PHYSICAL REVIEW E **70**, 046616 (2004)

Low-frequency spatial dispersion in wire media

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(Received 1 July 2004; published 29 October 2004)

This work is dedicated to the theoretical analysis of wire media, i.e., lattices of perfectly conducting wires consisting of two or three doubly periodic arrays of parallel wires which are orthogonal to one another. An analytical method based on the local field approach is used. The explicit dispersion equations are presented and studied. The possibility of introducing a dielectric permittivity is discussed. The theory is validated by comparison with the numerical data available in the literature.

DOI: 10.1103/PhysRevE.70.046616 PACS number(s): 41.20.Jb, 42.70.Qs, 77.22.Ch, 77.84.Lf

I. INTRODUCTION

In the recent years the periodic metallic lattices have found many applications in both optical and microwave ranges (see, for example, in Ref. [1,2]). However, some fundamental problems have not been resolved yet, even for typical metallic electromagnetic crystals. One of them is the problem of low-frequency spatial dispersion in wire media (WM). The low-frequency spatial dispersion of a simple wire medium (a doubly periodic regular array of parallel wires) has been studied only recently in Ref. [3]. In the present paper this theory is generalized for double and triple wire media. The study of spatial dispersion effects in the above-mentioned variants of WM has been started in Ref. [4]. However, this study (based on the numerical approach) is far from complete. Our theory significantly complements the results of Ref. [4]. It is an analytical one, and in order to validate it, a comparison to the results from Ref. [4] is carried out.

The unit cells of the lattices under study are shown in Fig. 1. They consist of two ($2d$, or double wire medium) or three ($3d$, or triple wire medium) doubly periodic regular arrays of parallel infinite wires which are orthogonal to one another. The wires are assumed to be perfectly conducting. The host medium is a uniform lossless dielectric with permittivity ε_0 and permeability μ_0. We denote the radii of wires directed along x, y, and z axes as r_x, r_y, and r_z, respectively. The periods of the lattice along x, y, and z axes are denoted a, b, and c, respectively. The lattices are spatially shifted with respect to each other by a half period (see Fig. 1). The wires axis positions in the chosen coordinate system are determined by the following equations:

(i) the x-directed wires: $y=bn+b/2$ and $z=cl+c/2$,

(ii) the y-directed wires: $x=am+a/2$ and $z=cl$,

(iii) the z-directed wires: $x=am$ and $y=bn$,

where m, n, and l are integers.

In order to model an electromagnetic response of a wire, we apply the local field approach. We assume that the wire diameters are small compared to the wavelength. Thus, every wire can be described in terms of effective linear current referred to the wire axis. The wire with radius r_0 oriented along a unit vector $\mathbf{d}(|\mathbf{d}|=1)$ can be characterized by a "polarizability" α, relating the complex amplitude I of the induced current and the local electric field $\mathbf{E}^{\rm loc}$,

$$I = \alpha(r_0,k,\mathbf{q}\cdot\mathbf{d})\mathbf{E}^{\rm loc}\cdot\mathbf{d}. \quad (1)$$

Here k is the wave number of the host medium and $\mathbf{q}\cdot\mathbf{d}=q_\parallel$ is the longitudinal component of the wave vector of the propagating mode. The following expression for α was obtained in Ref. [5]:

$$\alpha(r_0,k,q_\parallel) = \left[\frac{\eta(k^2-q_\parallel^2)}{4k}\left(1-j\frac{2}{\pi}\left\{\log\frac{\sqrt{k^2-q_\parallel^2}r_0}{2}+\gamma\right\}\right)\right]^{-1}, \quad (2)$$

where $\gamma\approx 0.5772$ is the Euler constant and $\eta=\sqrt{\mu_0/\varepsilon_0}$ is the wave impedance of the host medium.

