REDUCING BIAS IN BEAMSPACE METHODS FOR UNIFORM CIRCULAR ARRAY

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ABSTRACT

In this paper we characterize the error introduced by beamspace transform when it is applied to Uniform Circular Array (UCA). Several algorithms for Direction of Arrival (DoA) estimation employ this modal transform. In particular we focus on the UCA Unitary root-MUSIC algorithm. The performance of such estimator is degraded and bias occurs especially if the array has a small number of elements. Here we propose a novel technique for reducing the bias. This leads to practically bias-free DoA estimates.

1. INTRODUCTION

Circular arrays are of interest in a variety of applications, e.g. in multiantenna transceivers. Moreover, UCA's have uniform performance regardless of the angle of arrival and they can estimate both azimuth and elevation angles simultaneously. Several DoA estimators for UCA, such as MUSIC, root-MUSIC and ESPRIT [2]-[3] employ the beamspace transform in order to build a desired structure of the steering vectors (i.e. the Vandermonde structure) that is exploited in finding the DoA's. The beamspace transform is based on the phase-mode excitation principle [1]. However, the beamspace transform works properly only under certain conditions (depending on the number of array elements, radius and interelement spacing) that may be difficult to satisfy in some applications. For example, when a UCA with a small number of sensors (from 6 to 10 elements) is used, a residual error due to the beamspace transform leads to biased DoA estimates. Obviously, the Cramer-Rao Lower Bound (CRB) can not be achieved then.

A qualitative and quantitative analysis of the impact of the residual term on the signal and noise subspaces can be found in [4]. The analysis shows that the error may corrupt the DoA estimations by creating an error floor in the performance of algorithms such as UCA root-MUSIC [2]-[3].

A method for bias reduction was proposed in [5] where an optimal mapping from circular to linear manifold is employed. However, the method needs to split the azimuthal area of 360° in sectors of about 30° and then process each sector separately.

In this paper we propose a novel approach for removing the bias on the DoA estimates. This algorithm reduces the bias in three steps. No subdivision into sectors in the angular domain is required in the method as in [5]. Moreover, the proposed technique provides practically an error-free mapping between UCA and the ULA-like array (virtual array). This can be employed in techniques that exploit the Vandermonde structure in steering vectors, e.g., UCA root-MUSIC and spatial smoothing [2]-[3].

This paper is organized as follows. First, the UCA signal model is presented. In Section 3, the phase-mode excitation principle is described. In Section 4, we introduce both the Beamspace and the Generalized Beamspace Transforms. In Section 5, a novel technique for bias reduction is proposed. This method significatively improves the performance of algorithms employing rooting techniques for UCA. In Section 6 simulation results demonstrating the reduction in bias are shown. Finally, Section 7 concludes the paper.

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2. SIGNAL MODEL

Let us have a Uniform Circular Array of N sensors. There are P ($P < N$) uncorrelated narrow-band signal sources on the array plane, impinging the array from directions $\phi_1, \phi_2, \ldots, \phi_P$ (ϕ is the azimuth angle). Furthermore we assume that K snapshots are observed by the array. The $N \times K$ element-space array output matrix may be written:

$$\mathbf{X} = \mathbf{AS} + \mathbf{N}, \tag{1}$$

where $\mathbf{X}$ is the $N \times K$ element-space data matrix, $\mathbf{A}$ is the $N \times P$ element-space steering matrix, $\mathbf{S}$ is the $P \times K$ source matrix and $\mathbf{N}$ is the $N \times K$ noise matrix. The noise is modelled as a stationary, second-order ergodic, zero-mean spatially and temporally white circular complex Gaussian process.

