in its simplicity and robustness where the evaluation of the derivative is not, in principle, needed. Early in 1975 [1], nonlinear electromagnetic problems were formulated by the fixed-point technique, describing constitutive relations in the manner $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{x})$. Despite its slow convergence, the fixed-point technique has been particularly useful for solving problems of hysteretic magnetic media.

The conditions and criteria for the convergence of the fixed-point theory are well documented in mathematics. The fixed-point theory is based on contraction mapping principle. This principle guarantees the existence and uniqueness of fixed points, and provides a constructive method to find those fixed points. For 1D problems, the fixed point can be found in some interval $I = [a, b]$ if there exists a positive number α (called a contraction factor) such that

$$|G(a) - G(b)| \leq \alpha |a - b|, \qquad 0 < \alpha < 1. \tag{1}$$

This concept is used to make the iterated function $x_{k+1} = G(x_k)$ converge to a unique fixed point starting from any point $x_0 \in I$. The smaller the contraction factor α is, the faster the method converges to the fixed point.

The fixed-point iteration may converge *globally* whichever a starting value x_0 is given. Such a case, considered by Hantila [1], [2] and later used by many researchers [3], [4], guarantees convergence but very slow one. On the other hand, if the starting value x_0 is known in advance and close to the solution, the fixed-point iteration may be made to be *locally* convergent in a small interval where the convergence factor should be small in the vicinity of the solution and thus fast convergence is achieved [5]. In this paper, the locally convergent method of the fixed-point iteration proposed in [5], which has been devoted to nonlinear single-valued problems, is further extended to account for hysteretic functions. The focus will be devoted to the study of the 1D problem whereas the general case for the 2D and 3D problems can be similarly treated [6].

Although the fixed-point iteration is solely based on the principle of contraction maps, it will be shown in this paper that exploiting the derivative can accelerate the convergence significantly without affecting the stability of the iteration even for hysteretic functions.

Emad Dlala, Anouar Belahcen, and Antero Arkkio are with the Laboratory of Electromechanics, Helsinki University of Technology, P.O. Box 3000 FI-02015 TKK, Finland. E-mails: emad.dlala@tkk.fi, anouar.belahcen@tkk.fi, antero.arkkio@tkk.fi.

II. the Method

The magnetic field problem is formulated by Hantila [1] using the fixed-point theory in the following manner

$$M(B) = H - \nu_{\mathrm{FP}} B \tag{2}$$

where H is the magnetic field intensity, B is the magnetic flux density, and M is a magnetization-like quantity. The coefficient ν_{FP} is kept constant during iteration and should be properly chosen to ensure contraction. The nonlinearity between H and B can be expressed by the following relation

$$H = F(B) \tag{3}$$

where F may be a hysteretic or single-valued function and should be Lipschitz continuous.

The classical approach proposed by Hantila [1], [2] is based on computing the fixed-point coefficient once and for all as

$$\nu_{\mathrm{FP}} = \frac{\nu_{\mathrm{d_{min}}} + \nu_{\mathrm{d_{max}}}}{2} \approx \frac{\nu_0}{2} \tag{4}$$

where $\nu_{\mathrm{d_{min}}}$ and $\nu_{\mathrm{d_{max}}}$ are the minimum and maximum slopes (differential reluctivities) of the curve $F(B)$, and ν_0 is the reluctivity of the vacuum. This method is called the global-coefficient method (GCM) since it uses the same (global) coefficient in the whole B-H curve. The GCM is stable but produces very slow convergence because the contraction factor α resulted from the method is very close to one, and this is the main reason why the fixed-point method has been widely known to be inefficient for solving nonlinear field problems [2].

It is shown in [5] that the resultant iterative function $G(B)$ follows exactly the shape (and the derivative) of the function $M(B)$ while only a shift occurs to the function $G(B)$ at each time-step. Furthermore, the theory of contraction mappings proves that the fastest convergence takes place when the contraction factor of (1) is close to zero ($\alpha \approx 0$). To achieve this, the derivative of the function $M(B)$ with respect to B in equation (2) is taken and put to zero

$$\frac{\mathrm{d}G}{\mathrm{d}B} = \frac{\mathrm{d}M}{\mathrm{d}B} = \frac{\mathrm{d}H}{\mathrm{d}B} - \nu_{\mathrm{FP}} = 0, \tag{5}$$