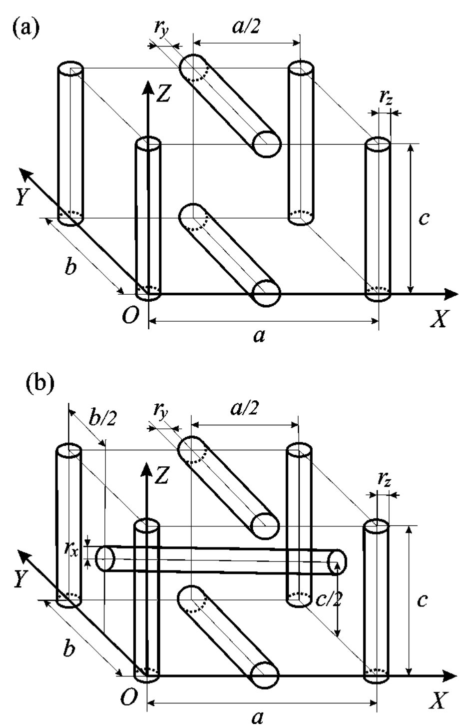

FIG. 1. Unit cells of double wire medium (a) and triple wire medium (b).

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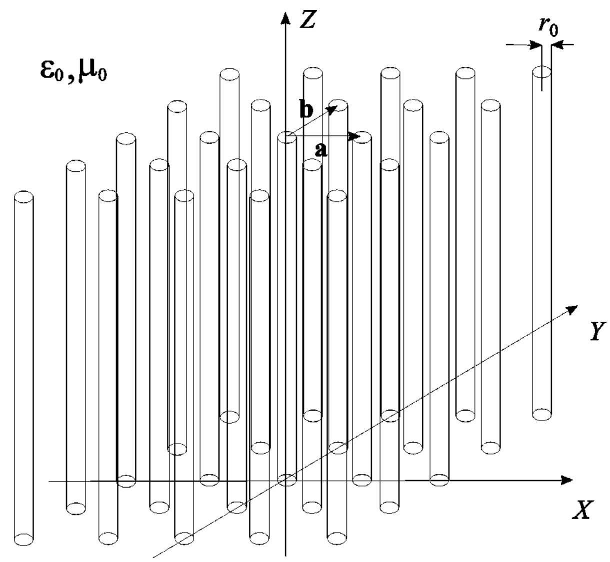

FIG. 2. Simple wire media: a doubly periodic lattice of parallel ideally conducting thin wires.

Let the eigenmode under consideration have the wave vector $\mathbf{q}=(q_x,q_y,q_z)^T$. Expressing the local field produced by all wires except the reference one through the current induced in the reference wire, we obtain the dispersion equation. It relates the components of $\mathbf{q}$ to k (i.e., to frequency ω). Then we can introduce effective material parameters of the wire medium which fit this dispersion equation. In this paper we widely use the results obtained in our preceding papers [3,5] for a simple WM. Therefore, in the next section a very short overview of those works is presented.

II. SIMPLE WIRE MEDIA

The geometry of a simple wire medium comprising the z-directed wires is shown in Fig. 2.

The currents in the wire with numbers (m,n) (counted along the x and y axes, respectively) are related to current I induced in the reference (zeroth) wire through the wave vector $\mathbf{q}$,

$$I_{m,n} = Ie^{j(q_x am+q_y bn)}. \tag{3}$$

Since the field re-radiated by the wire (m,n) is proportional to $I_{m,n}$, the local electric field acting on the zeroth wire can be expressed in terms of the so-called *dynamic interaction constant C*:

$$E_z^{\text{loc}} = C(k,q_x,q_y,q_z,a,b)I, \tag{4}$$

where (see in Refs. [3,5]):

$$C(k,q_x,q_y,q_z,a,b) = -\frac{\eta(k^2-q_z^2)}{4k} \times \sum_{(m,n)\neq(0,0)} [H_0^{(2)}(\sqrt{k^2-q_z^2}R_{m,n})e^{-j(q_x am+q_y bn)}], \tag{5}$$