The $N \times P$ element-space steering vector matrix may be written as $\mathbf{A} = [\mathbf{a}_1(\zeta,\phi), \mathbf{a}_2(\zeta,\phi), \ldots, \mathbf{a}_P(\zeta,\phi)]$ where each column is of the form

$$\mathbf{a}_p(\boldsymbol{\vartheta}) = [e^{j\zeta\cos(\phi_p-\gamma_0)}, e^{j\zeta\cos(\phi_p-\gamma_1)}, \ldots, e^{j\zeta\cos(\phi_p-\gamma_{(N-1)})}]^T \tag{2}$$

for $p = 1, 2, \ldots, P$. Here $\boldsymbol{\vartheta} = (\zeta, \phi)$ and $\zeta = \kappa r \sin\theta$, r is the radius, $\kappa = \frac{\omega}{c}$ is the wavenumber, c is the speed of light, $\omega = 2\pi f$ is the angular frequency and $\gamma_n = \frac{2\pi n}{N}$ ($n = 0, \ldots N-1$) is the sensor location. The elevation angle θ is measured down from the z-axis (assumed to be $\theta = 90^\circ$) and ϕ is the azimuth angle measured counterclockwise from the x-axis.

3. PHASE-MODE EXCITATION PRINCIPLE

The phase-mode excitation principle may be described using two different array configurations, the continuous and the discrete circular array (i.e. UCA) [1]-[2]. The principle forms the background for the Beamspace and Generalized Beamspace Transform.

The continuous array model can not be realized but it represents the ideal configuration for applying the principle and leads to an error-free scenario. By integrating the spatial harmonic of the array excitation $w_m(\gamma) = e^{jm\gamma}$ over the continuous circular array we can compute the normalized far-field pattern resulting from exciting the aperture with the m^{th} mode as [1]-[2]

$$f_m^c(\boldsymbol{\vartheta}) = \frac{1}{2\pi}\int_0^{2\pi} w_m(\gamma)e^{j\zeta\cos(\phi-\gamma)}\mathrm{d}\gamma = j^m J_m(\zeta)e^{jm\phi} \tag{3}$$

where $J_m(\zeta)$ is the Bessel function of the first kind of order m.

In case of discrete circular aperture (i.e. UCA) the normalized far-field pattern resulting from exciting the aperture with the m^{th} mode is

$$\begin{aligned} f_m^s(\boldsymbol{\vartheta}) =& j^m J_m(\zeta)e^{jm\phi} + \sum_{q=1}^{\infty}(j^g J_g(\zeta)e^{-jg\phi} + j^h J_h(\zeta)e^{jh\phi}) \\ =& j^m J_m(\zeta)e^{jm\phi} + \varepsilon_m \end{aligned} \tag{4}$$

where ε_m represents a sum on index $q = 1, 2, \ldots$ for defining the m^{th} excitation mode and the indices g and h are defined as $g = Nq - m$ and $h = Nq + m$, respectively.

Eq.(4) is composed of two terms. The first term is known as the *principal term*. The other quantity ε_m, called *residual term*, arises from sampling the continuous aperture by N sensors and represents an error. This higher-order distortion mode has to be minimized in order to get closer to the ideal (continuous) case [4].

4. BEAMSPACE TRANSFORMS

The beamspace transformation is done by employing a $\mathcal{M} \times N$ beamformer $\mathbf{F}_e^H$ (see [2] for details) as

$$\mathbf{a}_e(\boldsymbol{\vartheta}) = \mathbf{F}_e^H\mathbf{a}(\boldsymbol{\vartheta}) = \mathbf{C}_v\mathbf{V}^H\mathbf{a}(\boldsymbol{\vartheta}) \approx \sqrt{N}\mathbf{J}_\zeta\mathbf{d}(\phi) \tag{5}$$

where

$$\mathbf{C}_v = diag\Big\{j^{-M}, \ldots, j^{-1}, j^0, j^{-1}, \ldots, j^{-M}\Big\} \tag{6}$$

$$\mathbf{V} = \sqrt{N}\big[\mathbf{w}_{-M}\vdots\cdots\vdots\mathbf{w}_0\vdots\cdots\vdots\mathbf{w}_{-M}\big] \tag{7}$$

$$\mathbf{J}_\zeta = diag\Big\{J_M(\zeta), \ldots, J_0(\zeta), \ldots, J_M(\zeta)\Big\} \tag{8}$$

$$\mathbf{d}(\phi) = [e^{-jM\phi}, \ldots, e^{-j\phi}, 1, e^{j\phi}, \ldots, e^{jM\phi}]^T\,. \tag{9}$$

