Size effects on centrosymmetric anisotropic shear deformable beam structures

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Abstract
In this paper, the size effect on beam structures with centrosymmetric anisotropy is studied within strain gradient elasticity theory. Applying dimension reduction to the three dimensional anisotropic gradient elasticity, the third-order shear deformable (TSD) beam is analysed. A variational approach is used to determine the equilibrium equations of TSD beam together with consistent (classical and non-classical) boundary conditions. The TSD beam theory which is suitable for deep beam structures can be replaced by (less complicated) Euler-Bernoulli beam model for thin beam structures. The anisotropic Euler-Bernoulli beam model is also formulated within the framework of strain gradient theory. This anisotropic beam theory can be used to study size effects for any types of centrosymmetric anisotropy. To address the more practical cases of composite structures, the formulation is simplified for orthotropic and transversely isotropic materials. Finally, the analytical solutions are provided for bending of simply supported (TSD and Euler-Bernoulli) beams as well as clamped Euler-Bernoulli beams. The effect of the crystal orientation with respect to the beam geometry is investigated in these examples.

Keywords: anisotropy, strain gradient, shear deformable beam, orthotropy, centrosymmetric

1. Introduction
One of the main constituents of the mechanical systems are beam structures which have absorbed considerable attention in classical continuum mechanics. However, the emergence of new technologies with small-scale elements highlights the importance of the size effect on the miniature systems. Although plenty of investigations have been conducted on the analysis of micro- and nano-structures with isotropic material properties, the analysis of size effect on structures with general anisotropy has not been addressed in the literature so far. Due to the anisotropic nature of many nano-engineered materials, the analysis of anisotropic nano-beams is essential.

Generalized continua enable one to realise the size effect through introducing the internal length scales in a constitutive level [1]. Gradient elasticity is one of the popular candidates among other generalized continuum theories. This framework is presented by [2], and extensively has been applied to analyse mostly isotropic structures (e.g. Wang et al. [3], Lazopoulos and Lazopoulos [4], Ramezani [5], Akgöz and Civalek [6], Mousavi and Paavola [7], Akgöz and Civalek [8], Liang et al. [9], Wang et al. [10], Xu and Deng [11]). In particular, the gradient elasticity theory has been employed to study the behaviour of isotropic nanotubes (Ansari et al. [12], Zheng et al. [13]). Velocity gradient has also been considered within strain gradient theory to study the dynamic behaviour of beams [14, 15] and plates [16]. This framework has also been considered for the analysis of graded materials [17]. However, a general approach for the analysis of anisotropic micro- and nano-structures has not been developed so far. Consequently, it is quite motivating to extend the above-mentioned contributions toward an anisotropic version of strain gradient theory.

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The generalization of the isotropic gradient elasticity \[2, 18\] towards an anisotropic one has been recently studied by \[19\] and Auffray et al. \[20, 21\]. The strain energy within gradient theory for a general anisotropic material couples the strain and gradients of strain. This will result to even-order as well as odd-order tensors of material properties. In the special cases of centrosymmetric anisotropy, the odd-order tensors are vanished, and the strain energy includes only even-order tensors. These even-order tensors include a (classical) fourth-rank tensor of elastic constants for an anisotropic material, together with a sixth-rank tensor which incorporates both material anisotropy and anisotropic length scale effects. In order to separate the scale effects and material properties, it can be assumed that this sixth-rank tensor is decomposed as a combination of the classical fourth-rank tensor of elastic constants and a second-rank tensor of anisotropic length scale effects (Gitman et al. \[19\], Lazar and Po \[22\]). This assumption simplifies the formulation of centrosymmetric anisotropic structures. Recently, \[23\] applied the assumption proposed by \[19\] and \[22\] in order to study size effects in anisotropic plate structures.

In this paper, the (homogeneous and centrosymmetric) anisotropic shear deformable beam structure will be studied within strain gradient theory. As the specific cases, some practical examples including orthotropic as well as transversely isotropic beams will be discussed. This article complements an earlier contribution of the authors \[14\] dealing with the analysis of size-effect on isotropic shear deformable beams.

This paper is organized as follows. The variational formulation for a centrosymmetric anisotropic material is presented in section 2. In section 3, by employing the dimension reduction, the beam structures are investigated within third-order shear deformation and Euler-Bernoulli theories. Later in section 4, the general anisotropic beam is simplified to the most common cases including orthotropic and transversely isotropic beams. Solutions to some specific examples are presented in section 5. Finally, the conclusion is presented in section 6.

2. Anisotropic strain gradient elasticity

Within first strain gradient theory, Mindlin considers the strain energy density function \((U)\) to be a quadratic function in terms of strain and first-order gradient strain \[2\] as

\[
U = U(\varepsilon_{ij}, \partial_{k}\varepsilon_{ij}), \ i, j, k \in \{x, y, z\}.
\]

where comma denotes the partial derivative. The infinitesimal elastic strain components \(\varepsilon_{ij}\) in terms of displacement components \(u_{j}\) are

\[
\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2} (\partial_{j}u_{i} + \partial_{i}u_{j})
\]

In this framework, the Cauchy-like stress tensor components \(\tau_{ij}\) and double-stress tensor components \(\tau_{ijk}\) are given by

\[
\tau_{ij} = \frac{\partial U}{\partial \varepsilon_{ij}}, \quad \tau_{ijk} = \frac{\partial U}{\partial \varepsilon_{ij,k}}
\]

For the case of a homogeneous and centrosymmetric material, no coupling is considered between strain and strain-gradient terms, and the strain energy takes the form \[20, 22\]

\[
U = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} D_{ijklmn} \varepsilon_{ij,m} \varepsilon_{kl,n}.
\]

where \(C_{ijkl}\) is the standard fourth-rank tensor of elastic constants for an anisotropic material with symmetry properties

\[
C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}
\]

The sixth-rank constitutive tensor \(D_{ijklmn}\) incorporates material anisotropy and anisotropic length scale effects and has the following symmetries

\[
D_{ijklmn} = D_{jiklmn} = D_{ijlmkn} = D_{klnijm}
\]
Specific types of triclinic, monoclinic and orthorhombic crystals possess centrosymmetry. In order to simplify the formulation, it is assumed that the tensor $D_{ijklmn}$ can be decomposed as a combination of the fourth-rank tensor $C_{ijkl}$ and a second-rank tensor of anisotropic length scale effects $\Lambda_{mn}$ (Gitman et al. [19], Lazar and Po [22])

$$D_{ijklmn} = C_{ijkl}\Lambda_{mn}$$

In equation (7), tensor $\Lambda_{mn}$ presents the anisotropy of the gradient length scale parameter (weak non-local anisotropy [24]) and reflects the new physical anisotropic effects which appear when the material is studied in ultra small scales [22].

Due to the symmetry properties (6), and of the positive definiteness of $U$, the tensor $\Lambda_{mn}$ must be symmetric and positive definite. Consequently, by virtue of the decomposition, the 192 (21+171) independent material constants of a centrosymmetric triclinic material in Mindlin’s anisotropic gradient elasticity theory (6) are reduced to 21 elastic constants and 6 length scale parameters in Mindlin’s anisotropic gradient elasticity (7) with separable weak non-locality [22]. It is noted that the tensor $\Lambda_{mn}$ has the dimension [m$^2$]. Appendix A presents the tensor $\Lambda_{mn}$ for different classes of crystal symmetry.