$R_{m,n}=\sqrt{(am)^2+(bn)^2}$ and all (m,n) except $m=n=0$ are summed up. Expression (5) can be rewritten in the following form [5]:

$$C(k,q_x,q_y,q_z,a,b) = \frac{\eta(k^2-q_z^2)}{2jkb}\Bigg[\frac{1}{k_x^{(0)}}\frac{\sin k_x^{(0)}a}{\cos k_x^{(0)}a-\cos q_x a} + \sum_{n\neq 0}\left(\frac{1}{k_x^{(n)}}\frac{\sin k_x^{(n)}a}{\cos k_x^{(n)}a-\cos q_x a}-\frac{b}{2\pi|n|}\right) + \frac{b}{\pi}\left(\log\frac{\sqrt{k^2-q_z^2}b}{4\pi}+\gamma\right)+j\frac{b}{2}\Bigg], \tag{6}$$

where

$$k_x^{(n)} = -j\sqrt{\left(q_y+\frac{2\pi n}{b}\right)^2+q_z^2-k^2}, \tag{7}$$

and we choose $\text{Re}\{\sqrt{()}\}>0$. Those formulas physically correspond to the representation of the WM as a set of parallel grids (of z-directed wires) located parallel to one another with period a along the x axis (see also Fig. 4 for the case of 2d WM). Every grid radiates the spectrum of Floquet harmonics with wave vectors $(k_x^{(n)},q_y+2\pi n/b,q_z)$. The series with summation over n on the right-hand side of (6) describes the contribution of the high-order Floquet modes to the electromagnetic interaction of those grids. The dispersion equation follows from (1) and (4):

$$[\alpha^{-1}(r_0,k,q_z)-C(k,q_x,q_y,q_z,a,b)]I=0. \tag{8}$$

Taking into account expressions (2) and (6) one can rewrite (8) in the following form:

$$(k^2-q_z^2)\Bigg[\frac{1}{\pi}\log\frac{b}{2\pi r_0}+\frac{1}{bk_x^{(0)}}\frac{\sin k_x^{(0)}a}{\cos k_x^{(0)}a-\cos q_x a} + \sum_{n\neq 0}\left(\frac{1}{bk_x^{(n)}}\frac{\sin k_x^{(n)}a}{\cos k_x^{(n)}a-\cos q_x a}-\frac{1}{2\pi|n|}\right)\Bigg]I=0. \tag{9}$$

Note, that this is a real-valued dispersion equation. The real part of polarizability (2) which is responsible for radiation reaction is cancelled by real part of interaction constant (6).

Equation (9) has three types of solutions.

(1) Ordinary waves, in the case where $I=0$ in (9). They have no electric field component along wires ($E_z=0$) and propagate without interaction with the lattice. Their dispersion plot corresponds to the host medium and is shown in Fig. 3 by thin lines.

(2) Extraordinary waves, in the case where the expression in square parentheses in (9) equals zero. They correspond to the nonzero currents $I\neq 0$ and have the nonzero longitudinal component of electric field $E_z\neq 0$. Their dispersion properties are described in detail in Ref. [5]; their dispersion curves are presented in Fig. 3 by thick lines.

(3) Transmission-line modes (TLM), in the case where $(k^2-q_z^2)=0$ in (9). Those waves propagate along the wires; they are TEM waves ($E_z=0$), but $I\neq 0$. Their dispersion equation $q_z^2=k^2$ has no restriction for components q_x,q_y, and the phase shift of the currents in the adjacent wires can be arbitrary [3].

Under the quasistatic limit $ka\ll 2\pi$ and $|q|a\ll 2\pi$, the dispersion equation for extraordinary waves transforms to

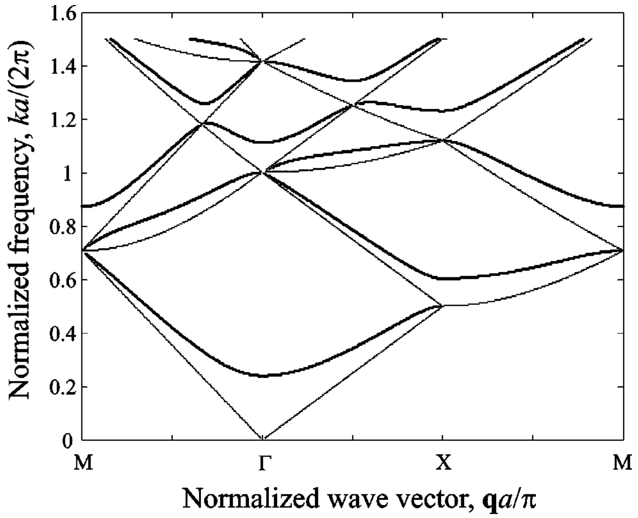

FIG. 3. Dispersion curves of wire media with filling ratio $f=\pi r_0^2/a^2=0.001$ (square lattice). Thin lines, ordinary waves; thick lines, extraordinary waves. TLM are not shown.