The modes that can be excited are $m \in [-M, M]$ and M is computed by considering the smallest integer that is close or equal to κr. Here we name $\mathcal{M} = 2M+1$ as the total number of excited modes. The matrices $\mathbf{C}_v$ and $\mathbf{J}_\zeta$ are $\mathcal{M} \times \mathcal{M}$ diagonal matrices, the vector $\mathbf{d}(\phi)$ has the size $\mathcal{M} \times 1$ and $diag\{\cdot\}$ and $(\cdot)^H$ denote a diagonal matrix and conjugate transposition, respectively. It is interesting to note that vector $\mathbf{d}(\phi)$ is of the form needed in Vandermonde matrix and it depends on the azimuth angle ϕ.

Equation (5) shows that the transformation is an approximation since the last equality holds only when certain conditions are fulfilled [2]-[4].

The Generalized Beamspace Transform (GBT) extends the original transform [2] so that also the residual error that arises by applying the beamspace transformation is taken into account. Mathematically it can be written (see [4] for details) as

$$\begin{aligned} \mathbf{a}_e(\boldsymbol{\vartheta}) =& \mathbf{F}_e^H\mathbf{a}(\boldsymbol{\vartheta}) \\ =& \mathbf{C}_v\mathbf{V}^H\mathbf{a}(\boldsymbol{\vartheta}) = \sqrt{N}\mathbf{J}_\zeta\mathbf{d}(\phi) + \sqrt{N}\mathbf{C}_v\boldsymbol{\varepsilon} \\ =& \sqrt{N}\mathbf{J}_\zeta\Big(\mathbf{d}(\phi) + \boldsymbol{\Delta}\mathbf{d}_1(\phi) + O(\boldsymbol{\Delta}\mathbf{d}_2(\phi))\Big) \end{aligned} \tag{10}$$

where $\boldsymbol{\Delta}\mathbf{d}_1(\phi) = \mathbf{J}_\zeta^{-1}\mathbf{C}_v\boldsymbol{\varepsilon}^{(1)}$ with $\boldsymbol{\varepsilon}^{(1)} = [\varepsilon_{-M}^{(1)}, \ldots, \varepsilon_M^{(1)}]^T$. Here $\boldsymbol{\varepsilon}^{(1)}$ is defined according to eq.(4) for $q = 1$ and $O(\boldsymbol{\Delta}\mathbf{d}_2(\phi))$ contains all the remaining terms of the sum $\boldsymbol{\varepsilon}^{(q)}$ for $q \in [2, +\infty]$.

Consequently, we can now decompose the UCA beamspace steering vector to a sum of three vectors: $\mathbf{d}(\phi)$ is the steering vector of the virtual array with Vandermonde structure, $\boldsymbol{\Delta}\mathbf{d}_1(\phi)$ is an additive perturbation with angular dependence on the nominal value of $\mathbf{d}(\phi)$ and $O(\boldsymbol{\Delta}\mathbf{d}_2(\phi))$ includes the remaining error which can be neglected without any loss of generality because the discussion can be extended.

As a result we can now consider the residual error of the beamspace transformation as a perturbation in the element positions of the true virtual array $\mathbf{d}(\phi)$. Therefore, the virtual array resulting from applying the beamspace transformation, can be considered to be an uncalibrated array with a misplacement of the element position proportional to the residual error $\boldsymbol{\Delta}\mathbf{d}(\phi) = \boldsymbol{\Delta}\mathbf{d}_1(\phi)$.

The residual error then introduces a bias into the DoA estimates regardless of the method we apply. Therefore, reducing the bias in the DoA estimates requires a method for reducing the residual error in advance or a procedure for calibrating the virtual array.

The bias in the DoA estimates is depicted in Fig.1. It is interesting to notice that the bias depends on the array configuration (either even or odd number of elements, array radius,...) and it also depends on the angle of arrival. For more details on the impact of the array configuration, see [4]-[6].

