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http://link.aps.org/abstract/pra/v75/p012302
Optimal control of coupled Josephson qubits

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1. INTRODUCTION

Aiming at Hamiltonian simulation and quantum computation recent years have seen an increasing array of quantum systems that can be coherently controlled. Next to natural microscopic quantum systems, a particularly attractive candidate for scalable setups are superconducting devices based on Josephson junctions [1–3]. Due to the ubiquitous bath degrees of freedom in the solid-state environment, the quantum coherence time remains limited, even in light of recent progress [4,5] approaching theoretical bounds. Thus it is a challenge to generate the gates fast and accurately enough to meet the error correction threshold. This poses fundamental questions, such as (i) to which extent are gate accuracies and speeds limited by the presence of nearby higher levels? (ii) does a constant and relatively strong interaction promote or hinder the gate performance and which parameter is limiting the gate time? and (iii) given the challenge in building control electronics, which properties do pulses for quantum gates in these pseudospin systems have to obey?

Recently, progress has been made in applying optimal control techniques to steer quantum systems [6] in a robust, relaxation-minimizing [7] or time-optimal way [8,9]. Spin systems are a particularly powerful paradigm of quantum systems [10]. N spins-1/2 are fully controllable, if (i) all spins can be addressed selectively by rf pulses and (ii) if the spins form an arbitrary connected graph of weak (Ising-type) coupling interactions. The optimal control techniques of spin systems can be extended to pseudospin systems, such as charge or flux states in superconducting setups, provided their Hamiltonian dynamics can be expressed to sufficient accuracy within a closed Lie algebra, e.g., sur(2n) in a system of N qubits.

II. CONTROLLING THE HAMILTONIAN DYNAMICS OF COUPLED CHARGE QUBITS

As a practically relevant and illustrative example, we consider two capacitively coupled charge qubits controlled by dc pulses as in Ref. [1]. The infinite-dimensional Hilbert space of charge states in the device can be mapped to its low-energy part defined by zero or one excess charge on the respective islands [2]. Identifying these charges as pseudospins, the Hamiltonian can be written as $H_{\text{qubit}} = H_{\text{drift}} + H_{\text{coupl}}$, where the drift or static part reads (for constants see caption to Fig. 1)

$$H_{\text{drift}} = -\frac{E_{c}}{2} [\sigma_{x}^{1} \otimes \sigma_{x}^{2}] - \frac{E_{c}}{2} [\sigma_{z}^{1} \otimes \sigma_{z}^{2}] + \frac{E_{c}}{4} [\sigma_{z}^{1} \otimes \sigma_{z}^{2}].$$

while the controls can be cast into

$$H_{\text{coupl}} = -\frac{E_{\text{dc}}}{2} [\sigma_{x}^{1} \otimes \sigma_{x}^{2}] - \frac{E_{\text{dc}}}{2} [\sigma_{z}^{1} \otimes \sigma_{z}^{2}] + \frac{E_{\text{dc}}}{4} [\sigma_{z}^{1} \otimes \sigma_{z}^{2}],$$

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PACS number(s): 03.67.Lx, 85.25.Cp, 82.56.Ja, 85.35.Gv

DOI: 10.1103/PhysRevA.75.012302
Note that the Pauli matrices involved constitute a minimal generating set of the Lie algebra su(4); hence the system is fully controllable. The control amplitudes $n_{ij}$, $i = 1, 2$ are gate charges controlled by external voltages via $n_{ij} = V_i C_j e^{i \theta}$. They are taken to be piecewise constant in each time interval $t_k$. This pseudospin Hamiltonian motivated by Ref. [1] also applies to other systems such as double quantum dots [11] and Josephson flux qubits [12], although in the latter case the controls are typically of pulses.

In a time interval $t_k$ the system thus evolves under $H_{\text{tot}}^{(t)} = H_{\text{Hamiltonian}}^{(t)} + H_{\text{control}}$. The task is to find a sequence of control amplitudes for the intervals $t_1, t_2, \ldots, t_{k-1}$ such as to maximize the overlap with the desired output state at $t_k$. This can be achieved by optimal-control based gradient flows as described in Refs. [13, 14].