$$q^2 = q_x^2 + q_y^2 + q_z^2 = k^2 - k_0^2, \tag{10}$$

where the following notations are used:

$$k_0^2 = \frac{2\pi/s^2}{\log\dfrac{s}{2\pi r_0} + F(r)}, \tag{11}$$

$s=\sqrt{ab}$, $r=a/b$, and $F(r)=F(1/r)$ is given by

$$F(r) = -\frac{1}{2}\log r + \sum_{n=1}^{+\infty}\left(\frac{\coth(\pi n r) - 1}{n}\right) + \frac{\pi r}{6}. \tag{12}$$

Parameter k_0 corresponds to the effective plasma frequency of the lattice $\omega_0 = k_0/\sqrt{\varepsilon_0\mu_0}$. For square lattices $a=b$ one has $F(1)=0.5275$. Comparing (10) with the well known dispersion equation of uniaxial dielectrics, we obtain an effective relative permittivity $\bar{\bar{\varepsilon}}$ of 1d WM in the following form:

$$\bar{\bar{\varepsilon}} = \varepsilon \mathbf{z}_0\mathbf{z}_0 + \mathbf{x}_0\mathbf{x}_0 + \mathbf{y}_0\mathbf{y}_0, \tag{13}$$

$$\varepsilon(k,q_z) = 1 - \frac{k_0^2}{k^2 - q_z^2}. \tag{14}$$

The dependence of dielectric permittivity on q_z given by (14) does not disappear until the frequency becomes zero. This means that wire media have low-frequency spatial dispersions. There is no low-frequency spatial dispersion for the extraordinary waves in the only case where the wave propagates across the wires ($q_z=0$). At low frequencies the propagation of those waves can be described in terms of plasma-like permittivity $\varepsilon = 1 - k_0^2/k^2$ (see also Ref. [6]). Relative to those waves the wire medium behaves as a cold, nonmagnetized plasma (a continuous dielectric medium). In other propagation directions the wire medium behaves differently. In Ref. [3] we discuss the importance of the low-frequency spatial dispersion in 1d wire media. Below, we show this phenomenon theoretically in 2d and 3d WM.

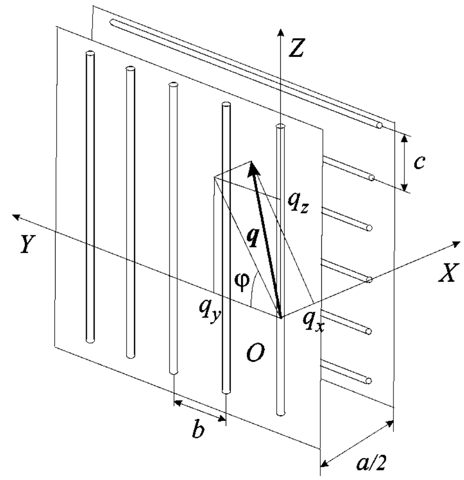

FIG. 4. The structure of a double wire medium is represented as a set of planar wire grids.

III. DOUBLE WIRE MEDIA

Now, let us consider a double wire medium which is comprised of y-directed and z-directed wires. It is shown in Fig. 4 as a set of parallel grids located along the x axis. Below, the local field approach is going to be applied taking the same approximation as was done in our earlier work [7] (where we have studied a doubly negative metamaterial in a similar way). Thus, the approximation is as follows: the electromagnetic field produced by a single grid of wires at a distance from the grid $a/2$ is considered as a field of a sheet of the average current $\mathbf{J}$. That approximation is accurate enough under the condition where the wavelength in the matrix is large compared to the grid periods ($kb \ll 2\pi$ and $kc \ll 2\pi$) and period a is not smaller than periods b, c. In this case the y-oriented grids interact with the z-oriented grids by the fundamental Floquet harmonic. Other harmonics are evanescent and their contribution to this cross-polarized interaction is negligible.