and this gives the most optimal fixed-point coefficient

$$\nu_{\mathrm{FP}} = \frac{\mathrm{d}H}{\mathrm{d}B}. \tag{6}$$

In time-stepping analysis, the initial value B_0 for time-step n is known from the *previous* time-step and should be sufficiently close to the fixed point. Therefore, a neighborhood interval U which contains B_0 and fulfils (1) is found by rewriting equation (6) as

$$\nu_{\mathrm{FP}}\big|^{n} = C\frac{\mathrm{d}H}{\mathrm{d}B}\bigg|^{n-1}, \quad C > 1 \tag{7}$$

where $\nu_{\mathrm{FP}}|^n$ refers to the fixed-point coefficient in the current time-step. It follows from (7) that in the neighborhood U, a fixed-point exists and the contraction factor α is sufficiently small to guarantee locally fast convergence. The convergence factor C is necessary and must be greater than one because in (5) we have enforced the contraction factor α to be zero while according to condition (1) α must be at least slightly greater than zero.

Method (7) computes a local coefficient ν_{FP} at each time step and produces a locally convergent iteration. Thus, the method is called the *local*-coefficient method (LCM).

In general, the idea behind the fast convergence of the LCM and the slow convergence of the GCM is briefly explained in Fig. 1. When the function $G(B)$ is computed from the LCM, the slope of $G(B)$ is close to zero and hence the iteration is fast (Fig. 1a.) However, the function $G(B)$ computed from the GCM in Fig. 1b has a slope close to one at the solution interval and hence the solution takes many iterations to converge for the same initial value B_0 used in the case of Fig. 1a. It is evident that the LCM creates smaller derivatives of the function $G(B)$ near the fixed-point solution as it is illustrated in Fig. 1a by the dotted circle. This feature allows significantly fast convergence.

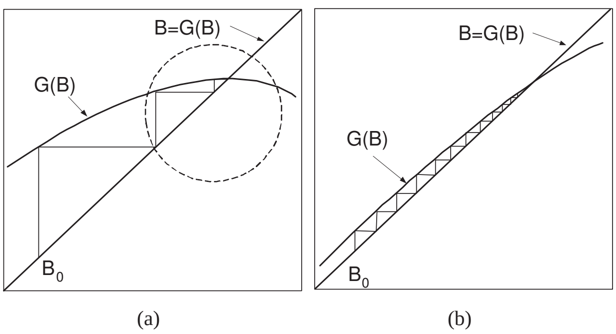

Fig. 1. A comparison of the speed of the fixed-point iteration between the two methods. (a) The local-coefficient method (LCM). (b) The global-coefficient method (GCM).

The convergence factor C is dependent on the time-step size and on the temporal behavior of the system involved. We have adopted a simple technique using linear search for finding the optimal value of C. A detailed discussion clarifying the role of C and how to find its optimal values is carried out in [5], [6].

What distinguishes the fixed-point method most is its independence from calculating the derivative; the method only requires contractive functions $M(B)$ to converge, no matter what technique was used to determine ν_{FP}. On the other hand, it is now known that *faster* iteration is certainly achieved by computing the derivative using the LCM. It is important to realize that unlike the Newton-Raphson method which fundamentally relies on computing the derivative, the fixed-point method would only need the derivative to speed up its iteration. In other words, the Newton-Raphson method uses the derivative to *find* the solution and thus the method is highly sensitive to the derivative and this is its main drawback. On the other hand, the LCM type of the fixed-point method uses the derivative to *accelerate* the solution, and therefore, the derivative can be even “approximated” if it was difficult to determine in the worst scenario.

III. Hysteresis within the fixed-point method

In hysteretic media, both the GCM and LCM must ensure the continuity of the multi-valued functions $F(B)$ at the reversal points in order to keep $M(B)$ Lipschitz continuous and hence contractive. Thus, any hysteresis model used to represent the function $F(B)$ must allow this feature. This problem occurs because the iteration could start outside the major loop from the initial value B_0 (computed from the previous time-step) and thus would diverge since the function $F(B)$ did not exist in that region. To tackle the problem, for example, the set of first-order reversal curves typically used for models such as the Preisach model must be extrapolated beyond the major loop and they should be treated as single-valued functions (see Fig. 2).