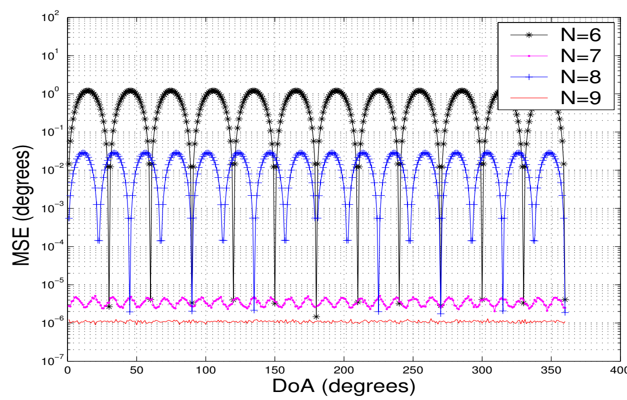

Fig. 1. MSE as function of the DoA's computed with UCA Unitary root-MUSIC when different number of sensors are used. The contribution of bias^2 to MSE is clearly visible due to high SNR. Settings: $K = 500$ snapshots, SNR=50 dB and interelement spacing $d = 0.3\lambda$. Note that array with odd number of elements has lower bias.

5. ALGORITHM FOR BIAS REDUCTION

In this section we first derive the two beamformers able to synthetize the dominant term of the residual error (for $q = 1$), see eq.(4) and (10). Hence we employ them in our novel technique that in only three steps can perform bias removal on the DoA's.

The proposed technique can be seen as a calibration procedure that first computes the calibration error and then modifies the original beamformer $\mathbf{F}_e^H$ for compensating the error itself.

The Generalized Beamspace Transform defined in Section 4 provides an expression of the residual errors $\sqrt{N}\mathbf{C}_v\boldsymbol{\varepsilon}^{(1)}$. In order to derive the two new beamformers $\mathbf{F}_{e1}^H$ and $\mathbf{F}_{e2}^H$, we define the term $\varepsilon^{(1)}$ as sum of two terms $\varepsilon^{(1)} = \varepsilon_1^{(1)} + \varepsilon_2^{(1)}$ where each component can be explicitly expressed for $m < |M|$ as [6]

$$\varepsilon_1^{(1)} = \begin{bmatrix} j^{(N+M)} J_{(N+M)}(\zeta) e^{-j(N+M)\phi} \\ \vdots \\ j^N J_N(\zeta) e^{-jN\phi} \\ \vdots \\ j^{(N-M)} J_{(N-M)}(\zeta) e^{-j(N-M)\phi} \end{bmatrix} \quad (11)$$

$$\varepsilon_2^{(1)} = \begin{bmatrix} j^{(N-M)} J_{(N-M)}(\zeta) e^{j(N-M)\phi} \\ \vdots \\ j^N J_N(\zeta) e^{jN\phi} \\ \vdots \\ j^{(N+M)} J_{(N+M)}(\zeta) e^{j(N+M)\phi} \end{bmatrix}. \quad (12)$$

We start by considering the continuous circular aperture case and, similarly to eq.(3), we define the normalized far-field patterns resulting from exciting the aperture with the mode $w_m^{(1)}(\gamma) = e^{j(m-N)\gamma}$ as

$$j^{(N-m)} J_{(N-m)}(\zeta) e^{-j(N-m)\phi} = \frac{1}{2\pi}\int_0^{2\pi} e^{j\zeta\cos(\phi-\gamma)} e^{j(m-N)\gamma} \mathrm{d}\gamma \quad (13)$$

and $w_m^{(2)}(\gamma) = e^{j(N+m)\gamma}$ as

$$j^{(N+m)} J_{(N+m)}(\zeta) e^{j(N+m)\phi} = \frac{1}{2\pi}\int_0^{2\pi} e^{j\zeta\cos(\phi-\gamma)} e^{j(N+m)\gamma} \mathrm{d}\gamma. \quad (14)$$