We can express the (m,n)-numbered z-directed current through the reference (zeroth) z-directed current I_z,

$$I_z^{(m,n)}(z) = I_z e^{j(q_x am + q_y bn + q_z z)}. \tag{15}$$

The same rule holds for y-directed currents,

$$I_y^{(m,l)}(z) = I_y e^{j(q_x am + q_z cl + q_y y)}. \tag{16}$$

The currents I_z and I_y are related to the local electric fields acting on the z- and y-directed reference wires $E_z^{\rm loc}$ and $E_y^{\rm loc}$ through polarizabilities $\alpha_{z,y}$,

$$I_z = \alpha_z E_z^{\rm loc}, \quad I_y = \alpha_y E_y^{\rm loc}, \quad \alpha_{y,z} \equiv \alpha(r_{y,z}, k, q_{y,z}), \tag{17}$$

which fit (2). Both $E_z^{\rm loc}$ and $E_y^{\rm loc}$ contain contributions of z and y arrays,

$$E_z^{\rm loc} = E_z^{(z)} + E_z^{(y)}, E_y^{\rm loc} = E_y^{(y)} + E_y^{(z)}. \tag{18}$$

Copolarized terms $E_z^{(z)}$ and $E_y^{(y)}$ could be expressed through I_z and I_y applying (4):

$$E_z^{(z)} = C_{zz} I_z, C_{zz} \equiv C(k, q_x, q_y, q_z, a, b), \tag{19}$$

$$E_y^{(y)} = C_{yy}I_y, C_{yy} \equiv C(k, q_x, -q_z, q_y, a, c). \qquad (20)$$

The cross components $E_z^{(y)}$ and $E_y^{(z)}$ could be expressed through I_y and I_z, respectively,

$$E_z^{(y)} = C_{zy}I_y, \quad E_y^{(y)} = C_{yz}I_z. \qquad (21)$$

The cross-polarized interaction factors $C_{yz,zy}$ are evaluated below. Substituting equations (19)–(21) into (17) we obtain a system of equations

$$(C_{yy} - \alpha_y^{-1})I_y + C_{yz}I_z = 0,$$

$$C_{zy}I_y + (C_{zz} - \alpha_z^{-1})I_z = 0. \qquad (22)$$

First of all, it should be noticed that the solution of (22) when $I_{y,z}=0$ corresponds to the ordinary waves with polarization along the x axis, propagating in the plane (y-z). The dispersion equation for such waves is $q_y^2+q_z^2=k^2$, $q_x=0$. Setting the determinant of (22) equal to zero, we obtain a dispersion equation for the extraordinary waves ($I_{y,z}\neq 0$),

$$(C_{yy} - \alpha_y^{-1})(C_{zz} - \alpha_z^{-1}) - C_{yz}C_{zy} = 0. \qquad (23)$$

It should be noted that the expressions in parentheses in (23) are exactly the dispersion equations for the simple wire media (from y wires and z wires, respectively).

In (23) coefficients $C_{yy,zz}$ are defined from (6), (19), and (20). Now, let us calculate coefficients C_{zy} and C_{yz} using the approximation of current sheets which has been mentioned above. The z component of the electric field produced by a sheet with surface current $\mathbf{J}_y(y,z)=\mathbf{y}_0(I_y/c)e^{j(q_y y+q_z z)}$ at the arbitrary distance x from the grid can be expressed by the formula [8],

$$E_{y,z}(x,y,z) = \frac{\eta q_y q_z}{2k_x k} J_{z,y}(y,z) e^{-jk_x|x|},$$

$$k_x \equiv -j\sqrt{q_y^2 + q_z^2 - k^2}. \qquad (24)$$

Summing up (24) over all m-numbered layers, we can write

$$C_{zy} = \frac{\eta q_y q_z}{2k_x kc} \sum_{m=-\infty}^{+\infty} e^{-jq_x am} e^{-j|m-\frac{1}{2}|k_x a}. \qquad (25)$$

