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CHARACTERIZATIONS OF SOBOLEV INEQUALITIES ON METRIC SPACES

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Abstract. We present isocapacitary characterizations of Sobolev inequalities in very general metric measure spaces.

1. Introduction

There is a well known connection between the isoperimetric and Sobolev inequalities. By the isoperimetric inequality, we have

$$|E|^{(n-1)/n} \le c(n)\mathcal{H}^{n-1}(\partial E), \tag{1.1}$$

where E is a smooth enough subset of $\mathbb{R}^n$, $|E|$ is the Lebesgue measure and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The constant $c(n)$ is chosen so that (1.1) becomes an equality when E is a ball. The Sobolev inequality states that

$$\left(\int_{\mathbb{R}^n} |u|^{n/(n-1)}\,\mathrm{d}x\right)^{(n-1)/n} \le c(n)\int_{\mathbb{R}^n} |\nabla u|\,\mathrm{d}x \tag{1.2}$$

for every $u \in C_0^\infty(\mathbb{R}^n)$. The smallest constant in (1.2) is the same as the constant in (1.1). The Sobolev inequality follows from the isoperimetric inequality through the co-area formula. On the other hand, the isoperimetric inequality can be deduced from the Sobolev inequality, see for example [5]. This shows that the isoperimetric and Sobolev inequalities are different aspects of the same phenomenon. Originally this observation is due to Federer and Fleming [6] and Maz′ya [11].

When the gradient is integrable to a power which is greater than one, the isoperimetric inequality has to be replaced with an isocapacitary inequality. When the exponent is one, capacity and Hausdorff content are equivalent and hence it does not matter which one we choose. In this case due to the boxing inequality, it is enough to have the isocapacitary inequality for balls instead of all sets. This elegant approach to Sobolev inequalities is due to Maz′ya, see [12] and [13]. Usually this characterization leads to descriptions of the best possible constants in Sobolev inequalities. However, the aim of the present work is not so much to study best possible constants but rather study necessary and

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sufficient conditions for Sobolev–Poincaré inequalities in a general metric space context. In weighted Euclidean spaces, these characterizations have been studied in [20].

Rather standard assumptions in analysis on metric measure spaces include a doubling condition for the measure and validity of some kind of Sobolev–Poincaré inequality. Despite the fact that plenty of analysis has been done in this general context, very little is known about the basic assumptions. Several necessary conditions are known, but unfortunately only few sufficient conditions are available so far. On Riemannian manifolds, Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality are not only sufficient but also necessary conditions for a scale invariant parabolic Harnack principle for the heat equation, see [15], [16] and [7]. It also known that Maz′ya type characterizations of Sobolev inequalities are available on Riemannian manifolds. The purpose of this work is to show that this is also the case on a very general metric measure spaces.

2. Preliminaries

We assume that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ such that $0 < \mu(B) < \infty$ for all balls $B = B(x, r) = \{y \in X : d(x, y) < r\}$. In what follows, Ω stands for an open bounded subset of X unless otherwise stated. The measure μ is said to be *doubling* if there exists a constant $c_D \geq 1$, called the *doubling constant*, such that

$$\mu(B(x, 2r)) \leq c_D \mu(B(x, r))$$

for all $x \in X$ and $r > 0$.

In this paper, a *path* in X is a rectifiable non-constant continuous mapping from a compact interval to X. A path can thus be parameterized by arc length.

By saying that a condition holds for *p-almost every path* with $1 \leq p < \infty$, we mean that it fails only for a path family with zero p-modulus. A family Γ of curves is of zero p-modulus if there is a non-negative Borel measurable function $\rho \in L^p(X)$ such that for all curves $\gamma \in \Gamma$, the path integral $\int_\gamma \rho \, \mathrm{d}s$ is infinite.

A nonnegative Borel function g on X is an *upper gradient* of an extended real valued function u on X if for all paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g \, \mathrm{d}s, \tag{2.1}$$

whenever both $u(x)$ and $u(y)$ are finite, and $\int_\gamma g\,\mathrm{d}s = \infty$ otherwise. If g is a nonnegative measurable function on X and if (2.1) holds for p-almost every path, then g is a *p-weak upper gradient* of u.

Let $1 \le p < \infty$. If u is a function that is integrable to power p in X, let

$$\|u\|_{N^{1,p}(X)} = \Big(\int_X |u|^p \,\mathrm{d}\mu + \inf_g \int_X g^p \,\mathrm{d}\mu \Big)^{1/p},$$

where the infimum is taken over all p-weak upper gradients of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u : \|u\|_{N^{1,p}(X)} < \infty\}/\!\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$. For properties of Newtonian spaces, we refer to [18].

Let E be a subset of Ω. We write $u \in \mathcal{A}(E,\Omega)$ if $u_{|E} = 1$ and $u_{|X\setminus\Omega} = 0$.