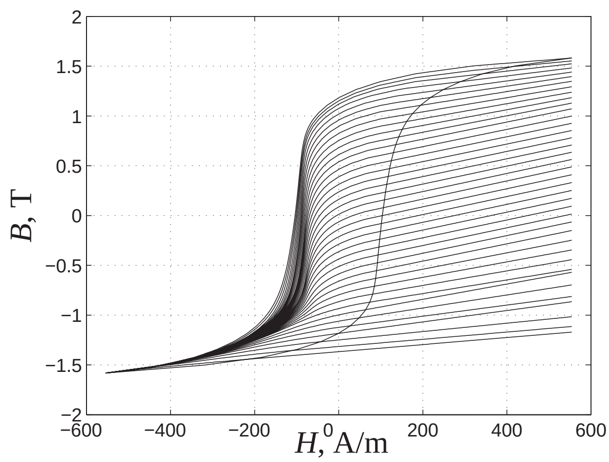

Fig. 2. First-order reversal curves $H = F(B)$ extrapolated to ensure continuity at the reversal points.

There is another problem that has to be handled when using the LCM within hysteretic media. Since the nature of hysteresis is branching, discontinuity of the derivative at

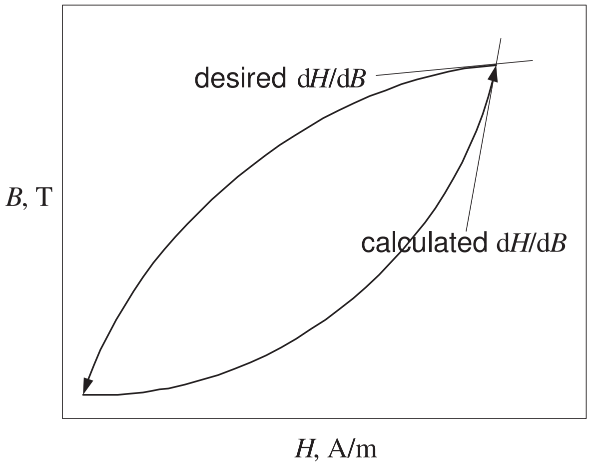

Fig. 3. The difference between the calculated slope and the desired one at a reversal point.

the reversals could become a problem for method (7). This problem arises because the derivative in (7) now calculated through the hysteresis model from the previous time step can be much smaller than the desired one (see Fig. 3). Although the reluctivity of vacuum ν_0 can be a good estimate to the desired slope at the reversals and will definitely guarantee convergence, it may in effect cause stability problems to the time-stepping scheme which can be vulnerable to large, sudden changes of $\nu_{\rm FP}$ in the system. It is therefore preferred to implement a predictive technique which can approximate the derivative smoothly at the reversals. It is generally well known in magnetism [7] that ferromagnetic materials exhibit the largest $\mathrm{d}H/\mathrm{d}B$ at the first-order reversal points and the trend of the derivatives change quite quadratically from saturation $B = B_{\rm s}$ until zero crossing (see Fig. 2). Consequently, we can say that, to approximate the derivative at the reversals, the following algorithm will ensure convergence as well as stability of the system

$$\frac{\mathrm{d}H}{\mathrm{d}B} = \frac{\nu_0}{4}\left[\frac{|B_{\rm r}| + B_{\rm s}}{B_{\rm s}}\right]^2 \tag{8}$$

where $B_{\rm r}$ is the flux density at the reversal point. Here we assume that at saturation, the differential reluctivity should be equal to the reluctivity of the vacuum ($\frac{\mathrm{d}H}{\mathrm{d}B} = \nu_0$.) Since the reversal points are detected by the hysteresis model, equation (8) can be conveniently used at the reversals.

IV. Finite-Element Eddy-Current Modeling

The nonlinear 1D magnetodynamic problem can be formulated by the fixed-point method as

$$\nu_{\rm FP}\frac{\partial^2 A}{\partial z^2} + \sigma\frac{\partial A}{\partial t} = -\frac{\partial M}{\partial z} \tag{9}$$

where σ is the conductivity of the ferromagnetic material. Here A is the magnetic vector potential alternating in the x-y plane, parallel to x and perpendicular to the lamination depth z. The nonlinearity in the lamination is handled by (2).