The summation result is the following:

$$C_{zy} = \frac{\eta q_y q_z}{k_x kc} \frac{je^{-jq_x a/2}\cos(q_x a/2)\sin(k_x a/2)}{\cos q_x a - \cos k_x a}. \qquad (26)$$

Taking into account the phase shift between z and y grids we obtain

$$C_{yz} = e^{jq_x a} C_{zy} c/b = \frac{\eta q_y q_z}{k_x kb} \frac{je^{jq_x a/2}\cos(q_x a/2)\sin(k_x a/2)}{\cos q_x a - \cos k_x a}. \qquad (27)$$

Substituting (2), (6), and (26) into (23), we derive an explicit dispersion equation,

$$(k^2 - q_y^2)\left[\frac{1}{\pi}\log\frac{c}{2\pi r_y} + \frac{1}{ck_x}\frac{\sin k_x a}{\cos k_x a - \cos q_x a} + \sum_{n\neq 0}\left(\frac{1}{c\beta_y^{(n)}}\frac{\sin \beta_y^{(n)} a}{\cos \beta_y^{(n)} a - \cos q_x a} - \frac{1}{2\pi|n|}\right)\right] \times (k^2 - q_z^2)\left[\frac{1}{\pi}\log\frac{b}{2\pi r_z} + \frac{1}{bk_x}\frac{\sin k_x a}{\cos k_x a - \cos q_x a} + \sum_{n\neq 0}\left(\frac{1}{b\beta_z^{(n)}}\frac{\sin \beta_z^{(n)} a}{\cos \beta_z^{(n)} a - \cos q_x a} - \frac{1}{2\pi|n|}\right)\right] = \frac{4q_y^2 q_z^2}{k_x^2 bc}\left(\frac{\cos(q_x a/2)\sin(k_x a/2)}{\cos q_x a - \cos k_x a}\right)^2, \qquad (28)$$

where

$$\beta_z^{(n)} = -j\sqrt{\left(q_y + \frac{2\pi n}{b}\right)^2 + q_z^2 - k^2},$$

$$\beta_y^{(n)} = -j\sqrt{\left(q_z + \frac{2\pi n}{c}\right)^2 + q_y^2 - k^2}.$$

The signs of all square roots are chosen so that $\mathrm{Re}\{\sqrt{()}\}>0$. The dispersion equation (28) cannot be simplified (except the quasistatic limit) even in a special case where $r_y=r_z$ and $a=b=c$. In fact, the perfect square can be obtained on the left-hand side of (28) if one neglects the contribution of high-order Floquet harmonics expressed by n series. However, this approximation leads to the wrong results for the shape of isofrequencies. Therefore we do not use it.

The preliminary analysis of (28) reveals some special solutions. There are two solutions which correspond to TLM: the first is $q_y=k, q_z=0$, q_x is arbitrary; the second one is $q_z=k, q_y=0$, q_x is arbitrary. Those waves propagate either along the y wires (when the electric field averaged over the lattice unit cell is polarized along z) or along the z wires (when the averaged electric field is polarized along y). They are TEM waves as well as TLM in simple WM [3]. The component q_x is a free parameter for TLM and plays the role of a phase shift between the currents in the adjacent grids of wires [3]. At first sight, it seems strange that the electric field with non-zero z component can propagate along y *across* the z-directed wires below the "plasma" frequency (which is the cutoff frequency for such waves in 1d WM). However, it is possible. When the TLM propagates along the y wires, all grids of z wires are excited; however, the superposition of their fields exactly vanishes in the planes $x=am+a/2$, where the grids of y wires are located. This result can easily be obtained analytically for arbitrary nonzero q_x and $q_t=k$. For the same reason it is also possible for a y-polarized TLM to propagate along z.

When $q_x = \pi/a$ or $q_y q_z = 0$, the right-hand side of (28) equals zero and the equation splits into two separate equations similar to (9) and describing the extraordinary waves in two simple WM. For $q_y = 0$ (or $q_z = 0$) the absence of the interaction between two simple WM's is trivial, since the propagation holds in the plane (x-z) [or (x-y)] and the electric field is polarized orthogonally to y-directed wires (or to z wires). However, the interaction between two 1d WM's is also absent when $q_x = \pi/a$. At low frequencies $ka < 1$ the equation $q_x = \pi/a$ corresponds to the excitation of TLM *in both y and z arrays* with polarization directions alternating along x. The existence of this kind of TLM (which does not transport energy at all) is specific for 2d WM.