Definition 2.1. Let $E \subset \Omega$. The *p–capacity* of E with respect to Ω is

$$\mathrm{cap}_p(E,\Omega) = \inf \int_\Omega g_u^p \,\mathrm{d}\mu,$$

where the infimum is taken over all continuous functions $u \in \mathcal{A}(E,\Omega)$ with p–weak upper gradients g_u. If there are no such functions, then $\mathrm{cap}_p(E,\Omega) = \infty$.

We say that a property regarding points in X holds *p–quasieverywhere* (p–q.e.) if the set of points for which the property does not hold has capacity zero.

To be able to compare the boundary values of Newtonian functions, we need a Newtonian space with zero boundary values. Let E be a measurable subset of X. The *Newtonian space with zero boundary values* is the space

$$N_0^{1,p}(E) = \{u|_E : u \in N^{1,p}(X) \text{ and } u = 0 \ p\text{–q.e. on } X \setminus E\}.$$

Note that if $\mathrm{cap}_p(X \setminus E, X) = 0$, then $N_0^{1,p}(E) = N^{1,p}(X)$. The space $N_0^{1,p}(E)$ equipped with the norm inherited from $N^{1,p}(X)$ is a Banach space, see Theorem 4.4 in [19].

Definition 2.2. We say that X supports a *weak* $(1,p)$*–Poincaré inequality* if there exist constants $c_p > 0$ and $\tau \ge 1$ such that for all balls $B(x,r)$ of X, all integrable functions u on X and all upper gradients g_u of u, we have

$$⨍_{B(x,r)} |u - u_{B(x,r)}| \,\mathrm{d}\mu \le c_p r \left(⨍_{B(x,\tau r)} g_u^p \,\mathrm{d}\mu \right)^{1/p}, \tag{2.2}$$

where

$$u_B = ⨍_B u \,\mathrm{d}\mu = \frac{1}{\mu(B)} \int_B u \,\mathrm{d}\mu.$$

3. Functions with zero boundary values

In this section, we give necessary and sufficient conditions for Sobolev inequalities of type

$$\left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{1/q} \le c_S \left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{1/p},$$

where the constant c_S is independent of $u \in N_0^{1,p}(\Omega)\cap C(\Omega)$, and μ and ν are Borel regular outer measures. We consider two ranges of indices separately.

We have chosen to study continuous Newtonian functions, but our arguments do not depend on the choice of the function space. For example, similar results also hold for all Newtonian functions and Lipschitz functions if the definition of p–capacity is adjusted accordingly.

Remark 3.1. In the standard versions of Poincaré inequality, the inequality depends on the diameter of the set. Therefore the constant c_S also depends strongly on the diameter of the set in many cases.

3.1. **The case** $1 \le p \le q < \infty$**.** Let $u : \Omega \to [-\infty, \infty]$ be a μ–measurable function. By the well-known Cavalieri principle

$$\int_\Omega |u|^p \,\mathrm{d}\mu = p \int_0^\infty \lambda^{p-1} \mu(E_\lambda) \,\mathrm{d}\lambda,$$

where

$$E_\lambda = \{x \in \Omega : |u(x)| > \lambda\}.$$

The following simple integral inequality will be useful for us. Notice that the equality occurs when $p = q$.

Lemma 3.2. *If* $u : \Omega \to [-\infty, \infty]$ *is* μ*–measurable and* $0 < p \le q < \infty$*, then*

$$\left(\int_\Omega |u|^q \, d\mu\right)^{1/q} \le \left(p \int_0^\infty \lambda^{p-1} \mu(E_\lambda)^{p/q} \, d\lambda\right)^{1/p}. \tag{3.1}$$

Proof. We have

$$\left(\int_X (|u|^p)^{q/p} \,\mathrm{d}\mu\right)^{p/q} = \sup_{\|f\|_s \le 1} \int_X |u|^p f \,\mathrm{d}\mu,$$

where $s = q/(q-p)$ is the Hölder conjugate of q/p. By $\|f\|_s$ we denote the $L^s(\mu)$–norm of f. Define a measure $\widetilde{\mu}$ as

$$\widetilde{\mu}(A) = \int_A |f| \,\mathrm{d}\mu$$

for every μ–measurable set $A \subset X$. If $\mu(E) > 0$ and $\|f\|_s \le 1$, we conclude that

$$\begin{aligned}\widetilde{\mu}(E) &= \int_E |f| \,\mathrm{d}\mu \le \mu(E)^{1-1/s} \left(\int_E |f|^s \,\mathrm{d}\mu \right)^{1/s} \\ &\le \mu(E)^{1-1/s} = \mu(E)^{p/q}.\end{aligned}$$

Here we used the Hölder inequality. Hence

$$\begin{aligned}\int_X |u|^p f \,\mathrm{d}\mu \le \int_X |u|^p \,\mathrm{d}\widetilde{\mu} &= p \int_0^\infty \lambda^{p-1} \widetilde{\mu}(E_\lambda) \,\mathrm{d}\lambda \\ &\le p \int_0^\infty \lambda^{p-1} \mu(E_\lambda)^{p/q} \,\mathrm{d}\lambda.\end{aligned}$$

Taking supremum over all functions f with $\|f\|_s \le 1$ completes the proof. □

Next we prove a strong type inequality for the capacity. When $p = 1$ the obtained estimate reduces to the co-area formula. The proof is based on a general truncation argument, see page 110 in [12]. A similar argument has been used for example in [3], [15] and [8].