Throughout the work, we take the parameters from the experiment [1]. Figure 1 shows the fastest decompositions obtained by numerical optimal control for the CNOT gate into evolutions controlled by the desingularized control fields $E_{\text{control}} = E_{\text{control}}^{(t)}$. The total time $T = 255$ ps is taken to be the pertinent overall decay time. Assuming independent errors, the quality factor is $Q = \text{Fe}^{2} \text{TC}$, where the error rate $1 - q = 0.001$ is an estimate for the error-correction threshold. With the pulses presented here, the total error rate amounts to $1 - q = 0.0055$, instead of $1 - q = 0.5917$ in the pioneering setting [1].
III. PULSE SHAPING HARDWARE

In the pertinent time scale, commercial devices for generating arbitrary wave forms are not available. Yet high-end pulse generators [20,21] or ultrafast classical Josephson electronics [19,22] are close to the necessary specifications.

A. Overview

As a proof of principle, it is important to note on a general scale how to generate these pulses experimentally, which can readily be exemplified using the well-established technique of pulse shaping in Laplace space [23]. One starts with an input current pulse \( I_s(t) \) shorter than the desired one. Its shape may be arbitrary as long as it contains enough spectral weight at the harmonics necessary for the desired pulse. Such pulses can be generated optically or electrically [21]. They serve as input to a discrete electrical two-terminal element with transfer function \( Z_{21} \) to be designed for the desired output shape. In Laplace space, the output signal takes the form \( V_{21}(s) = Z_{21}(s)I_s(s) \). So the gate voltages \( V_g(t) \) (as in Fig. 1) are Laplace transformed to \( V_g(s) \) in order to determine the transfer function \( Z_{21}(s) \) by fitting \( V_g(s) \) to \( V_{21}(s) \) given the input pulse \( I_s(s) \). However, here in the special case of palindromic pulse shapes expressed by a cosine Fourier series [see Eq. (3)], \( Z_{21} \) is already directly given by a series of Lorentzians, viz. the Laplace transform of the cosine series. This results in the particularly simple circuit networks of 20 reactive LC filters shown in Fig. 5. They match the desired pulse shapes extremely well (see Fig. 1) giving a trace fidelity of \( 1-10^{-6} \) for the entire CNOT. However, compensating for a frequency-dependent transfer function from the generator to the sample, which has to be precisely determined for the respective experimental setting, requires the above more general approach.

B. General approach

Apart from giving details, this section generalizes the pulse shaping schemes outlined above. The network of filters can be obtained by Laplace transforming the gate voltages \( V_g(t) \) obtained from optimal control theory to \( V_g(s) \). Then the transfer function \( Z_{21}(s) \) is determined by fitting a \( V_{21}(s) \) to \( V_g(s) \) in Laplace space, where \( V_{21}(s) = Z_{21}(s)I_s(s) \).

Now criteria of realizability of standard network synthesis [23,24] apply. A standard two-terminal element is created by input and output impedances \( Z_1 \) and \( Z_2 \) as well as transfer functions \( Z_{12} \) and \( Z_{21} \). A pulse shape can be realized by a passive LCR two-terminal element (where, henceforth, where 20 components (listed in Table I) suffice to give high accuracy. Alternatively, the pulse shapes can be generated by superimposing short Gaussian, single-flux quantum (SFQ) pulses [18] or rapid SFQ pulses [19], where the coupling strength determines the minimal pulse length.

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L.C.R. denote inductivity, capacity and resistance, respectively, if in Laplace space (with $s = \sigma + \alpha$) the pertinent transfer function $T(s)$ is of the form

$$T(s) = \frac{\alpha}{s + \beta},$$

where $\alpha$ is a real-valued rational function, $2\beta$ is a Hurwitz polynomial (42), (3) the degree of $p(s)$ does not exceed the degree of $q(s), and (4) |\beta| = 1$.

Conditions (2) and (3) exclude that the $H(s)$ has poles in the right half-plane. Note that many important time-domain functions, such as, e.g., the trigonometric functions, the Heaviside function, the Dirac delta function, and the exponential function are expressed as rational functions in Laplace space. Thus a wide range of pulse shapes in time domain is accessible by circuit synthesis. Importantly, the transmission functions of typical coaxial cables used to interconnect the different parts of the experimental setup give rise to damping and thus introduce dissipative (i.e., resistive) elements in the circuit, the main physical limitation being that the bandwidth of the output cannot be greatly enhanced relative to the input. The maximum enhancement originates from a series inductor with $L = \alpha$. Thus the spectral content required at the output must be contained in the input as this scheme is essentially subtractive synthesis.