The detailed derivation of equations (13)-(14) as well as a practical numerical evaluation of the same integrals can be found in ref.[6]. However, under an implementation point of view, the above integrals can be numerically evaluated on $Q = 24$ points as [6]

$$\frac{1}{Q}\sum_{q=0}^{Q-1} e^{j\zeta\cos(\phi-\gamma_q)} e^{j(m-N)\gamma_q} = $$
$$= j^{(N-m)} J_{(N-m)}(\zeta) e^{-j(N-m)\phi} + O(||\gamma_q^2||) \quad (15)$$
$$\frac{1}{Q}\sum_{q=0}^{Q-1} e^{j\zeta\cos(\phi-\gamma_q)} e^{j(N+m)\gamma_q} = $$
$$= j^{(N+m)} J_{(N+m)}(\zeta) e^{j(N+m)\phi} + O(||\gamma_q^2||) \quad (16)$$

where $\gamma = \frac{2\pi q}{Q}$ for $q = 0, \ldots, Q-1$ and $O(||\gamma_q^2||)$ are small terms that can be neglected without any consequences.

Equations (13)-(14), as well as eq.(15)-(16), allow us to find a closed-form expression of the dominant term (for $q = 1$) of the residual error after the beamspace transform is applied, see eq.(4). Observe that for $m \in [-M, M]$ the equations describe the vectors in equations (11) and (12).

Let us assume that we have successfully evaluated the integrals in (13)-(14), the general idea in the algorithm for bias reduction is the following. The proposed algorithm proceeds in three steps: use the conventional beamspace transform for mapping the data from the element-space to the beamspace domain. Then compute the residual error through the introduction of two new beamformers, $\mathbf{F}_{e1}^H$ and $\mathbf{F}_{e2}^H$. Finally correct the original beamformer $\mathbf{F}_e^H$ by subtracting the new beamformers from the original one.

The $N \times K$ element-space data matrix $\mathbf{X}$ is mapped into the $\mathcal{M} \times K$ beamspace data matrix $\mathbf{Y}$ by using the beamspace transform [2]-[3]. Combining equations (1), (10)-(12) we can write

$$\mathbf{Y} = \mathbf{F}_e^H \mathbf{X} = \mathbf{F}_e^H \mathbf{A}\mathbf{S} + \mathbf{F}_e^H \mathbf{N} \quad (17)$$
$$= \big(\sqrt{N}\mathbf{J}_\zeta \mathbf{d}(\phi) + \sqrt{N}\mathbf{C}_v \varepsilon^{(1)}\big)\mathbf{S} + \mathbf{F}_e^H \mathbf{N} \quad (18)$$
$$= \big(\sqrt{N}\mathbf{J}_\zeta \mathbf{d}(\phi)\big)\mathbf{S} + \sqrt{N}\mathbf{C}_v\big(\varepsilon_1^{(1)} + \varepsilon_2^{(1)}\big)\mathbf{S} + \mathbf{F}_e^H \mathbf{N} \quad (19)$$

in which the two terms $\big(\sqrt{N}\mathbf{C}_v\varepsilon_1^{(1)}\big)\mathbf{S}$ and $\big(\sqrt{N}\mathbf{C}_v\varepsilon_2^{(1)}\big)\mathbf{S}$ related to the residual term of the beamspace transform have been rewritten separately. The term $\big(\sqrt{N}\mathbf{J}_\zeta\mathbf{d}(\phi)\big)\mathbf{S}$ forms the ideal beamspace data matrix given as a product of the ideal steering vector with Vandermonde structure $\mathbf{d}(\phi)$ and the signal $\mathbf{S}$. Moreover the noise term $\mathbf{F}_e^H\mathbf{N}$ is still a complex circular Gaussian process since the beamformer matrix $\mathbf{F}_e^H$ is unitary.

Our goal is to remove $\big(\sqrt{N}\mathbf{C}_v\varepsilon_1^{(1)}\big)\mathbf{S}$ and $\big(\sqrt{N}\mathbf{C}_v\varepsilon_2^{(1)}\big)\mathbf{S}$ from eq.(19) because they cause the bias in the DoA estimates. Clearly we do not know the terms since the signal matrix $\mathbf{S}$ is not known. An alternate solution is to construct two other beamformers $\mathbf{F}_{e1}^H$ and $\mathbf{F}_{e2}^H$ in order to synthesize the bias term and then simply cancel it out. The two new beamformers are computed as follows

$$\mathbf{F}_{e1}^H \mathbf{A} = \sqrt{N}\mathbf{C}_v \varepsilon_1^{(1)} \quad (20)$$
$$\mathbf{F}_{e2}^H \mathbf{A} = \sqrt{N}\mathbf{C}_v \varepsilon_2^{(1)}. \quad (21)$$