Lemma 3.3. *Let $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$ and $1 \le p < \infty$. Then*

$$\int_0^\infty \lambda^{p-1} \operatorname{cap}_p(E_\lambda, \Omega)\, d\lambda \le 2^{2p-1} \int_\Omega g_u^p \, d\mu,$$

where g_u is a p-weak upper gradient of u.

Proof. A straightforward calculation shows that

$$\begin{aligned}&\int_0^\infty \lambda^{p-1} \operatorname{cap}_p(E_\lambda, \Omega) \,\mathrm{d}\lambda \\ &= \sum_{j=-\infty}^{\infty} \int_{2^{j-1}}^{2^j} \lambda^{p-1} \operatorname{cap}_p(E_\lambda, \Omega) \,\mathrm{d}\lambda \\ &\le \sum_{j=-\infty}^{\infty} (2^j - 2^{j-1}) 2^{j(p-1)} \operatorname{cap}_p(E_{2^{j-1}}, \Omega) \\ &= \frac{1}{2} \sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_p(E_{2^{j-1}}, \Omega) \\ &= 2^{p-1} \sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_p(E_{2^j}, \Omega).\end{aligned}$$

Let

$$u_j = \begin{cases} 1, & \text{if} \quad u \ge 2^j, \\ 2^{1-j}|u| - 1, & \text{if} \quad 2^{j-1} < u < 2^j, \\ 0, & \text{if} \quad u \le 2^{j-1}. \end{cases}$$

Then $u_j \in \mathcal{A}(E_{2^j}, \Omega)$. This implies that

$$\operatorname{cap}_p(E_{2^j}, \Omega) \le 2^{p(1-j)} \int_{E_{2^{j-1}} \setminus E_{2^j}} g_u^p \, \mathrm{d}\mu$$

and consequently

$$\begin{aligned} \sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_p(E_{2^j}, \Omega) &\le \sum_{j=-\infty}^{\infty} 2^{jp+p(1-j)} \int_{E_{2^{j-1}} \setminus E_{2^j}} g_u^p \, \mathrm{d}\mu \\ &\le 2^p \int_\Omega g_u^p \, \mathrm{d}\mu. \end{aligned}$$

The claim follows from this. □

The following result gives a necessary and sufficient condition for a Sobolev inequality in terms of an isocapacitary inequality. This is a metric space version of a corollary on page 113 of [12].

Remark 3.4. We do not need the doubling condition in this theorem.

Theorem 3.5. *Suppose that* $1 \le p \le q < \infty$.

(i) *If there is a constant* γ *such that*

$$\nu(E)^{p/q} \le \gamma \operatorname{cap}_p(E, \Omega) \tag{3.2}$$

for every $E \subset \Omega$, *then*

$$\left(\int_\Omega |u|^q \, d\nu \right)^{1/q} \le c_S \left(\int_\Omega g_u^p \, d\mu \right)^{1/p} \tag{3.3}$$

for every $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$ *with* c_S *depending only on* γ *and* p.

(ii) *If* (3.3) *holds for every* $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$ *and if the constant* c_S *is independent of* u, *then* (3.2) *holds for every* $E \subset \Omega$ *with* $\gamma = c_S$.

Proof. (i) By Lemma 3.2, (3.2) and Lemma 3.3, we obtain

$$\begin{aligned} \left(\int_\Omega |u|^q \, \mathrm{d}\nu \right)^{1/q} &\le \left(p \int_0^\infty \lambda^{p-1} \nu(E_\lambda)^{p/q} \, \mathrm{d}\lambda \right)^{1/p} \\ &\le \left(\gamma p \int_0^\infty \lambda^{p-1} \operatorname{cap}_p(E_\lambda, \Omega) \, \mathrm{d}\lambda \right)^{1/p} \\ &\le \left(\gamma p 2^{2p-1} \right)^{1/p} \left(\int_\Omega g_u^p \, \mathrm{d}\mu \right)^{1/p}. \end{aligned}$$

(ii) If $u \in \mathcal{A}(E, \Omega)$ is continuous, then by (3.3), we have

$$\nu(E)^{1/q} \le \left(\int_\Omega |u|^q \, \mathrm{d}\nu \right)^{1/q} \le c_S \left(\int_\Omega g_u^p \, \mathrm{d}\mu \right)^{1/p}.$$

The claim follows by taking the infimum on the right-hand side. □

Remark 3.6. The previous theorem gives a necessary and sufficient condition for the Hardy inequality

$$\int_\Omega \left(\frac{|u(x)|}{\operatorname{dist}(x, X \setminus \Omega)} \right)^p \mathrm{d}\mu \le c_H \int_\Omega g_u^p \,\mathrm{d}\mu, \tag{3.4}$$

where the constant c_H is independent of $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$. Indeed, (3.4) holds if and only if

$$\int_E \frac{1}{\operatorname{dist}(x, X \setminus \Omega)^p} \,\mathrm{d}\mu \le \gamma \operatorname{cap}_p(E, \Omega)$$

for every $E \subset \Omega$. Thus Theorem 3.5 is a generalization of Theorem 4.1 in [10]. In the metric space context, the Hardy inequality has also been studied in [4].