With the corresponding decomposition, there is a number of ways for designing a lumped circuit for a given transfer function, e.g., the method of Gewertz [25] that systematically eliminates poles and introduces loops in the electrical circuit. Iteratively the circuit is synthesized from basic building blocks: One LCR loop for each pair of complex conjugate poles, and one RC filter for each pole on the real axis. Note that in Laplace space the degree of both the numerator and the denominator polynomial of the transfer function $H(s)$ approaches the same limiting value for large values of $s$.

1. Cauer synthesis for controls with time reversal symmetry

In the special case of a real symmetric Hamiltonian allowing for a palindromic pulse sequence, the transfer function $Z(s)$ is directly obtained by the Laplace transform of the cosine series representation of the control pulses thus circuit elements in the circuit, the main physical limitation being that the bandwidth of the output cannot be greatly enhanced relative to the input. The maximum enhancement originates from a series inductor with $L = \alpha$. Thus the spectral content required at the output must be contained in the input as this scheme is essentially subtractive synthesis.

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desired. We take the Heaviside-type rectangular input pulse \( I = I_0 / H_20851 / H_9008 / H_20849 T / H_20850 \) in Table I.

**FIG. 5.** Circuit network for pulse shaping by Cauer synthesis via the Laplace transform of the cosine Fourier series of Eq. (3), so \( n_{22, \text{filter}} / H_20850 \). The method applies to all top-gate-style pulses of the form \( I = I_0 / H_20849 T / H_20850 \), which is the input current pulse (here 55 ps) and \( V_{\text{out}}(s) / H_20850 / H_20850 \) relates to gate voltages \( n_{22, \text{filter}} / H_20849 s / H_20850 / H_20850 \) on qubits \( s = 1 \). All the values \( a(s) / H_20849, L_s / H_20850 \) and \( C_s / H_20850 \) are tabulated in Table I. Desired values. We take the Heaviside-type rectangular input pulse \( I = I_0 / H_20851 / H_9008 / H_20849 T / H_20850 \) in Table I.

**FIG. 6.** Circuit network for pulse shaping by Cauer synthesis via the Laplace transform of the cosine Fourier series of Eq. (3), so \( n_{22, \text{filter}} / H_20850 \). The method applies to all top-gate-style pulses of the form \( I = I_0 / H_20849 T / H_20850 \), which is the input current pulse (here 55 ps) and \( V_{\text{out}}(s) / H_20850 / H_20850 \) relates to gate voltages \( n_{22, \text{filter}} / H_20849 s / H_20850 / H_20850 \) on qubits \( s = 1 \). All the values \( a(s) / H_20849, L_s / H_20850 \) and \( C_s / H_20850 \) are tabulated in Table I. Desired values. We take the Heaviside-type rectangular input pulse \( I = I_0 / H_20851 / H_9008 / H_20849 T / H_20850 \) in Table I.

**FIG. 7.** Circuit network for pulse shaping by Cauer synthesis via the Laplace transform of the cosine Fourier series of Eq. (3), so \( n_{22, \text{filter}} / H_20850 \). The method applies to all top-gate-style pulses of the form \( I = I_0 / H_20849 T / H_20850 \), which is the input current pulse (here 55 ps) and \( V_{\text{out}}(s) / H_20850 / H_20850 \) relates to gate voltages \( n_{22, \text{filter}} / H_20849 s / H_20850 / H_20850 \) on qubits \( s = 1 \). All the values \( a(s) / H_20849, L_s / H_20850 \) and \( C_s / H_20850 \) are tabulated in Table I.
leave the state space of the working qubits: At no time do the projections onto the leakage space exceed 0.6%. Clearly, optimizations including explicit leakage levels could improve the quality even further in systems where necessary [26].

As illustrated in Fig. 6, in simplified terms, the high quality can be understood by relating the limited bandwidth to the transitions between the eigenstates of the local parts of $H_{\text{system}}$ in Eq. (1). While one-charge transitions to leakage levels like $|1\rangle\rightarrow|0\rangle$ and $|2\rangle\rightarrow|1\rangle$ are allowed, two-charge transitions like $|1\rangle\rightarrow|1\rangle$ and $|2\rangle\rightarrow|0\rangle$ are forbidden in terms of the transition-matrix elements $\langle|l\rangle|H_{\text{system}}|l+1\rangle$. Note the charge control on gate 2 in Fig. 1 is around $\Delta t_{\text{prep}}=0.2$ thus driving the working transition $|0\rangle\rightarrow|1\rangle$, while the “spectral overlap” of the Fourier transform of the time course in both controls with energy differences corresponding to one-charge leakage transitions is small. Hence simple spectroscopic arguments underpin the high fidelity.