Considering the assumption (7), the strain energy density takes the form

$$U = \frac{1}{2}C_{ijkl}\varepsilon_{kl} + \frac{1}{2}\Lambda_{mn}C_{ijkl}\varepsilon_{ij,m}\varepsilon_{kl,n}.$$  (8)

Substituting the strain energy (8) into equations (3), the stress tensor $\tau_{ij}$ and double-stress tensor $\tau_{ijk}$ are expressed as

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}, \quad \tau_{ijm} = \Lambda_{mn}C_{ijkl}\varepsilon_{ij,m}\varepsilon_{kl,n} = \Lambda_{mn}\tau_{ij,n}.$$  (9)

The strain energy density reads

$$U = \frac{1}{2}\varepsilon_{ij}\tau_{ij} + \frac{1}{2}\varepsilon_{ij,k}\tau_{ijk}.$$  (10)

Accordingly, the strain energy $U_\Omega$ in a region $\Omega$ occupied by the elastically deformed material reads

$$U_\Omega = \int_\Omega U\,dv = \frac{1}{2}\int_\Omega (\varepsilon_{ij}\tau_{ij} + \varepsilon_{ij,k}\tau_{ijk})\,dv.$$  (11)

The variation of the strain energy ($U_\Omega$) is

$$\delta U_\Omega = \int_\Omega (\tau_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\varepsilon_{ij,k})\,dv = \int_\Omega (\tau_{ij}\delta u_{i,j} + \Lambda_{kl}\tau_{ij,l}\delta u_{i,k})\,dv.$$  (12)

Moreover, the variation of the external work reads

$$\delta W_\Omega = \int_\Omega f_i\delta u_i\,dv + \int_{\partial\Omega} (t_i\delta u_i + q_i n_j\delta u_{i,j})\,da$$  (13)

where $\partial\Omega$ is the bounding (closed) surface of $\Omega$, $f_i$ is body force and $t_i$ and $q_i$ are Cauchy traction vector and double stress traction vector on the boundary, respectively. The principle of virtual work reads

$$\delta U_\Omega = \delta W_\Omega.$$  (14)

Substitution of the variations of the strain energy and external work (12, 13) into equation (14) and application of the fundamental lemma of calculus of variation will lead to the governing equilibrium equations and boundary conditions in three dimensional form. In order to simplify the 3-D formulation for specific cases such as beams, dimension reduction is applied to the general formulation.

3. Beam structures within anisotropic strain gradient elasticity

In this section, the size effect on beam structures is studied within anisotropic strain gradient elasticity theory. For the analysis of deep beam structures, the TSD beam theory will be employed, while the Euler-Bernoulli beam theory will be used for the analysis of shallow beam structures.
3.1. Anisotropic third-order shear deformable beam

Consider a beam with a rectangular cross-section of height \( h \) and width \( b \) (figure (1)). The beam is considered to be made of homogeneous and centrosymmetric anisotropic material and is subjected to lateral load \( t_z(x) \) on its upper surface. According to the TSD beam theory ([25], [26], [27]), the displacement field of the TSD beam is

\[
\begin{align*}
  u_x(x, z) &= z\beta(x) - \alpha z^3 \left[ \beta(x) + \frac{\partial w(x)}{\partial x} \right] \\
  u_z(x, z) &= w(x)
\end{align*}
\]

where

\[ \alpha = \frac{4}{3h^2}. \]

Here, \( u_x(x, z) \) and \( u_z(x, z) \) denote the total displacements along the coordinates \( x \) and \( z \), respectively. In equation (15), \( w(x) \) represents the transverse deflection of a point on the beam axis and \( \beta(x) \) denotes the rotation of the beam cross section.

![Figure 1: Beam with rectangular cross section subjected to lateral load \( t_z(x) \).](image)

By substituting the displacement field (15) into the strain-displacement relation (2), the nonzero components of the strain tensor are obtained as (\( x \) and \( z \) not summable)

\[
\begin{align*}
  \varepsilon_{xx} &= z\beta_x - \alpha z^3 (\beta_x + w_{xx}) \\
  \varepsilon_{xz} &= \frac{1}{2} \left( 1 - 3\alpha z^2 \right) (w_x + \beta)
\end{align*}
\]

According to equations (9) and (17), the Cauchy and higher stress components read

\[
\begin{align*}
  \tau_{ij} &= C_{ijxx} \left[ (z - \alpha z^3) \beta_{xx} - \alpha z^3 w_{xx} \right] + C_{ijxz} \left( 1 - 3\alpha z^2 \right) (w_x + \beta) \\
  \tau_{ijk} &= \Lambda_{kx} C_{ijxx} \left[ (z - \alpha z^3) \beta_{xxx} - \alpha z^3 w_{xxx} \right] \\
  &\quad + \Lambda_{kx} C_{ijxz} \left( 1 - 3\alpha z^2 \right) (w_{xx} + \beta) \\
  &\quad + \Lambda_{kz} C_{ijxx} \left[ (1 - 3\alpha z^2) \beta_{x} - 3\alpha z^2 w_{xx} \right] \\
  &\quad - 6\alpha z \Lambda_{kz} C_{ijxz} (w_x + \beta)
\end{align*}
\]
Using equations (17) and (18), the first variation of the strain energy (12) takes the form

\[
\delta U_D = \int_{\Omega} \left\{ \tau_{zz} \left[ (z - \alpha z^3) \delta \beta_x - \alpha z^3 \delta w_{xx} \right] + \tau_{zr} \left[ (1 - 3\alpha z^2) (\delta w_{xx} + \delta \beta) \right] + [\Lambda_{xx} \tau_{xx,z} + \Lambda_{zz} \tau_{zz,z}] \left[ (z - \alpha z^3) \delta \beta_x - \alpha z^3 \delta w_{xx} \right] + [\Lambda_{xx} \tau_{xx,z} + \Lambda_{zz} \tau_{zz,z}] \left[ (1 - 3\alpha z^2) (\delta w_{xx} + \delta \beta_x) \right] + [\Lambda_{xx} \tau_{xx,z} + \Lambda_{zz} \tau_{zz,z}] \left[ (1 - 3\alpha z^2) \delta \beta_x - 3\alpha z^2 \delta w_{xx} \right] - [\Lambda_{xx} \tau_{xx,z} + \Lambda_{zz} \tau_{zz,z}] \left[ 6\alpha z (\delta w_{xx} + \delta \beta) \right] \right\} \, dv.
\]

(19)

In order to apply the dimension reduction, the stress resultants are defined as

\[
\{ N_{xx}, M_{xx}, R_{xx}, P_{xx} \} = \int_A \{ 1, z, z^2, z^3 \} \tau_{xx} \, dA,
\]

\[
\{ N_{xx}, M_{xx}, R_{xx} \} = \int_A \{ 1, z, z^2 \} \tau_{xx} \, dA.
\]

(20)

Moreover, the gradient-of-stress resultants are defined as

\[
\{ N_{xx}^z, M_{xx}^z, R_{xx}^z, P_{xx}^z \} = \int_A \{ 1, z, z^2, z^3 \} \tau_{xx,z} \, dA,
\]

\[
\{ N_{xx}^z, M_{xx}^z, R_{xx}^z \} = \int_A \{ 1, z, z^2 \} \tau_{xx,z} \, dA.
\]

(21)