3.2. **The case $p = 1$.** When $p = 1$ and $\Omega = X$, the isocapacitary inequalities reduce to isoperimetric inequalities. Moreover, in this case we can improve Theorem 3.5 under the additional assumptions that the measure is doubling and the space supports a Poincaré inequality. Indeed, it is enough that condition (3.2) is satisfied for all balls. To prove that, we will need equivalence of the capacity of order one and the Hausdorff content of co-dimension one

$$\mathcal{H}_\infty^h(K) = \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \ : \ K \subset \bigcup_{i=1}^{\infty} B(x_i, r_i) \right\}.$$

Theorem 3.7. *Let X be a complete metric space with a doubling measure μ. Suppose that X supports a weak $(1,1)$–Poincaré inequality. Let K be a compact subset of X. Then*

$$\frac{1}{c} \operatorname{cap}_1(K) \le \mathcal{H}_\infty^h(K) \le c \operatorname{cap}_1(K),$$

where c depends only on the doubling constant and the constants in the weak $(1,1)$–Poincaré inequality.

The proof is based on co-area formula and a metric space version of so–called boxing inequality. For more details, see [9]. A similar result has been studied in Mäkäläinen [14].

Theorem 3.8. *Let X be a complete metric space with a doubling measure μ. Suppose that X supports a weak $(1,1)$–Poincaré inequality. Suppose that $1 \le q < \infty$. If there is a constant γ such that*

$$\nu(B)^{1/q} \le \gamma \operatorname{cap}_1(B, X) \tag{3.5}$$

for every ball $B \subset X$, then

$$\left(\int_X |u|^q \, d\nu \right)^{1/q} \le c_S \int_X g_u \, d\mu, \tag{3.6}$$

where c_S is independent of $u \in N^{1,1}(X) \cap C_0(X)$.

Proof. First we prove that if the space satisfies (3.5) for all balls in X, then it satisfies the same condition for all compact sets with a different constant.

Let $K \subset X$ be compact, $\varepsilon > 0$ and $\{B(x_i, r_i)\}_{i=1}^{\infty}$ be a covering of K such that

$$\mathcal{H}^h_\infty(K) \geq \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} - \varepsilon.$$

Since $q \geq 1$, we have

$$\nu(K)^{1/q} \leq \sum_{i=1}^{\infty} \nu(B(x_i, r_i))^{1/q}.$$

Because

$$u_i(x) = (1 - \operatorname{dist}(x, B(x_i, r_i))/r_i)_+$$

belongs to $\mathcal{A}(B(x_i, r_i), X)$, and $g_i = \chi_{B(x_i, 2r_i)}/r_i$ is an upper gradient of u_i, we have

$$\operatorname{cap}_1(B(x_i, r_i)) \leq \int_X g_i \, \mathrm{d}\mu \leq c_D \frac{\mu(B(x_i, r_i))}{r_i}.$$

By combining the above estimates and (3.5), we conclude

$$\begin{aligned} \nu(K)^{1/q} &\leq \sum_{i=1}^{\infty} \nu(B(x_i, r_i))^{1/q} \\ &\leq \gamma \sum_{i=1}^{\infty} \operatorname{cap}_1(B(x_i, r_i)) \\ &\leq \gamma c_D \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} \\ &\leq \gamma c_D (\mathcal{H}^h_\infty(K) + \varepsilon). \end{aligned}$$

The claim follows by Theorem 3.7 as $\varepsilon \to 0$.

Now the theorem follows as in the proof of Theorem 3.5. Note that since u has compact support and is continuous, we can as well consider compact level sets $\{|u| \geq t\}$ instead of open sets. □

3.3. **The case** $1 \leq q < p < \infty$**.** In the case $1 \leq q < p < \infty$, the isocapacitary inequality takes a different form. Let E_j, $j = -N, -N+1, \dots, N, N+1$ be such that $E_j \subset \Omega$ and $E_j \subset E_{j+1}$ for $j = -N, -N+1, \dots, N$. We define

$$\gamma = \sup \left[\sum_{j=-N}^{N} \left(\frac{\nu(E_j)^{p/q}}{\operatorname{cap}_p(E_j, E_{j+1})} \right)^{q/(p-q)} \right]^{(p-q)/q}, \tag{3.7}$$

where the supremum is taken over all sequences of sets as above. The following result is a metric space version of a theorem on page 120 of [12].

Theorem 3.9. *Suppose that* $1 \le q < p < \infty$.