The minimum-norm solution to (20)-(21) which defines the beamformers can be found as

$$\mathbf{F}_{e1} = \mathbf{A}^\dagger \sqrt{N} \varepsilon_1^{(1)H} \mathbf{C}_v^H \quad (22)$$
$$\mathbf{F}_{e2} = \mathbf{A}^\dagger \sqrt{N} \varepsilon_2^{(1)H} \mathbf{C}_v^H \quad (23)$$

where $\mathbf{A}$ is the UCA element-space steering matrix with left pseudo-inverse $\mathbf{A}^\dagger = \mathbf{A}\big(\mathbf{A}^H\mathbf{A}\big)^{-1}$ and $\mathbf{C}_v$ is defined in eq.(6). Notice that $\mathbf{A}$ can not be directly computed because it would require knowledge of the DoA's. Equations (22)-(23) are computed using an estimated steering matrix $\mathbf{A}_R$. For more details see Table 1.

At this point we can form the $\mathcal{M} \times K$ correction data matrices $\tilde{\mathbf{Y}}_1$ and $\tilde{\mathbf{Y}}_2$ as

$$\tilde{\mathbf{Y}}_1 = \mathbf{F}_{e1}^H \mathbf{X} = \big(\sqrt{N}\mathbf{C}_v\varepsilon_1^{(1)}\big)\mathbf{S} + \mathbf{F}_{e1}^H \mathbf{N} \quad (24)$$
$$\tilde{\mathbf{Y}}_2 = \mathbf{F}_{e2}^H \mathbf{X} = \big(\sqrt{N}\mathbf{C}_v\varepsilon_2^{(1)}\big)\mathbf{S} + \mathbf{F}_{e2}^H \mathbf{N} \quad (25)$$

where $\mathbf{X}$ is the $N \times K$ UCA element-space data matrix.

The bias cancellation is performed by substituting the original beamspace data matrix $\mathbf{Y}$ by the two correction data matrices $\tilde{\mathbf{Y}}_1$ and $\tilde{\mathbf{Y}}_2$. Mathematically we have

$$\hat{\mathbf{Y}} = \mathbf{Y} - \tilde{\mathbf{Y}}_1 - \tilde{\mathbf{Y}}_2 = \big(\mathbf{F}_e^H - \mathbf{F}_{e1}^H - \mathbf{F}_{e2}^H\big)\mathbf{X} \quad (26)$$
$$= \big(\sqrt{N}\mathbf{J}_\zeta\mathbf{d}(\phi)\big)\mathbf{S} + \big(\mathbf{F}_e^H - \mathbf{F}_{e1}^H - \mathbf{F}_{e2}^H\big)\mathbf{N} \quad (27)$$

where $\sqrt{N}\mathbf{J}_\zeta\mathbf{d}(\phi)$ is the ideal steering vector of the virtual array with Vandermonde structure. Notice that the matrix $\big(\mathbf{F}_e^H - \mathbf{F}_{e1}^H - \mathbf{F}_{e2}^H\big)$ is close to unitary because the norms of $\mathbf{F}_{e1}^H, \mathbf{F}_{e2}^H$ are very small. Therefore we can say the statistics of the noise does not change significantly and we can still consider the noise to be a complex circular Gaussian process. Table 1 summarizes the novel technique for bias removal.