Here, \( A \) represents the cross section area of the beam. These resultants can be written in terms of the displacement field as

\[
N_{xx} = \hat{A}_{zz} (w_x + \beta), \quad M_{xx} = \hat{D}_{zz} \beta_x - \alpha F_{xx} w_{xx},
\]

\[
R_{xx} = \hat{D}_{zz} (w_x + \beta), \quad P_{xx} = \hat{F}_{xx} \beta_x - \alpha H_{xx} w_{xx},
\]

\[
N_{zz} = \hat{A}_{zz} (w_x + \beta), \quad M_{zz} = \hat{D}_{zz} \beta_x - \alpha F_{zz} w_{xx},
\]

\[
R_{zz} = \hat{D}_{zz} (w_x + \beta), \quad N_{zz}^z = \hat{A}_{zz} \beta_x - 3\alpha D_{xx} w_{xx},
\]

\[
M_{zz}^z = -6\alpha D_{zz} (w_x + \beta), \quad R_{zz}^z = \hat{D}_{zz} \beta_x - 3\alpha F_{zz} w_{xx},
\]

\[
P_{zz}^z = -6\alpha F_{zz} (w_x + \beta), \quad N_{zz} = \hat{A}_{zz} \beta_x - 3\alpha D_{zz} w_{xx},
\]

\[
M_{zz}^z = -6\alpha D_{zz} (w_x + \beta), \quad R_{zz}^z = \hat{D}_{zz} \beta_x - 3\alpha F_{zz} w_{xx}.
\]

(22)

In resultants (22), the coefficients are

\[
\hat{A}_{zz} = A_{xx} - 3\alpha D_{xx}, \quad \hat{A}_{zz} = A_{zz} - 3\alpha D_{zz}, \quad \hat{A}_{zz} = A_{zz} - 3\alpha D_{zz},
\]

\[
\hat{D}_{xx} = D_{xx} - \alpha F_{xx}, \quad \hat{D}_{zz} = D_{zz} - 3\alpha F_{zz}, \quad \hat{D}_{zz} = D_{zz} - \alpha F_{zz},
\]

\[
\hat{D}_{zz} = D_{zz} - 3\alpha F_{zz}, \quad \hat{D}_{zz} = D_{xx} - 3\alpha F_{xx}, \quad \hat{F}_{xx} = F_{xx} - \alpha H_{xx},
\]

(23)

where

\[
(A_{xx}, D_{xx}, F_{xx}, H_{xx}) = \int_A C_{xxxx} (1, z^2, z^4, z^6) \, dA,
\]

\[
(A_{zz}, D_{zz}, F_{zz}) = \int_A C_{zzzz} (1, z^2, z^4) \, dA,
\]

\[
(A_{zz}, D_{zz}, F_{zz}) = \int_A C_{zzzz} (1, z^2, z^4) \, dA.
\]

(24)
Using definitions of the resultants (20) and (21), the variation of the strain energy of the beam (19) takes the form

\[ \delta U = \int_0^L \left( M_{xx} \delta \beta_{,x} - \alpha P_{xx} (\delta \beta_{,x} + \delta w_{,xx}) + (N_{xx} - 3 \alpha R_{xx}) (\delta \beta + \delta w_{,x}) ight) + \lambda_{xx} [M_{xx,xx} \delta \beta_{,xx} - \alpha P_{xx,xx} (\delta \beta_{,xx} + \delta w_{,xxxx})] \\
+ (N_{xx,xx} - 3 \alpha R_{xx,xx}) (\delta \beta_{,xx} + \delta w_{,xxxx}) \right] \delta \beta \, dx \\
+ \lambda_{zz} \left[ N_{zz} \delta \beta_{,x} - 3 \alpha R_{zz}^z (\delta \beta_{,x} + \delta w_{,xx}) - 6 \alpha M_{zz}^z (\delta \beta + \delta w_{,x}) \right] \delta \beta_{,x} \, \delta \beta \, dx, \]  

(25)

where \( L \) is the length of the beam. Applying Green’s theorem to the variation of strain energy (25) results in

\[ \delta U = \int_0^L \left[ -M_{xx} + N_{zz} + \Lambda_{xx} (M_{xx,xx} + N_{xx}) \right] \delta \beta \, dx \]

(26)

Assuming that only the transverse load \( t_z(x) \) acts on the upper surface of the beam (along the centroidal axis), the variation of the external work (13) is simplified to

\[ \delta W_{\Omega} = \int_0^L t_z(x) \delta w_{,x} \, dx. \]  

(28)
Substitution of the variation of the strain energy (26) and the variation of external work (28) in the principle of virtual work (14) results in

$$
\int_0^L \left[ -\tilde{M}_{xx,x} + \tilde{N}_{zz} + \Lambda_{xx} \left( \tilde{M}_{xx,xxx} - \tilde{N}_{zz,xx} \right) \\
+ \Lambda_{zz} \left( \tilde{M}_{zz,xx} - \tilde{N}_{zz,xx} - 6\alpha M_{zz,z} \right) \\
- \Lambda_{zz} \left( \tilde{N}_{zz,xx} + 6\alpha M_{zz,z} \right) \right] \delta \beta \, dx = 0,
$$

(29)

Moreover, the following conditions should be satisfied at both ends of the beam (i.e. $x = 0$ and $x = L$)

$$
\left[ \tilde{M}_{xx} + \Lambda_{xx} \left( \tilde{M}_{xx,xx} + \tilde{N}_{zz,x} \right) \\
+ \Lambda_{zz} \left( \tilde{M}_{zz,xx} + \tilde{N}_{zz,xx} + \tilde{N}_{zz,xx} \right) + \Lambda_{zz} \tilde{N}_{zz,xx} \right] \delta \beta = 0
$$

(30)

Due to the fundamental lemma of calculus of variation, the variational equations (29) give the following equilibrium equations

$$
\begin{align*}
-\tilde{M}_{xx,x} + \tilde{N}_{zz} + \Lambda_{xx} \left( \tilde{M}_{xx,xxx} - \tilde{N}_{zz,xx} \right) \\
+ \Lambda_{zz} \left( \tilde{M}_{zz,xx} - \tilde{N}_{xx,xx} - 6\alpha M_{zz,z} \right) \\
- \Lambda_{zz} \left( \tilde{N}_{xx,xx} + 6\alpha M_{zz,z} \right) \\
- \alpha P_{xx,xx} + \tilde{N}_{zz} + \Lambda_{xx} \left( \alpha P_{xx,xxx} + \tilde{N}_{zz,xx} \right) \\
+ \Lambda_{zz} \left( 3\alpha R_{xx,xx} - 6\alpha M_{zz,z} \right) \\
- \alpha P_{xx} + \Lambda_{xx} \left( \alpha P_{xx,xx} + \tilde{N}_{zz,xx} \right) \\
+ \Lambda_{zz} \left( 3\alpha R_{xx,xx} - 3\alpha R_{xx,xx} - 3\alpha R_{xx,xx} \right) - 3\alpha \Lambda_{zz} R_{xx} \right] \delta w = 0
\end{align*}
$$

(31a)

According to equation (30), the equilibrium equations (31) are accompanied by the following boundary conditions

$$
\begin{align*}
-\tilde{M}_{xx,x} + \tilde{N}_{zz} + \Lambda_{xx} \left( \tilde{M}_{xx,xxx} - \tilde{N}_{zz,xx} \right) \\
+ \Lambda_{zz} \left( \tilde{M}_{zz,xx} - \tilde{N}_{xx,xx} - 6\alpha M_{zz,z} \right) \right), \\
- \Lambda_{zz} \left( \tilde{N}_{xx,xx} + 6\alpha M_{zz,z} \right) = 0,
\end{align*}
$$