(i) *If* $\gamma < \infty$, *then*

$$\left(\int_\Omega |u|^q \, d\nu\right)^{1/q} \le c_S \left(\int_\Omega g_u^p \, d\mu\right)^{1/p}, \tag{3.8}$$

where c_S *is independent of* $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$.

(ii) *If* (3.8) *holds for every* $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$ *and if the constant* c_S *is independent of* u, *then* $\gamma < \infty$.

Proof. (i) We have

$$\begin{aligned}\int_\Omega |u|^q \, \mathrm{d}\nu &= \sum_{j=-\infty}^{\infty} q \int_{2^j}^{2^{j+1}} \lambda^{q-1} \nu(E_\lambda) \, \mathrm{d}\lambda \\ &\le q \sum_{j=-\infty}^{\infty} 2^{(j+1)(q-1)} 2^j \nu(E_{2^j}) \\ &= q 2^{q-1} \sum_{j=-\infty}^{\infty} 2^{jq} \nu(E_{2^j}).\end{aligned}$$

By the Hölder inequality,

$$\begin{aligned}&\sum_{j=-\infty}^{\infty} 2^{jq} \nu(E_{2^j}) \\ &\quad = \sum_{j=-\infty}^{\infty} \left(\frac{\nu(E_{2^j})^{p/q}}{\operatorname{cap}_p(E_{2^j}, E_{2^{j-1}})}\right)^{q/p} \left(2^{jp} \operatorname{cap}_p(E_{2^j}, E_{2^{j-1}})\right)^{q/p} \\ &\quad \le \left(\sum_{j=-\infty}^{\infty} \left(\frac{\nu(E_{2^j})^{p/q}}{\operatorname{cap}_p(E_{2^j}, E_{2^{j-1}})}\right)^{q/(p-q)}\right)^{(p-q)/p} \\ &\qquad \times \left(\sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_p(E_{2^j}, E_{2^{j-1}})\right)^{q/p}.\end{aligned}$$

Let

$$u_j = \begin{cases} 1, & \text{if } |u| > 2^j, \\ \dfrac{|u| - 2^{j-1}}{2^{j-1}}, & \text{if } 2^{j-1} < |u| \le 2^j, \\ 0, & \text{if } |u| \le 2^{j-1}. \end{cases}$$

Then

$$\operatorname{cap}_p(E_{2^j}, E_{2^{j-1}}) \le \int_\Omega g_{u_j}^p \, \mathrm{d}\mu \le 2^{-(j-1)p} \int_{E_{2^{j-1}} \setminus E_{2^j}} g_u^p \, \mathrm{d}\mu$$

It follows that

$$\begin{aligned}\sum_{j=-\infty}^{\infty} 2^{jp}\operatorname{cap}_p(E_{2^j},E_{2^{j-1}}) &\le 2^p \sum_{j=-\infty}^{\infty}\int_{E_{2^{j-1}}\setminus E_{2^j}} g_u^p\,\mathrm{d}\mu\\ &= 2^p\int_\Omega g_u^p\,\mathrm{d}\mu\end{aligned}$$

and consequently

$$\int_\Omega |u|^q\,\mathrm{d}\nu \le c\left(\int_\Omega g_u^p\,\mathrm{d}\mu\right)^{q/p}.$$

(ii) Let E_j be as as in the statement of the theorem, and define

$$\lambda_j=\sum_{i=j}^{N}\left(\frac{\nu(E_i)}{\operatorname{cap}_p(E_i,E_{i+1})}\right)^{1/(p-q)},\quad j=-N,-N+1,\dots,N,$$

and $\lambda_{N+1}=0$. Let $u_j\in\mathcal{A}(E_j,E_{j+1})$ be continuous, and define

$$u=\begin{cases}(\lambda_j-\lambda_{j+1})u_j+\lambda_{j+1} & \text{in}\quad E_{j+1}\setminus E_j,\\ \lambda_{-N} & \text{in}\quad E_{-N},\\ 0 & \text{in}\quad \Omega\setminus E_{N+1}.\end{cases}$$

Then $u\in N_0^{1,p}(\Omega)\cap C(\Omega)$. By the Cavalieri principle

$$\begin{aligned}\int_\Omega |u|^q\,\mathrm{d}\nu =&q\int_0^\infty \lambda^{q-1}\nu(E_\lambda)\,\mathrm{d}\lambda=\sum_{j=-N}^{N} q\int_{\lambda_{j+1}}^{\lambda_j}\lambda^{q-1}\nu(E_\lambda)\,\mathrm{d}\lambda\\ &\ge\sum_{j=-N}^{N}\nu(E_j)(\lambda_j^q-\lambda_{j+1}^q).\end{aligned}$$