Moreover, our controls are notably robust with regard to ±5% variation of the tunneling frequencies $E_{\text{tun}}$, and the coupling term $E_{\text{c}}$ as well as to Gaussian noise on the control amplitudes and time intervals as shown in Fig. 7. Variations of the tunneling energies $E_{\text{tun}}$ ($\sigma=1.2$) may result from imperfections in the junction oxide as well as deviating coupling strength $E_{\text{c}}$. These parameters have to be determined spectroscopically, where the relative error normally does not exceed 5%. Even the time-optimized controls as short as $T=55$ ps cope with such variations. Significant improvement of the broadband behavior, however, could not be obtained by pulse sequences up to a total duration of $T=75$ ps, thus suggesting that broadband CNOT controls tailored for the special (and rare) instances with ill-defined experimental parameters will require considerably longer pulse schemes. Similar robustness is observed against Gaussian noise on the control amplitudes or time units.

V. TOFFOLI GATE FOR THREE LINEARLY COUPLED QUBITS

Likewise, in a system of three linearly coupled charge qubits, we determined a realization of the TOFFOLI gate with experimentally available controls (Fig. 8), where the speed-up against a circuit of nine CNOT gates is by a factor of 2.8 with our CNOT and by 13 with the CNOT gates of Ref. [1].

In a linear chain of three coupled qubits, a TOFFOLI gate needs nine CNOT gates, which gives an error rate of $1-\text{fidel}=1-0.4083^{a}=0.9997$ using the CNOT of Ref. [1].

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an error rate of $1 - q_{\text{cnot}} = 1 - 0.99459 = 0.0483$ with nine of our CNOT, while the error rate of the TOFFOLI gate is $1 - q_{\text{toff}} = 0.0178$, assum- ing for the moment that the $T_2$ in a coupled three-qubit Josephson system would also be in the order of 10 ns.

Due to the quite strong qubit-qubit interactions in multi-qubit setups, generating three-qubit gates directly is much faster than by universal gates. This also holds in simple algorithms [27] on superconducting qubit setups: A minimization algorithm for searching control amplitudes in coupled Cooper pair boxes was applied in [28], where the optimization was restricted to very few values. In Ref. [29], a rf-pulse sequence for a CNOT with fixed couplings was introduced, to which optimal control could be applied likewise: The sequence is longer using more of the available decoherence time, which is partly (but not fully) compensated by the longer $T_2$ at the optimum point. For the charge-qubit setting here, the control techniques lead to a time-optimized gate that can be performed some 200 times within a nonoptimum point $T_2$ of 10 ns.

VI. TOWARD THE ERROR-CORRECTION THRESHOLD: GUIDELINES AND FRONTIERS AHEAD

It is the main purpose of this section to make a strong case for the next generation of fast pulse shapers. Actually we regard them as paramount for reaching the goal of scalable quantum computation with superconducting Josephson elements. Let $F$ denote the fidelity of a gate of duration $T$, and let $T_2$ be the pertinent overall decay time. Assuming independent errors, the quality of a gate is roughly determined by $q = F = e^{-T_2/4T/2}$, where the error rate $1 - q = 10^{-5}$ is an estimate for the error-correction threshold (see, e.g., [30]).

This goal can be met by improvements on three frontiers: (1) Fighting decoherence by making $T_2$ longer, (2) cutting gate times by making $T$ shorter, and (3) improving fidelity by making $F$ larger, where this work shows how to exploit optimal control for getting to the limits in the latter two.

In fact the Josephson devices known today [1, 3, 4] have already undergone a great deal of hardware optimization bringing decoherence down close to its theoretical limits. The observed decoherence times in charge qubits are on the scale of $T_2 = 0.5$ to 2.5 ns for two-qubit dynamics [31, 32] and 10 ns for single qubits [32]. Both can be improved by using echo techniques [33], which hints at $1/\tau$ noise as the limiting factor. Other improvements of $T_2$ rely on operating with microwave pulses [3, 34] at an optimum bias point at the expense of much slower pulses limited by the Rabi fre- quency. Although our technique may incorporate both strat- egies, echo and microwave pulses, we base our technological estimate in the next section on an optimistic $T_2$ of 10 ns, which appears to be accessible in a charge qubit setup as in [1].