(31b)
conditions

\[
\begin{align*}
\{ \dot{M} & + \Lambda_{zz} \left( -\dot{M}_{xx,xx} + \dot{N}_{xx,x} \right) \\
+ \Lambda_{xx} \left( -\dot{M}_{xx,xx} + \dot{N}_{xx,x} + \dot{N}_{xx,xx} \right) + \Lambda_{zz} \dot{N}_{xx,xx} = 0 \} \quad \text{or} \quad \delta \beta = 0 \\
\Lambda_{xx} M_{xx,x} + \Lambda_{xx} \dot{M}_{xx} = 0 \} \quad \text{or} \quad \delta \beta_x = 0 \\
\alpha P_{xx,x} + \dot{N}_{xx} + \dot{N}_{xx} \left( -\alpha P_{xx,xx} - \dot{N}_{xx,xx} \right) \\
+ \Lambda_{xx} \left( -\alpha P_{xx,xx} - \dot{N}_{xx,xx} + 3\alpha R_{xx,xx} - 6\alpha M_{xx,x} \right) \\
+ \Lambda_{zz} \left( 3\alpha R_{xx,xx} - 6\alpha M_{xx,x} \right) = 0 \\
- \alpha P_{xx} + \Lambda_{xx} \left( \alpha P_{xx,xx} + \dot{N}_{xx,xx} \right) \\
+ \Lambda_{xx} \left( \alpha P_{xx,xx} + \dot{N}_{xx,xx} - 3\alpha R_{xx,xx} \right) - 3\alpha \Lambda_{zz} R_{xx,xx}^z = 0 \\
\{- \alpha \Lambda_{xx} P_{xx,xx} - \alpha \Lambda_{xx} P_{xx}^z = 0 \} \quad \text{or} \quad \delta w_{xx} = 0 
\end{align*}
\]

For a simply supported edge, the boundary conditions (32) are selected to be

\[
\begin{align*}
1: & \quad \{ \dot{M}_{xx} + \Lambda_{xx} \left( -\dot{M}_{xx,xx} + \dot{N}_{xx,x} \right) \\
& + \Lambda_{xx} \left( -\dot{M}_{xx,xx} + \dot{N}_{xx,x} + \dot{N}_{xx,xx} \right) + \Lambda_{zz} \dot{N}_{xx,xx} = 0 \} \\
2: & \quad \{ \Lambda_{xx} M_{xx,x} + \Lambda_{xx} \dot{M}_{xx} = 0 \} \quad \text{or} \quad \delta \beta_x = 0 \\
3: & \quad w = 0 \\
4: & \quad \{ - \alpha P_{xx} + \Lambda_{xx} \left( \alpha P_{xx,xx} + \dot{N}_{xx,xx} \right) \\
& + \Lambda_{xx} \left( \alpha P_{xx,xx} + \dot{N}_{xx,xx} - 3\alpha R_{xx,xx} \right) - 3\alpha \Lambda_{zz} R_{xx,xx}^z = 0 \} \\
5: & \quad \{- \alpha \Lambda_{xx} P_{xx,xx} - \alpha \Lambda_{xx} P_{xx}^z = 0 \} \quad \text{or} \quad \delta w_{xx} = 0 
\end{align*}
\]

while for a clamped boundary, we have

\[
\begin{align*}
1: & \quad \beta = 0 \\
2: & \quad \{ \Lambda_{xx} M_{xx,x} + \Lambda_{xx} \dot{M}_{xx} = 0 \} \quad \text{or} \quad \delta \beta_x = 0 \\
3: & \quad w = 0 \\
4: & \quad w_x = 0 \\
5: & \quad \{- \alpha \Lambda_{xx} P_{xx,xx} - \alpha \Lambda_{xx} P_{xx}^z = 0 \} \quad \text{or} \quad \delta w_{xx} = 0 
\end{align*}
\]

It is noted that the selection of higher-order boundary conditions, i.e. second and fifth equations in (33) and (34) are arbitrary.

Using (23), (27), (31), the governing equilibrium equations can be written in terms of deflections and rotations as

\[
\begin{align*}
- \left( \ddot{D}_{xx} \beta_x - \alpha \dddot{F}_{xx} w_{xx} \right)_{,x} + \dddot{A}_{xx} (w_x + \beta) \\
+ \Lambda_{xx} \left[ \dddot{D}_{xx} \beta_x - \alpha \dddot{F}_{xx} w_{xx} \right]_{,xxx} - \dddot{A}_{xx} (w_x + \beta)_{,x} \\
+ \Lambda_{xx} \left[ - \left( \dddot{A}_{xx} + 6\alpha \dddot{D}_{zz} \right) (w_x + 2\beta)_{,x} + 3\alpha \dddot{D}_{zz} w_{xxx} \right] \\
- \Lambda_{zz} \left[ \dddot{A}_{xx} \beta_x - 3\alpha \dddot{D}_{zz} w_{xx} \right]_{,x} - 36\alpha^2 \dddot{D}_{zz} (w_x + \beta) 
= 0,
\end{align*}
\]
\[- \alpha \left( \tilde{F}_{xx} \beta_{,x} - \alpha H_{xx} w_{,xx} \right)_{,xx} - \tilde{A}_{xx} (w_{,x} + \beta)_{,x} \]
\[+ \Lambda_{xx} \left[ \alpha \left( \tilde{F}_{xx} \beta_{,x} - \alpha H_{xx} w_{,xx} \right)_{,xxx} + \tilde{A}_{xx} (w_{,x} + \beta)_{,xxx} \right] \]
\[+ \Lambda_{zz} \left[ -6 \alpha D_{zz} w_{,xxx} + \left( \tilde{A}_{zz} + 3 \alpha \tilde{D}_{zz} \right) \beta_{,xxx} \right] \]
\[+ \Lambda_{xz} \left[ -3 \alpha \left( \tilde{D}_{xz} \beta_{,x} - 3 \alpha F_{xz} w_{,xx} \right)_{,xxx} - 36 \alpha^2 D_{xz} (w_{,x} + \beta)_{,x} \right] - t_z = 0. \tag{36} \]

where
\[
\tilde{D}_{xx} = \tilde{D}_{xx} - \alpha \tilde{F}_{xx}, \quad \tilde{A}_{zz} = \tilde{A}_{zz} - 3 \alpha \tilde{D}_{zz}, \quad \tilde{A}_{xz} = \tilde{A}_{xz} - 3 \alpha \tilde{D}_{xz}. \tag{37} \]

By considering the properties of isotropic material, the anisotropic TSD beam formulation is simplified to the one for isotropic TSD beam [14]. Moreover, the current gradient elastic TSD beam formulation is reduced to the classical TSD beam theory once the anisotropic length scale tensor \((\Lambda_{mn})\) is set equal to zero.