From this we conclude that

$$\begin{aligned}\left(\sum_{j=-N}^{N}\nu(E_j)(\lambda_j-\lambda_{j+1})^q\right)^{p/q} &\le\left(\sum_{j=-N}^{N}\nu(E_j)(\lambda_j^q-\lambda_{j+1}^q)\right)^{p/q}\\ &\le\left(\int_\Omega|u|^q\,\mathrm{d}\nu\right)^{p/q}\le c_S\int_\Omega g_u^p\,\mathrm{d}\nu=c_S\sum_{j=-N}^{N}\int_{E_{j+1}\setminus E_j} g_u^p\,\mathrm{d}\mu\\ &\le c_S\sum_{j=-N}^{N}(\lambda_j-\lambda_{j+1})^p\int_{E_{j+1}\setminus E_j} g_{u_j}^p\,\mathrm{d}\mu.\end{aligned}$$

Taking the infimum on the right-hand side, we arrive at

$$\left(\sum_{j=-N}^{N}\nu(E_j)(\lambda_j-\lambda_{j+1})^q\right)^{p/q}\le c_S\sum_{j=-N}^{N}(\lambda_j-\lambda_{j+1})^p\operatorname{cap}_p(E_j,E_{j+1}).$$

Since

$$\lambda_j - \lambda_{j+1} = \left(\frac{\nu(E_j)}{\operatorname{cap}_p(E_j, E_{j+1})} \right)^{1/(p-q)},$$

we obtain

$$\left[\sum_{j=-N}^{N} \left(\frac{\nu(E_j)^{p/q}}{\operatorname{cap}_p(E_j, E_{j+1})} \right)^{q/(p-q)} \right]^{p/q} \le c_S \sum_{j=-N}^{N} \left(\frac{\nu(E_j)^{p/q}}{\operatorname{cap}_p(E_j, E_{j+1})} \right)^{q/(p-q)},$$

and the claim follows. □

Next we present an integral version of Theorem 3.9. See also page 30 in [13] for the Euclidean case.

Theorem 3.10. *Let $q < p$ and $\mu(\Omega) < \infty$ and*

$$\lambda_p(s) = \inf\{\operatorname{cap}_p(G) : G \subset \Omega \text{ and } \nu(G) \ge s\}.$$

Then

$$\int_0^{\nu(\Omega)} \left(\frac{t^{p/q}}{\lambda_p(t)} \right)^{q/(p-q)} \frac{dt}{t} \le c_I < \infty \tag{3.9}$$

if and only if the Sobolev inequality (3.8) *holds for every $u \in N_0^{1,p}(\Omega) \cap C(\Omega)$ with a constant c_S that is independent of u.*

Proof. First, assume that (3.9) holds. Let $s = p/q$ and $s' = p/(p-q)$ be the Hölder conjugate of s. Then

$$\begin{aligned}
\int_\Omega u^q \,\mathrm{d}\nu &\le 2^q \sum_{j=-\infty}^{\infty} 2^{jq} \nu(\{2^j < u < 2^{j+1}\}) \\
&\le 2^q \left(\sum_{j=-\infty}^{\infty} 2^{jp} \operatorname{cap}_p(\{u > 2^j\}) \right)^{q/p} \\
&\quad \times \left(\sum_{j=-\infty}^{\infty} \frac{(\nu(\{u > 2^j\}) - \nu(\{u > 2^{j+1}\}))^{s'}}{\operatorname{cap}_p(\{u > 2^j\})^{s'/s}} \right)^{1/s'} \\
&\le c \left(\int_\Omega g_u^p \,\mathrm{d}\mu \right)^{q/p} \left(\int_0^{\nu(\Omega)} \frac{t^{s'-1}}{\lambda_p(t)^{s'/s}} \,\mathrm{d}t \right)^{1/s'} \\
&\le c_I^{1/s'} c \left(\int_\Omega g_u^p \,\mathrm{d}\mu \right)^{q/p}.
\end{aligned}$$

Here we used the Hölder inequality, monotonicity of $\lambda_p(t)$ and the fact that

$$\operatorname{cap}_p(\{u > 2^j\}) \le 2^{-jp+p} \int_{\{2^{j-1}<u<2^j\}} g_u^p \, \mathrm{d}\mu.$$

Assume now that (3.8) holds. For every $j \in \mathbb{Z}$, let $u_j \in N_0^{1,p}(\Omega) \cap C(\Omega)$ be a function such that $0 \le u_j \le 1$, $\nu(\{u_j = 1\}) \ge 2^j$ and

$$\int_\Omega g_{u_j}^p \, \mathrm{d}\mu \le \lambda_p(2^j) + \varepsilon_j,$$

with $0 \le \varepsilon_j \le \lambda_p(2^j)$. Let

$$u = \sup_j \beta_j u_j,$$

where

$$\beta_j = \left(\frac{2^j}{\lambda_p(2^j)} \right)^{1/(p-q)}.$$

Now

$$\int_\Omega u^q \, \mathrm{d}\nu \ge \frac{1}{2} \sum_{j=-\infty}^{\infty} \beta_j^q 2^j \tag{3.10}$$

and

$$\int_\Omega g_u^p \, \mathrm{d}\mu \le \left(\sum_j \beta_j^p (\lambda_p(2^j) + \varepsilon_j) \right) \le 2 \sum_j \beta_j^p \lambda_p(2^j). \tag{3.11}$$