In the lamination, the dynamic, hysteretic relation is described by a viscosity-based model [8]

$$H(z,t) = F(B(z,t)) \pm \left|\frac{1}{R}\frac{\mathrm{d}B(z,t)}{\mathrm{d}t}\right|^{1/p}. \tag{10}$$

The first term of (10) can be, in principle, computed by any static hysteresis model. In this work, the history-dependent model [7], which employs a static family of first-order reversal curves, has been used. The second term of (10) represents the excess field through the time delay of the magnetic flux from the field strength. The dynamic magnetic resistivity R is a material property and the constant p is related to the dependency of the excess loss on the frequency. The signs $\pm$ are switched according to whether the field is increasing or decreasing.

Equation (9) is discretized using first-order finite-elements and solved by the Crank-Nicholson time-stepping scheme. The 1D model is subjected to a known magnetic flux per unit length $\phi = Bd$ where d is the thickness of the lamination. Thus, the magnetic vector potential on the boundary is determined accordingly.

Substantial computation time is expected to be saved as a result from using the LCM. Instead of keeping a global coefficient $\nu_{\rm FP}$ fixed in the entire mesh, we will adaptively compute a local coefficient $\nu_{\rm FP}$ in each element or more specifically in each integration point in an element. However, there is a distinctive feature which merits the GCM over the LCM. In finite-element analysis, the global matrix can be factored once and for all when using the GCM, whereas it has to be factored at each time step when using the LCM because $\nu_{\rm FP}$ is being updated. This feature may become noticeable only for large problems in which the number of nodes would exceed 30000. In smaller problems (less than 10000 nodes) such as in a 2D electrical machine mesh, most of the computation effort is rather used in the construction and assembly of the matrix and thus the benefits of the LCM outweigh its drawbacks. On the other hand, in rotating machine applications, the global matrix has to be modified in the airgap in any case due to the rotor motion, and thus, the advantage of the GCM goes in vain. In such an application, the matrix is updated at each time-step anyway and is fixed during iteration for both the LCM and GCM, but yet these two methods bear the advantage over the standard Newton-Raphson method which updates its matrix at each iteration step.

V. Numerical Results

In this section, the numerical procedures introduced in sections II, III, and IV are implemented to test the efficiency of the proposed methods. The magnetodynamic field problem in the lamination of a soft magnetic material is solved in which a sinusoidal source was applied on the boundary of the considered geometry.

The LCM and GCM are examined where, for the LCM, the global matrix was allowed to be updated at each time step but it was fixed for the GCM. All simulations were performed in Fortran by the same computer using the same stopping criterion for the iteration. In all simulations, the number of time steps per period was 400, the frequency was 200 Hz, and the number of first-order finite-elements was 50. The convergence factor C of the LCM was optimally found to be 5, allowing fast convergence.

TABLE I
THE COMPUTATION TIME RESULTS OBTAINED

Test results	LCM	GCM
Iterations per time-step	25	181
CPU-time per time-step (sec)	0.0055	0.0271
Total CPU-time (sec)	2.21	10.82

Table I shows comparative results obtained by implementing the methods for one period. The average number of iterations per time step, the average CPU-time spent on a time step, and the total CPU-time employed in the entire simulation are tabulated. Clearly, the LCM was converging with remarkable speed and although the GCM kept its matrix constant along the simulation, the method yet is much slower. In fact the number of iterations here played a significant role because most of the computation time was reserved to construct the matrix (including the time spent on the hysteresis model) and not on factoring the matrix. Fig. 4 depicts the development of the coefficient $\nu_{\rm FP}(z)$ in some selected elements of the lamination; the corresponding flux density waveforms are shown in Fig. 5. The graphs illustrate the effectiveness of the LCM in adapting the fixed-point coefficient $\nu_{\rm FP}$ according to the changes of the flux density. Furthermore, the discontinuity of the derivative (indicated by the dotted ellipse in Fig. 4) has been dealt with properly. Only the jump at the reversal is noticed which is natural and does not harm the stability of the Crank-Nicholson scheme in solving the temporal problem as it is evidenced by the smoothness of the trajectories of the flux density in the lamination (Fig. 5.)