### 3.2. Anisotropic Euler-Bernoulli beam

The displacement field of the Euler-Bernoulli beam is assumed as
\[
u_{x}(x, z) = -zw_{,x}(x),
\]
\[u_{z}(x) = w(x) \tag{38} \]

According to equation (2), the only nonzero component of the strain tensor is
\[
\varepsilon_{xx} = -zw_{,xx}. \tag{39} \]

Following a similar procedure explained in the previous section, the variation of the strain energy for Euler-Bernoulli beam reads
\[
\delta U_{\Omega} = - \int_{0}^{L} \left[ (M_{xx} + \Lambda_{zz} N_{xx}^z) \delta w_{,xx} + \Lambda_{xx} M_{xx,xx} \delta w_{,xxx} \right] dx, \tag{40} \]

where the resultant \(M_{xx}\) and \(N_{xx}^z\) are defined in equations (20) and (21). Employing the principle of virtual work (14), the governing equation for anisotropic Euler-Bernoulli beam within the framework of strain gradient elasticity theory is obtained as
\[
-M_{xx,xx} + \Lambda_{xx} M_{xx,xxx} - \Lambda_{zz} N_{xx,xx}^z - t_z = 0. \tag{41} \]

Furthermore, the boundary conditions at \(x = 0\) and \(x = L\) are
\[
\begin{align*}
\{ M_{xx,x} - \Lambda_{xx} M_{xx,xxx} + \Lambda_{zz} N_{xx,xx}^z = 0 \} & \quad \text{or} \quad \delta w = 0 \\
\{ -M_{xx} + \Lambda_{xx} M_{xx,xx} - \Lambda_{zz} N_{xx,x}^z = 0 \} & \quad \text{or} \quad \delta w_{,x} = 0 \\
\{ -\Lambda_{xx} M_{xx,x,xx} = 0 \} & \quad \text{or} \quad \delta w_{,xx} = 0
\end{align*} \tag{42} \]

For a simply supported edge, the boundary conditions of the Euler-Bernoulli beam are
\[
\begin{align*}
1: \quad w &= 0 \\
2: \quad -M_{xx} + \Lambda_{xx} M_{xx,xx} - \Lambda_{zz} N_{xx,x}^z &= 0 \\
3: \quad \{ -\Lambda_{xx} M_{xx,x,xx} = 0 \} & \quad \text{or} \quad \delta w_{,xx} = 0
\end{align*} \tag{43} \]

whereas for a clamped edge, the boundary conditions take the form
\[
\begin{align*}
1: \quad w &= 0 \\
2: \quad w_{,x} &= 0 \\
3: \quad \{ -\Lambda_{xx} M_{xx,x} = 0 \} & \quad \text{or} \quad \delta w_{,xx} = 0.
\end{align*} \tag{44} \]
The governing equation of the Euler-Bernoulli beam (41) can indeed be obtained from the governing equations of the shear-deformable beam (31) after setting $\alpha = 0, \beta(x) = -w_x$ [28]. However, the boundary conditions of the Euler-Bernoulli beam (42) cannot be derived directly by simplifying the boundary conditions of the shear-deformable beam (32). In fact, $\alpha = 0, \beta(x) = -w_x$ should be substituted in the variation of the strain energy (26), and after application of the Green theorem on $w_x$, the correct form of the boundary conditions for the Euler-Bernoulli beam (42) can be obtained.

In terms of deflection, the differential equation (41) reads

$$
(D_{xx} w_{xx})_{xx} - \Lambda_{xx} (D_{xx} w_{xx})_{xxxx} + \Lambda_{zz} (A_{xx} w_{xx})_{xx} - t_x = 0. 
$$

(45)

By setting $\Lambda_{xx} = \Lambda_{zz} = l^2$, equilibrium equation (45) and boundary conditions (42) are reduced to those for isotropic beam [14]. The coefficients $A_{xx}$ and $D_{xx}$ are given in equation (24).

4. Special cases

In this section, the general anisotropic beam formulation is simplified to special types of anisotropy (including orthotropy and transverse isotropy) which are generally of more practical use. In order to simplify the general anisotropy to the specific cases, the Voigt notation is employed [29]

$$
C_{ijkl} \rightarrow C_{st}, \ s,t \rightarrow 1,2,...,6 : 11 \rightarrow 1, \ 22 \rightarrow 2, \ 33 \rightarrow 3, \ 23 \rightarrow 4, \ 13 \rightarrow 5, \ 12 \rightarrow 6. 
$$

(46)

This "shorthand" notation leads to more simplified expressions.

4.1. Orthotropic beams

Orthotropic materials, can be assumed to be composed of orthorhombic crystals. The matrix of anisotropic length scale for an orthorhombic crystal reads (Appendix A)

$$
\Lambda_{mn} = \begin{bmatrix}
\Lambda_{xx} & 0 & 0 \\
0 & \Lambda_{yy} & 0 \\
0 & 0 & \Lambda_{zz}
\end{bmatrix}
\Lambda_{xx} > 0, \ \Lambda_{yy} > 0, \ \Lambda_{zz} > 0. 
$$

(47)

Using Voigt notation, the elastic modulus tensor for orthotropic material is given by

$$
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66}
\end{bmatrix}
$$

(48)

Consequently, the components of the stress tensors (9) are simplified to

$$
\tau_{xx} = C_{11}\varepsilon_{xx}, \ \tau_{xz} = \tau_{zx} = 2C_{55}\varepsilon_{xz}, \\
\tau_{xxz} = \Lambda_{xx}\tau_{xx,x}, \ \tau_{xxz} = \Lambda_{xz}\tau_{xx,z}, \ \tau_{xz} = \Lambda_{zz}\tau_{xz,z}, \ \tau_{xz} = \Lambda_{zz}\tau_{xz,z}.
$$

(49)

The stress resultants are given by equations (22), while

$$
(A_{xxz}, D_{xxz}, F_{xxz}, H_{xxz}) = \int_A C_{11} (1, z^2, z^4, z^6) \ dA, \ A_{zz} = D_{zz} = F_{zz} = 0, \\
(A_{zz}, D_{zz}, F_{zz}) = \int_A C_{55} (1, z^2, z^4) \ dA
$$

(50)
For an orthotropic TSD beam, the governing equilibrium equations (35) and (36) are simplified to

\begin{align*}
&- \left( \tilde{D}_{xx} \beta_{,x} - \alpha \tilde{F}_{xx} w_{,xx} \right)_{,x} + \tilde{A}_{zz} (w_{,x} + \beta) \\
&+ \Lambda_{xx} \left[ \left( \tilde{D}_{xx} \beta_{,x} - \alpha \tilde{F}_{xx} w_{,xx} \right)_{,xxx} - \tilde{A}_{zz} (w_{,x} + \beta)_{,xxx} \right] \\
&- \Lambda_{zz} \left[ \tilde{A}_{zz} \beta_{,x} - 3 \alpha \tilde{D}_{xx} w_{,xx} \right]_{,x} - 36 \alpha^2 D_{zz} (w_{,x} + \beta) = 0,
\end{align*}

(51)

\begin{align*}
&- \alpha (\tilde{F}_{xx} \beta_{,x} - \alpha H_{xx} w_{,xx})_{,xx} - \tilde{A}_{zz} (w_{,x} + \beta)_{,x} \\
&+ \Lambda_{xx} \left[ \alpha (\tilde{F}_{xx} \beta_{,x} - \alpha H_{xx} w_{,xx})_{,xxx} + \tilde{A}_{zz} (w_{,x} + \beta)_{,xxx} \right] \\
&+ \Lambda_{zz} \left[ -3 \alpha \left( \tilde{D}_{xx} \beta_{,x} - 3 \alpha \tilde{F}_{xx} w_{,xx} \right)_{,xxx} - 36 \alpha^2 D_{zz} (w_{,x} + \beta)_{,x} \right] - t_x = 0
\end{align*}