As

$$\beta_j^q 2^j = \beta_j^p \lambda_p(2^j),$$

it follows by (3.8), (3.10) and (3.11) that

$$\sum_j \frac{(2^j)^{p/(p-q)}}{\lambda_p(2^j)^{q/(p-q)}} = \sum_j \beta_j^q 2^j \le c,$$

and

$$\int_0^{\nu(\Omega)} \left(\frac{t^{p/q}}{\lambda_p(t)} \right)^{q/(p-q)} \frac{\mathrm{d}t}{t} \le c.$$

by monotonicity of λ_p. □

4. Functions with general boundary values

In this section, we obtain necessary and sufficient conditions for Sobolev–Poincaré inequalities of type

$$\inf_{a \in \mathbb{R}} \left(\int_\Omega |u-a|^q \, \mathrm{d}\nu \right)^{1/q} \le c \left(\int_\Omega g_u^p \, \mathrm{d}\mu \right)^{1/p}$$

where $u \in N^{1,p}(\Omega) \cap C(\Omega)$ and $1 \le p \le q < \infty$. To this end, we shall need the concept of conductivity, see Chapter 4 in [12].

Let $\Omega \subset X$ be a bounded open set. Let F be a closed subset of Ω and let G be an open subset of Ω such that $F \subset G$. The open set $C = G \setminus F$ is called a *conductor* and

$$\mathcal{B}(F, G, \Omega) = \{u \in N^{1,p}(\Omega) \cap C(\Omega) \,:\, u \geq 1 \text{ in } F \text{ and } u \leq 0 \text{ in } \Omega \setminus G\}$$

is the set of admissible functions. The number

$$\operatorname{con}_p(F, G, \Omega) = \inf_{u \in \mathcal{B}(F,G,\Omega)} \int_\Omega g_u^p \,\mathrm{d}\mu$$

is called the *p–conductivity* of C.

The next result can be proved in the same way as Lemma 3.3.

Lemma 4.1. *Let $G \subset \Omega$ be open and $1 \leq p < \infty$. Suppose that $u \in N^{1,p}(\Omega) \cap C(\Omega)$ such that $u = 0$ in $\Omega \setminus G$. Then*

$$\int_0^\infty \lambda^{p-1} \operatorname{con}_p(E_\lambda, G, \Omega)\, d\lambda \leq 2^{2p-1} \int_\Omega g_u^p \, d\mu.$$

The following result is a metric space version of a theorem on page 210 of [12].

Theorem 4.2. *Suppose that $1 \leq p \leq q < \infty$ and let $G \subset \Omega$ be open.*

(i) *If there is a constant γ such that*

$$\nu(F)^{p/q} \leq \gamma \operatorname{con}_p(F, G, \Omega) \tag{4.1}$$

for every $F \subset G$, then

$$\left(\int_\Omega |u|^q \, d\nu\right)^{1/q} \leq c \left(\int_\Omega g_u^p \, d\mu\right)^{1/p} \tag{4.2}$$

for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$ such that $u = 0$ in $\Omega \setminus G$. Here the constant c is independent of u.

(ii) *If (4.2) holds for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$ such that $u = 0$ in $\Omega \setminus G$, then (4.1) holds for every $F \subset G$ with $\gamma = c$.*

Proof. (i) We conclude by Lemma 3.2, (4.1) and Lemma 4.1 that

$$\begin{aligned}
\left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{1/q} &\leq \left(p \int_0^\infty \lambda^{p-1} \nu(E_\lambda)^{p/q} \,\mathrm{d}\lambda\right)^{1/p} \\
&\leq \left(p\gamma \int_0^\infty \lambda^{p-1} \operatorname{con}_p(E_\lambda, G, \Omega) \,\mathrm{d}\lambda\right)^{1/p} \\
&\leq \left(p\gamma 2^{2p-1} \int_\Omega g_u^p \,\mathrm{d}\mu\right)^{1/p}.
\end{aligned}$$

(ii) If $u \in \mathcal{B}(F, G, \Omega)$, then inequality (4.2) implies that

$$\nu(F)^{1/q} \leq \left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{1/q} \leq c \left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{1/p}.$$

The claim follows by taking the infimum on the right-hand side. □

Theorem 4.3. *Suppose that $1 \le p \le q < \infty$ and that $\Omega \subset X$ is a bounded open set.*

(i) *If there is a constant γ such that*
$$\nu(F)^{p/q} \le \gamma \operatorname{con}_p(F, G, \Omega) \tag{4.3}$$
for every conductor $G \setminus F$ with $\nu(G) \le \nu(\Omega)/2$, then
$$\inf_{a\in\mathbb{R}} \left(\int_\Omega |u-a|^q \, d\nu \right)^{1/q} \le c \left(\int_\Omega g_u^p \, d\mu \right)^{1/p} \tag{4.4}$$
for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$.