(2) The pulse controls currently available are too slow to fully exploit the potential of the experimental setting: Within a decay time of 10 ns, just 40 CNOT gates of the current duration of 255 ps can be performed (with the rise times in the order of 35 ps). On the other hand, the capacitively coupled Josephson hardware elements themselves have large intrinsic frequency scales allowing for fast operation and may well reach the decoherence-limited thresholds—provided gates could be executed some 100 times faster than in the current experimental setting, where we have shown that, within 10 ns, approximately 200 time-optimized high-fidelity CNOT can be run.

(3) For obtaining sufficiently high fidelities experimen- tally, an important part of the future challenge will be the accurate determination of the experimental system response: Once this can be done, a nonideal system response can easily be incorporated into our algorithms thus allowing for getting fidelities that are essentially limited by the robustness of the experimental setting. With fidelities of $F$ up to $10^{-5}$ being ideally accessible by our pulses, the total error rate is then entirely limited by decoherence ($1/T_2$).

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Our results make a strong case for faster pulse generation, both shorter in total length and with the possibility of shaping the external structure. This is a cornerstone for future progress and needs to be combined with the current strategies such as decoherence engineering and the optimal working point. In particular, even though the current experimental controls [1] could further be optimized for higher fidelity, a simple estimate shows that this will not suffice for significant improvements even if the same scales of pulse shaping technology: In the case of a CNOT, the quality would always be limited by $\epsilon^2 < 0.975$ even at fidelities of $T=1$. On the same note, if higher fidelity is achieved by additional compensation pulses [35–38], the total sequence becomes longer and the quality again deteriorates. The optimal working point strategy works excellently for single qubits [34] but becomes difficult for two-qubit operations, which also appear to be slow [29,39]. Rather, by making the Josephson hardware system even faster without introducing higher $T_1$ decay rates, high quality gates can be achieved by optimized fast control alone, even if the optimal point is not invoked.

Realistically, a combination of optimal control, optimal point, and refocusing may be most powerful and accessible. Clearly, this technological frontier has not been really explored so far, yet the time scales needed are not excessively short compared to what has been realized with electro-optical methods involving pulsed lasers and switches [40]. For getting sufficiently high fidelities experimentally, it will be crucial to accurately determine the experimental system response, which should then be included into the numerical algorithms.

ACKNOWLEDGMENTS

We thank N. Khaneja for stimulating scientific exchange. We gratefully acknowledge discussion on experimental issues with M. Mariantoni, as well as Y. Nakamura and the NEC group, J. M. Martinis, A. Ustinov, L.C.L. Hollenberg, T. Cubitt, and D. van der Weide. This work was supported by EPSRC in SPP-1707 and SFB 631, by the EU integrated project QAP, by the Finnish Cultural Foundation, by ARDA, and by NSA (ARO Grant No. F-43385-FH-QC).


We have shown how to take pulse controls for realizing quantum gates in pseudospin systems from fidelity-limited pioneering stages to the decoherence limit of near time optimal high-fidelity controls. In superconducting charge qubits, the progress towards the error-correction threshold is by a factor of 100. Limiting the optimal-control based shapes to low bandwidth allows for nonadiabatic pulses with remarkably low leakage to higher states thus justifying the two-level truncation to the low-energy part of the spectrum. Moreover, shapes could be kept simple enough to be realized by Cauer synthesis or a few LCR circuits. So the approach will find wide application, in particular for the next generation of fast pulse-shaping devices.

We expect $T_2$ times dominated by $1/\tau_0$ contributions will not change largely under the pulses, so time optimal controls provide a significant step towards the accuracy threshold for quantum computing, even if cutting decoherence times reaches its intrinsic limits.

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It was sufficient to run the GRAPE algorithm [13] on a MATLAB platform with 10 randomized initial control sequences for each fixed final time $T$ through 10,000 iterations taking some 250 sec CPU time on a 2.7 MHz 512 MB RAM Athlon processor to give trace fidelities $>1-10^{-9}$. The final pulse shape leading to a fidelity $>1-10^{-9}$ for the shortest $T$ was then obtained in an overnight run with no restriction on the number of iterations.

A polynomial (with real coefficients) of the variable $\sigma \rightarrow \sigma + i \omega$ is termed a Hurwitz polynomial if its zeros are strictly in the left half-plane [i.e., $\text{Re}(\omega)<0$], while a modified Hurwitz polynomial may also have zeros that are purely imaginary [i.e., $\text{Re}(\omega)=0$].