(52)

and the boundary conditions at \( x = 0 \) and \( x = L \) read

\begin{align*}
\{ \hat{M}_{xx} + \lambda_{xx} \left( -\hat{M}_{xx,xx} + \hat{N}_{zz,xx} \right) + \Lambda_{zz} \hat{N}_{zz} = 0 \} \quad \text{or} \quad \delta \beta = 0 \\
\{ \lambda_{xx} \hat{M}_{xx,xx} = 0 \} \quad \text{or} \quad \delta \beta_{,x} = 0 \\
\{ \alpha \hat{F}_{xx,xx} + \hat{N}_{zz} + \lambda_{xx} \left( -\alpha \hat{P}_{xx,xxx} - \hat{N}_{zz,xx} \right) \} \quad \text{or} \quad \delta w = 0 \\
\{ + \Lambda_{zz} \left( 3 \alpha R_{zz,xx,xx} - 6 \alpha M_{zz,xx} \right) = 0 \} \\
\{ -\alpha \hat{P}_{xx} + \lambda_{xx} \left( \alpha \hat{P}_{xx,xx} + \hat{N}_{zz,xx} \right) - 3 \alpha \Lambda_{zz} R_{zz} = 0 \} \quad \text{or} \quad \delta w_{,x} = 0 \\
\{ -\alpha \lambda_{xx} \hat{P}_{xx,xx} = 0 \} \quad \text{or} \quad \delta w_{,xx} = 0
\end{align*}

(53)

Proper selection of the boundary conditions for simply supported and clamped edges are explained in the previous section.

### 4.2. Transversely isotropic beams

For transversely isotropic materials, length scale effects can be simulated by considering hexagonal crystal. The matrix of anisotropic length scale of hexagonal crystal reads (Appendix A)

\[
\Lambda_{mn} = \begin{bmatrix}
\Lambda_{xx} & 0 & 0 \\
0 & \Lambda_{xx} & 0 \\
0 & 0 & \Lambda_{zz}
\end{bmatrix} \quad \Lambda_{xx} > 0, \quad \Lambda_{zz} > 0.
\]

(54)

The elastic modulus tensor for a transversely isotropic material is

\[
C = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & \alpha (C_{11} - C_{12})/2
\end{bmatrix}
\]

(55)

Comparing the elastic modulus tensors of orthotropic and transversely isotropic materials (equations (48) and (55)) and the matrices of anisotropic length scale for these materials (equations (47) and (54)) shows that the differential equations and boundary conditions for a transversely isotropic TSD beam can be obtained by simply setting \( C_{55} = C_{44} \) in equations (51), (52) and (53).
4.3. Isotropic beams

For an isotropic material, the elastic modulus tensor is

\[
C = \begin{bmatrix}
\lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\]  
(56)

where \(\lambda\) and \(\mu\) are Lamé constants. For an isotropic beam with a large aspect ratio, the Poisson effect may be neglected for the sake of simplicity [30]. Consequently, the coefficients (24) take the form

\[
(A_{xx}, D_{xx}, F_{xx}, H_{xx}) = \int_A E \left(1, z^2, z^4, z^6\right) dA, \quad A_{zz} = D_{zz} = F_{zz} = 0,
\]

\[
(A_{xz}, D_{xz}, F_{xz}) = \int_A \mu \left(1, z^2, z^4\right) dA
\]

(57)

where \(E\) is the modulus of elasticity. The matrix of length scale effect for isotropic material is (Appendix A)

\[
\Lambda_{ij} = l^2 \delta_{ij}, \quad i, j \in \{x, y, z\}
\]

(58)

where \(\delta_{ij}\) denote the components of the unit second-order tensor (i.e. Kronecker delta). Using equations (57) and (58), the equilibrium equations of an isotropic TSD beam read [14]

\[
- \left(\tilde{D}_{xx} \beta_{,x} - \alpha \bar{F}_{xx \beta_{,xx} w_{,xx}}\right)_{,x} + \bar{A}_{xz} (w_{,x} + \beta) + l^2 \left[\left(\tilde{D}_{xx} \beta_{,x} - \alpha \bar{F}_{xx \beta_{,xx} w_{,xx}}\right)_{,xx} - \bar{A}_{xz} (w_{,x} + \beta)_{,xx}\right] - \left(\bar{A}_{xz} \beta_{,x} - 3\alpha \tilde{D}_{xx} w_{,xx}\right)_{,x} + 36\alpha^2 D_{xz} (w_{,x} + \beta) = 0,
\]

\[
- \left(\bar{F}_{xx} \beta_{,x} - \alpha H_{xx \beta_{,xx} w_{,xx}}\right)_{,xx} - \bar{A}_{xz} (w_{,x} + \beta)_{,x} + l^2 \left[\alpha \left(\bar{F}_{xx} \beta_{,x} - \alpha H_{xx \beta_{,xx} w_{,xx}}\right)_{,xxx} + \bar{A}_{xz} (w_{,x} + \beta)_{,xxx}\right] - 3\alpha \left(\tilde{D}_{xx} \beta_{,x} - 3\alpha \bar{F}_{xx \beta_{,xx} w_{,xx}}\right)_{,xx} - 36\alpha^2 D_{xz} (w_{,x} + \beta)_{,x} - t_z = 0.
\]

(59)

(60)

The corresponding boundary conditions are given in equations (53) in which the simplified coefficients (57) and the matrix of length scale effect (58) should be taken into account.

5. Examples and discussion

In the previous sections, the general anisotropic beams as well as some specific anisotropies are formulated within strain gradient elasticity. In order to elaborate the size effects on beam structures, it is interesting to solve the aforementioned system of governing equations. For many cases, the solution can only be obtained by using numerical methods. However for some specific cases, the analytical solution is feasible. In this section, analytical solution to a simply supported TSD and Euler-Bernoulli beam and a clamped Euler-Bernoulli beam are presented.

In the following examples, a beam composed of crystal TiSi\(_2\) is studied. For simplification, we normalize all lengths with the length of the beam. Therefore, the results demonstrate the behaviour of uni-length beams composed of TiSi\(_2\) crystal with the assumption of different internal length scales.
5.1. Simply supported anisotropic beam

We consider a simply supported anisotropic TSD beam which is subjected to lateral load \( t_z(x) \) on its upper surface. The boundary conditions (33) for a simply supported beam read

\[
\begin{align*}
1 : & \quad \hat{M}_{xx} + \Lambda_{xx} \left( -\hat{M}_{xx,x} + \hat{N}_{xz,x} \right) \\
2 : & \quad \beta_x = 0 \\
3 : & \quad w = 0 \\
4 : & \quad \begin{cases} \\
\alpha P_{xx} + \Lambda_{xx} \left( \alpha P_{xx,xx} + \hat{N}_{xz,x} \right) \\
+ \Lambda_{xz} \left( \alpha P_{xz,xx} + \hat{N}_{xz} - 3\alpha R_{xz,x} \right) - 3\alpha \Lambda_{zz} \hat{R}_{xz} = 0
\end{cases} \\
5 : & \quad w_{,xx} = 0
\end{align*}
\]