(ii) *If (4.4) holds for every $u \in N^{1,p}(\Omega) \cap C(\Omega)$, then (4.3) holds for every conductor $G \setminus F$ with $\nu(G) \le \nu(\Omega)/2$.*

Proof. (i) Let $\alpha \in \mathbb{R}$ be such that
$$\nu(\{x \in \Omega : u(x) \ge \alpha\}) \ge \frac{1}{2}\nu(\Omega)$$
and
$$\nu(\{x \in \Omega : u(x) > \alpha\}) \le \frac{1}{2}\nu(\Omega).$$
Now $(u-\alpha)_+ \in N^{1,p}(\Omega) \cap C(\Omega)$ and $u = 0$ in $\Omega \setminus G$, where
$$G = \{x \in \Omega : u(x) > \alpha\}.$$
Clearly $\nu(G) \le \frac{1}{2}\nu(\Omega)$ and by (4.3),
$$\nu(F)^{p/q} \le \gamma \operatorname{con}_p(F, G, \Omega)$$
for every $F \subset G$. By Theorem 4.2, we have
$$\left(\int_\Omega (u-\alpha)_+^q \, \mathrm{d}\nu \right)^{1/q} \le c \left(\int_{\{x\in\Omega : u(x)>\alpha\}} g_u^p \, \mathrm{d}\mu \right)^{1/p}.$$
Similarly,
$$\left(\int_\Omega (\alpha-u)_+^q \, \mathrm{d}\nu \right)^{1/q} \le c \left(\int_{\{x\in\Omega : u(x)<\alpha\}} g_u^p \, \mathrm{d}\mu \right)^{1/p}.$$
A combination of these estimates implies that
$$\begin{aligned}
&\left(\int_\Omega |u-\alpha|^q \, \mathrm{d}\mu \right)^{1/q} \\
&\quad \le \left(\int_\Omega (\alpha-u)_+^q \, \mathrm{d}\mu \right)^{1/q} + \left(\int_\Omega (u-\alpha)_+^q \, \mathrm{d}\mu \right)^{1/q} \\
&\quad \le c \left(\int_{\{x\in\Omega : u(x)>\alpha\}} g_u^p \, \mathrm{d}\mu \right)^{1/p} + c \left(\int_{\{x\in\Omega : u(x)<\alpha\}} g_u^p \, \mathrm{d}\mu \right)^{1/p} \\
&\quad \le c \left(\int_\Omega g_u^p \, \mathrm{d}\mu \right)^{1/p}.
\end{aligned}$$

(ii) Let $G \setminus F$ be a conductor with $\nu(G) \le \frac{1}{2}\nu(\Omega)$ and suppose that $u \in N^{1,p}(\Omega) \cap C(\Omega)$ such that $u = 0$ in $\Omega \setminus G$ and $u = 1$ on F. Since

$$\begin{aligned}\left(\int_\Omega |u-u_\Omega|^q \,\mathrm{d}\nu\right)^{1/q} &\\ &\le \left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q} + |a-u_\Omega|\nu(\Omega)^{1/q}\\ &\le \left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q} + \nu(\Omega)^{1/q} \fint_\Omega |u-a| \,\mathrm{d}\nu\\ &\le \left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q} + \nu(\Omega)^{1/q} \left(\fint_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q}\\ &= 2\left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q},\end{aligned}$$

we have

$$\begin{aligned}\inf_{a\in\mathbb{R}}\left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q} &\le \left(\int_\Omega |u-u_\Omega|^q \,\mathrm{d}\nu\right)^{1/q}\\ &\le 2\inf_{a\in\mathbb{R}}\left(\int_\Omega |u-a|^q \,\mathrm{d}\nu\right)^{1/q}.\end{aligned}$$

Now

$$\begin{aligned}c\left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{q/p} &\ge \int_\Omega |u-u_\Omega|^q \,\mathrm{d}\nu\\ &= \int_G |u-u_\Omega|^q \,\mathrm{d}\nu + |u_\Omega|^q \nu(\Omega\setminus G).\end{aligned}$$

This and the fact that $\nu(\Omega \setminus G) \ge \frac{1}{2}\nu(\Omega)$ imply that

$$|u_\Omega|^q \nu(\Omega) \le c\left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{q/p}.$$

Since

$$\begin{aligned}\left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{1/q} &\le |u_\Omega|\nu(\Omega)^{1/q} + \left(\int_\Omega |u-u_\Omega|^q \,\mathrm{d}\nu\right)^{1/q}\\ &\le c\left(\int_\Omega g_u^q \,\mathrm{d}\mu\right)^{1/p},\end{aligned}$$

we have

$$\nu(F)^{1/q} \le \left(\int_\Omega |u|^q \,\mathrm{d}\nu\right)^{1/q} \le c\left(\int_\Omega g_u^p \,\mathrm{d}\mu\right)^{1/p}$$

and by taking an infimum over all functions u, we have

$$\nu(F)^{p/q} \le c\,\mathrm{con}_p(F,G,\Omega).$$

□

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