Table 1. Algorithm for bias reduction

Step 1: Form the array observation matrix $\mathbf{X}$ as in eq.(1),

a) compute the initial beamspace data matrix $\mathbf{Y}$ as $\mathbf{Y} = \mathbf{F}_e^H \mathbf{X}$

b) estimate the initial set of DoA's by using UCA root-MUSIC algorithm [2]-[3]

Step 2: with the initial DoA's computed on step 1

c) evaluate eq.(13)-(14) by using the approximation in (15)-(16) and form $\varepsilon_1^{(1)}$ and $\varepsilon_2^{(1)}$ as in (11)-(12),

d) form the estimated steering matrix $\mathbf{A}_R$ as in (2),

e) compute the beamformers $\mathbf{F}_{e1}^H$ and $\mathbf{F}_{e2}^H$ by using equations (22)-(23),

f) compute the new beamspace data matrix $\tilde{\mathbf{Y}}$ starting from the initial observation matrix $\mathbf{X}$ as in eq.(26),

g) using UCA root-MUSIC algorithm [2]-[3], estimate the second set of DoA's.

Step 3: using the DoA's computed on step 2

h) repeat points (c), (d), (e) and (f),

i) estimate the DoA's by using the UCA root-MUSIC algorithm [2]-[3]

6. SIMULATION RESULTS

In this sections some simulation results are presented. They clearly demonstrate how the bias in the DoA estimates is reduced such that no error floor occurs and variance close to CRB is achieved.

In Fig.2 we illustrate the reduction in the RMS error by using the UCA Unitary root-MUSIC algorithm [3] during each of the three steps. The original DoA estimates are significantly biased and clear error floors are visible in the figure. After the second step the error floors get lowered indicating that the bias is reduced. Finally, after the third step the bias is practically completely removed. In fact, the RMS curves are parallel to the CRB but the CRB is not reached since the beamspace transform adds extra variance on the DoA estimates [4].

In Fig.3 we depict the RMS error of UCA Unitary root-MUSIC using the bias cancellation technique proposed in this paper in comparison to a conventional method.

7. CONCLUSIONS

In this paper we derived an algorithm for removing the bias in the DoA estimates. Bias occurs as a consequence of applying the beamspace transform when using a UCA with a small number of sensors. Practically bias-free performance is achieved. The beamspace transform, however, adds extra variance to the estimates which explains the gap to the CRB.

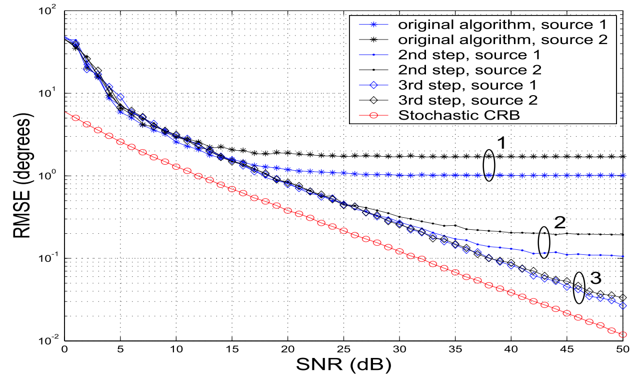

Fig. 2. Cancelling steps for: the original UCA Unitary root MUSIC algorithm, after the second step and after the third step. Settings: $N = 8$, $r = \frac{\lambda}{2.6}$, $K = 256$ and $(\phi_1, \phi_2) = (15°, 25°)$.The bias is removed but excess variance remains.

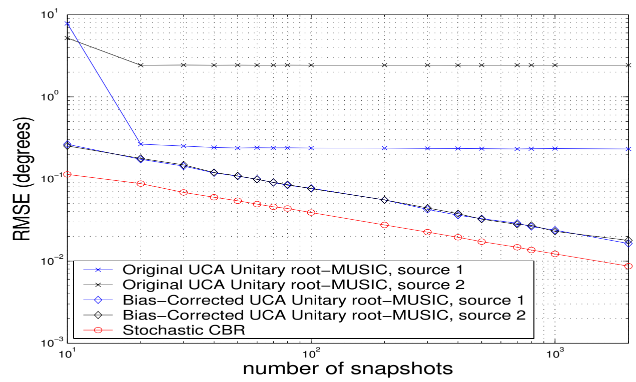

Fig. 3. Performances of the UCA Unitary root-MUSIC. Settings: $N = 8$, $r = \frac{\lambda}{2.6}$, SNR=50 dB and $(\phi_1, \phi_2) = (10°, 15°)$. The bias is reduced but the CRB is not attained because of the additional variance.

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