The governing equations (35) and (36) together with the boundary conditions (61) can have a solution in the form of

\[
w(x) = \sum_{n=1}^{\infty} w_n \sin\left( \frac{n\pi x}{L} \right), \quad \beta(x) = \sum_{n=1}^{\infty} \beta_n \cos\left( \frac{n\pi x}{L} \right)
\]

The load \( t_z(x) \) can similarly be expanded as

\[
t_z(x) = \sum_{n=1}^{\infty} t_n \sin\left( \frac{n\pi x}{L} \right)
\]

where the coefficients \( t_n \) are

\[
t_n = \frac{2}{L} \int_0^L t_z(x) \sin\left( \frac{n\pi x}{L} \right) dx.
\]

Substituting (62) and (63) into (35) and (36) leads to

\[
\begin{bmatrix} k_1 & k_2 & k_3 \\ k_2 & k_3 & k_1 \end{bmatrix} \begin{bmatrix} \beta_n \\ w_n \end{bmatrix} = \begin{bmatrix} 0 \\ t_n \end{bmatrix}
\]

The \( k_i \) components in the matrix equation (65) are

\[
\begin{align*}
k_1 &= (1 + \Lambda_{xx} \gamma^2) a_1 + a_2 + a_3 + 2a_8, \\
k_2 &= (1 + \Lambda_{xx} \gamma^2) a_4 + \gamma a_3 + a_5 + \gamma a_8 + a_9, \\
k_3 &= (1 + \Lambda_{xz} \gamma^2) a_6 + \gamma^2 a_3 + a_7 + 2\gamma a_9
\end{align*}
\]

while \( \gamma = \frac{n\pi}{L} \) and

\[
\begin{align*}
a_1 &= \gamma^2 \bar{D}_{xz} + \bar{A}_{xz}, \\
a_2 &= \Lambda_{zz} \gamma^2 \bar{A}_{xx}, \\
a_3 &= 3\alpha^2 \Lambda_{zz} D_{xz}, \\
a_4 &= -\alpha \gamma^3 \bar{F}_{xx} + \gamma \bar{A}_{xz}, \\
a_5 &= -3\alpha \Lambda_{zz} \gamma^3 \bar{D}_{xx}, \\
a_6 &= \alpha^2 \bar{H}_{xx} \gamma^4 + \gamma^2 \bar{A}_{xz}, \\
a_7 &= 9\alpha^2 \Lambda_{zz} \gamma^4 \bar{F}_{xx}, \\
a_8 &= \Lambda_{xz} \gamma^2 (\bar{A}_{xz} + 6\alpha \bar{D}_{zz}), \\
a_9 &= -3\alpha \Lambda_{zz} \gamma^3 \bar{D}_{zz}.
\end{align*}
\]

The amplitudes \( w_n \) and \( \beta_n \) can then be readily determined as

\[
\begin{align*}
w_n &= \frac{k_1}{k_1 k_3 - k_2^2} t_n, \\
\beta_n &= \frac{-k_2}{k_1 k_3 - k_2^2} t_n.
\end{align*}
\]

Substitution of the amplitudes (68) into equations (62) gives the analytical solution for the simply supported beam.
In a same manner, the analytical solution for a simply supported Euler-Bernoulli beam can be obtained. The boundary conditions (43) for a simply supported Euler-Bernoulli beam are selected to be

1: \( w = 0 \)
2: \(-M_{xx} + \Lambda_{xx}M_{xx,xx} - \Lambda_{zz}N_{xx}^2 = 0\)
3: \( w_{xx} = 0 \).

Using the same procedure as used for the TSD beam, \( w_n \) reads

\[
w_n = \frac{t_n}{\gamma^4 D_{xx} + \Lambda_{xx} \gamma^6 D_{xx} + \Lambda_{zz} \gamma^4 A_{xx}}.
\]

Displacement field \( w(x) \) can be determined by substitution of equation (70) into (62).

To shed more light on the effect of the internal length parameters on the behaviour of the beam, a specific case is studied. A simply supported beam which is made of orthorhombic crystal TiSi\(_2\) is subjected to a sinusoidal load \( t_z(x) = t_1 \sin\left(\frac{\pi x}{L}\right) \) on its upper surface. The elastic constants of the beam are [31]

\[
C_{11} = 317.5 \text{ GPa}, \quad C_{55} = 75.8 \text{ GPa}
\]

The dimensions of the beam are assumed to be

\[
L = 20 \text{ nm}, \quad h = b = 0.1L.
\]

Literature lacks any quantified information about the value of the anisotropic internal length scales. The length scales used here are of the order of Angstrom which is selected to demonstrate the size effect in the gradient theory. Proper experimental observations or numerical experiments such as molecular dynamics can shed light on the value of these internal length scales.

The normalized deflection and rotation of a TSD beam versus the ratio \( x/L \) for different length scale parameters are illustrated in figures (2) and (3), respectively. The classical solution (i.e. for \( \Lambda_{xx} = \Lambda_{zz} = 0 \))
is also plotted for comparison. It is observed that the nonzero length scale parameter results in the the reduction of the magnitude of deformation and rotation of the beam. This trend is in line with the observation of the behaviour of the isotropic beams (Polizzotto [18], Wang et al. [10]). The differences between the classical and non-classical solutions are remarkable which necessitates the study of size effect on anisotropic nano-structures.

Comparison of the results of two sets of internal length scales (i.e. \((\Lambda_{xx}, \Lambda_{zz}) = (0.0005L^2, 0.0001L^2)\) and \((\Lambda_{xx}, \Lambda_{zz}) = (0.0001L^2, 0.0005L^2)\)) demonstrate the importance of the crystal orientation with respect to the beam geometry. In particular, \(\Lambda_{xx} > \Lambda_{zz}\) leads to higher displacement and rotations comparing to the case of \(\Lambda_{xx} < \Lambda_{zz}\). In other words, once \(\Lambda_{xx} > \Lambda_{zz}\), the behaviour of the beam is closer to the classical prediction. Accordingly, for the manufacturing of the nano-structures, the orientation of the crystal material with respect to the specimen geometry should be taken into account.

Figures (4) and (5) also compare the effect of internal length scales of \(x\) and \(z\) direction on the behaviour of the beam. Within gradient elasticity, size effect occurs once the ratio of the structural dimension to the internal length scale is not negligible. Consequently as expected, since the ratio \(\Lambda_{zz}/h\) is higher than the ratio \(\Lambda_{xx}/L\), the displacement and rotation of the beam is more sensitive to \(\Lambda_{zz}\).
Figure 4: Variation of normalized deflection of a simply supported TSD beam versus ratio $x/L$

Figure 5: Variation of normalized rotation of a simply supported TSD beam versus ratio $x/L$

Figure (6) depicts the effect of the length scale parameters on the normalized deflection $w(x)/L$ of a simply supported Euler-Bernoulli beam. It is observed that the behaviour of the Euler-Bernoulli beam is in
line with the observation of the behaviour of the TSD beam. However, it is noted that for the aspect ratio of \( h/L = b/L = 0.1 \), TSDT should be employed.