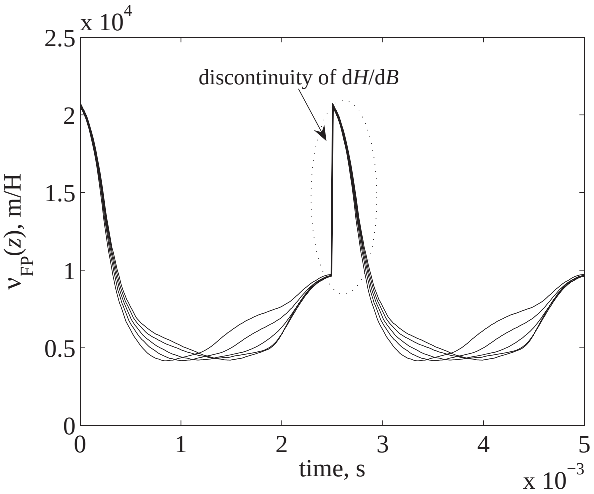

Fig. 4. The variation of the local-coefficient $\nu_{\rm FP}(z)$ in certain points of the lamination at 200 Hz.

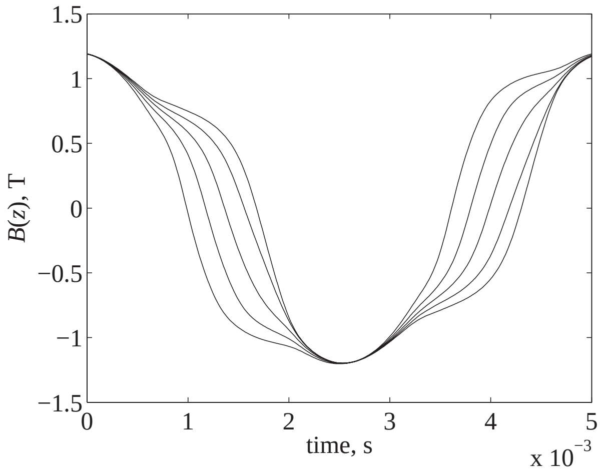

Fig. 5. The waveforms of the flux density corresponding to Fig. 4.

VI. CONCLUSION

The local-coefficient method (LCM) of the fixed-point technique has shown a remarkable success in accelerating the convergence of the fixed-point iteration used for solving time-stepping hysteretic field problems. Certain issues regarding applying the LCM in hysteretic media have been addressed in this paper. The LCM is proven simple, fast, and potential to be a leading technique in this area. The LCM ensures locally convergent iteration and produces a small contraction factor for fast convergence.

ACKNOWLEDGMENT

This work was supported by the Academy of Finland, Fortum Corporation, and the Finnish Cultural Foundation.

REFERENCES

[1] F. Hantila, “A method of solving stationary magnetic field in non-linear media,” *Revue Roumaine des Sciences Techniques, Électrotechnique et Énergétique*., vol. 20, no. 3 pp. 397-407, 1975.
[2] F. Hantila, G. Preda, and M. Vasiliu, “Polarization method for static fields,” *IEEE Trans. Magn.*, vol. 36, no. 4, pp. 672-675, Jul. 2000.
[3] O. Bottauscio, M. Chiampi, D. Chiarabaglio, and M. Repetto, “Preisach-type hysteresis models in magnetic field computation,” *J. Phys B: Condensed Matter*, vol. 275, Iss 1-3, pp. 34-39, Jan 2000.
[4] E. Dlala, J. Saitz, and A. Arkkio, “Inverted and forward Preisach models for numerical analysis of electromagnetic field problems,” *IEEE Trans. Magn.*, vol. 42, no. 8, pp. 1963-1973, Aug. 2006.
[5] E. Dlala, A. Belahcen, and A. Arkkio, “Locally convergent fixed-point method for solving time-stepping nonlinear field problems,” *IEEE Trans. Magn.*, To be published.
[6] E. Dlala and A. Arkkio, “Analysis of the convergence of the fixed-point method used for solving nonlinear rotational magnetic field problems,” *IEEE Trans. Magn.*, Submitted for Publication.
[7] S. E. Zirka, Y. Moroz, P. Marketos, and A. J. Moses, “Congruency-based hysteresis models for transient simulation”, *IEEE Trans. Magn.,* Vol 40, pp. 390-399, Mar. 2004.
[8] S. E. Zirka, Y. I. Moroz, P. Marketos, and A. J. Moses, “Viscosity-based magnetodynamic model of soft magnetic materials”, *IEEE Trans. Magn.*, vol. 42, pp. 2121 - 2132, Sep. 2006.