![Graph](image)

Figure 6: Variation of normalized deflection of a simply supported Euler-Bernoulli beam versus ratio \( x/L \)

### 5.2. Clamped anisotropic Euler-Bernoulli beam

We assume an anisotropic Euler-Bernoulli beam which is clamped at both ends and is subjected to a uniform lateral load \( q \) on its upper surface. The equilibrium equation (45) is rewritten in the following dimensionless form

\[
W^{(6)}(X) - r W^{(4)}(X) + \bar{q} = 0. \tag{73}
\]

where \( X = x/L \), \( W(X) = w(x)/L \), \( W^{(n)}(X) = \frac{d^nw(X)}{dX^n} \), and

\[
r = (1 + \frac{\dot{\Lambda}_{zz} A_{xx}}{D_{xx}} L^2)/\dot{\Lambda}_{xx}, \quad \bar{q} = \frac{q L^5}{\dot{\Lambda}_{xx} D_{xx}}. \tag{74}
\]

In equation (74), \( \dot{\Lambda}_{xx} = \frac{\Lambda_{xx}}{L^2} \) and \( \dot{\Lambda}_{zz} = \frac{\Lambda_{zz}}{L^2} \). The ordinary differential equation (73) has the solution of the form

\[
W(X) = b_1 + b_2 X + b_3 X^2 + b_4 X^3 + K X^4 + b_5 e^{\sqrt{r}X} + b_6 e^{-\sqrt{r}X}. \tag{75}
\]

Here, the coefficient \( K \) reads

\[
K = \frac{\bar{q}}{24r} \tag{76}
\]

and \( b_i \) \((i = 1, 2, \ldots, 6)\) are constants which should be determined using the boundary conditions. According to equation (44), the lower-order boundary conditions for the clamped Euler-Bernoulli beam are

\[
W(0) = W(1) = 0, \quad W_X(0) = W_X(1) = 0 \tag{77}
\]
and the higher-order (nonstandard) boundary conditions are assumed as

\[ W_{XX}(0) = W_{XX}(1) = 0 \] (78)

Using the classical and higher-order boundary conditions, the constants \( b_i \) can be simply obtained by the equation

\[ b = A^{-1}B \] (79)

Here, the matrix \( A \) is

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & e^{\sqrt{r}} & e^{-\sqrt{r}} \\
0 & 1 & 0 & 0 & \sqrt{r} & -\sqrt{r} \\
0 & 1 & 2 & 3 & \sqrt{r}e^{\sqrt{r}} & -\sqrt{r}e^{-\sqrt{r}} \\
0 & 0 & 2 & 0 & r & r \\
0 & 0 & 2 & 6 & re^{\sqrt{r}} & re^{-\sqrt{r}}
\end{bmatrix}
\] (80)

and the matrix \( B \) reads

\[
B = \begin{bmatrix}
0 & -K & 0 & -4K & 0 & -12K
\end{bmatrix}^T.
\] (81)

Substitution of the coefficients \( b_i \) (79) into the expression (75) gives the analytical solution for the beam.

In order to study the effect of the length scale parameters on the behaviour of the beam, an example is presented here. A clamped Euler-Bernoulli beam is assumed to be made of orthorhombic crystal TiSi\(_2\) with elastic constant \( C_{11} = 317.5 \) GPa [31] and is subjected to a uniform load \( q \) on its upper surface. The dimensions of the beam are considered to be

\[ L = 20 \text{ nm}, \quad h = b = 0.1L, \] (82)

The normalized deflection of the beam versus the ratio \( x/L \) for different length scale parameters is plotted in figure (7). Similar to the case of simply supported beam, the size effect is significant for clamped beam subjected to uniform loading. Additionally, \( \Lambda_{xx} > \Lambda_{zz} \) results in higher displacement comparing to the case of \( \Lambda_{xx} < \Lambda_{zz} \).

Figure 7: Variation of normalized deflection of a clamped Euler-Bernoulli beam versus ratio \( x/L \)
The examples presented in this section address the problems for which the analytical solutions are feasible. The formulation derived in this paper can be accompanied with proper numerical techniques (such as finite element method [32] or isogeometric finite element method [33]) to study arbitrary anisotropic beam structures.

6. Conclusion
Anisotropic centrosymmetric third-order shear deformable (TSD) beam is studied within strain gradient elasticity theory. Higher-order differential equations together with classical and non-classical boundary conditions are derived using a variational approach. Euler-Bernoulli beam theory is also formulated for anisotropic gradient elastic beams. The resulting beam model is capable of capturing the size effect. This is essential in the new emerging technologies due to the application of nano-engineered anisotropic materials. The general anisotropic beam model is also simplified to specific cases of anisotropy including orthotropy and transverse isotropy. Any other cases of centrosymmetric anisotropy can be realized by substituting the corresponding material properties in the general formulation presented in this paper.

To study the size effects on anisotropic beam structures, the problem of a simply supported anisotropic shear deformable and Euler-Bernoulli beams and an anisotropic clamped Euler-Bernoulli beam are solved analytically. Similar to the isotropic case, it is observed that increasing the values of the internal length scales decreases the deformation of the anisotropic beam.

Moreover, the orientation of the crystal structure of the material with respect to the beam geometry is proved to be of considerable importance in the behaviour of the beam. In particular, once the internal length scale of the axial direction of the beams is higher than the length scale of the transverse direction, higher displacement and rotation are predicted. This observation can be considered for the design and manufacturing of the miniature structures from anisotropic materials.

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References
Appendix A

For different classes of crystal symmetry [34], the tensor of anisotropic length scale $\Lambda_{mn}$ and the corresponding conditions for positive definiteness are [22]:

Triclinic crystal:

$$\Lambda_{mn} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{bmatrix}$$

(83)

$$\Lambda_{11} > 0, \quad \begin{vmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{12} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{vmatrix} > 0.$$

Monoclinic crystal (standard orientation 2||b):

$$\Lambda_{mn} = \begin{bmatrix} \Lambda_{11} & 0 & \Lambda_{13} \\ 0 & \Lambda_{22} & 0 \\ \Lambda_{13} & 0 & \Lambda_{33} \end{bmatrix}$$

(84)

$$\Lambda_{11} > 0, \quad \Lambda_{22} > 0, \quad \Lambda_{33} > 0, \quad \Lambda_{11}\Lambda_{33} - \Lambda_{13}^2 > 0.$$

Monoclinic crystal (orientation 2||c):

$$\Lambda_{mn} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & 0 \\ \Lambda_{12} & \Lambda_{22} & 0 \\ 0 & 0 & \Lambda_{33} \end{bmatrix}$$

(85)

$$\Lambda_{11} > 0, \quad \Lambda_{22} > 0, \quad \Lambda_{33} > 0, \quad \Lambda_{11}\Lambda_{22} - \Lambda_{12}^2 > 0.$$

Orthorhombic crystal:

$$\Lambda_{mn} = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{22} & 0 \\ 0 & 0 & \Lambda_{33} \end{bmatrix}$$

(86)

$$\Lambda_{11} > 0, \quad \Lambda_{22} > 0, \quad \Lambda_{33} > 0.$$

Tetragonal, hexagonal, and trigonal crystal:

$$\Lambda_{mn} = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ 0 & \Lambda_{11} & 0 \\ 0 & 0 & \Lambda_{33} \end{bmatrix}$$

(87)

$$\Lambda_{11} > 0, \quad \Lambda_{33} > 0.$$
Cubic crystal:

\[
\Lambda_{mn} = \begin{bmatrix}
\Lambda_{11} & 0 & 0 \\
0 & \Lambda_{11} & 0 \\
0 & 0 & \Lambda_{11}
\end{bmatrix} \quad \Lambda_{11} > 0.
\]  

(88)