More detailed study of (28) requires numerical calculations and their results are presented below.

IV. TRIPLE WIRE MEDIA

Analysis of triple wire media can be carried out in the same way as described above. Similarly to (22), we obtain

$$(C_{xx} - \alpha_x^{-1})I_x + C_{xy}I_y + C_{xz}I_z = 0,$$

$$C_{yx}I_x + (C_{yy} - \alpha_y^{-1})I_y + C_{yz}I_z = 0,$$

$$C_{zx}I_x + C_{zy}I_y + (C_{zz} - \alpha_z^{-1})I_z = 0, \qquad (29)$$

where $C_{yy,zz,yz}$ are determined by (19), (20), (26), (27), and (6) and

$$C_{xx} = C(k, q_y, q_z, q_x, b, c). \qquad (30)$$

Here α_i is denoted as $\alpha_i = \alpha(r_i, k, q_i)$ and the subscript i means the Cartesian components (x, y, z). Other interaction factors are as follows:

$$C_{xy} = j\frac{\eta q_x q_y}{k_z ka}\frac{\cos(q_z c/2)\sin(k_z c/2)}{\cos q_z c - \cos k_z c}e^{jq_z c/2}, \qquad (31)$$

$$C_{xz} = j\frac{\eta q_x q_z}{k_y ka}\frac{\cos(q_y b/2)\sin(k_y b/2)}{\cos q_y b - \cos k_y b}e^{jq_y b/2}, \qquad (32)$$

$$C_{yx} = j\frac{\eta q_x q_y}{k_z kb}\frac{\cos(q_z c/2)\sin(k_z c/2)}{\cos q_z c - \cos k_z c}e^{-jq_z c/2}, \qquad (33)$$

$$C_{zx} = j\frac{\eta q_x q_z}{k_y kc}\frac{\cos(q_y b/2)\sin(k_y b/2)}{\cos q_y b - \cos k_y b}e^{-jq_y b/2}, \qquad (34)$$

$$k_y \equiv -j\sqrt{q_x^2 + q_z^2 - k^2}, \quad k_z \equiv -j\sqrt{q_x^2 + q_y^2 - k^2}.$$

There are no ordinary waves in that medium since there are no vectors orthogonal to all wires simultaneously. The determinant of (29) gives the dispersion equation:

$$(C_{xx} - \alpha_x^{-1})(C_{yy} - \alpha_y^{-1})(C_{zz} - \alpha_z^{-1})$$

$$- (C_{xx} - \alpha_x^{-1})C_{yz}C_{zy} - (C_{yy} - \alpha_y^{-1})C_{xz}C_{zx}$$

$$- (C_{zz} - \alpha_z^{-1})C_{xy}C_{yx} + C_{xy}C_{yz}C_{zx} + C_{xz}C_{zy}C_{yx} = 0. \qquad (35)$$

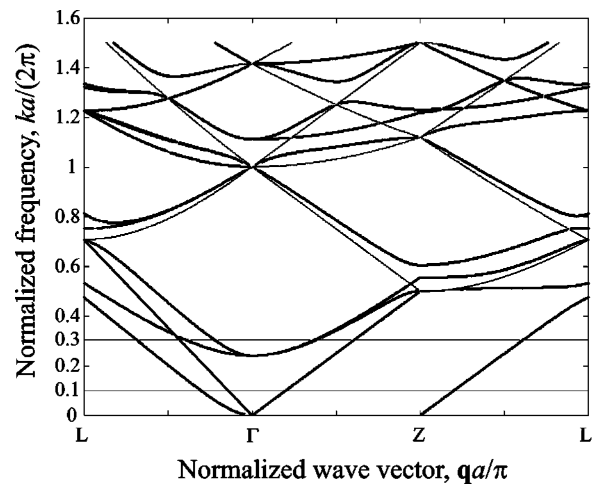

FIG. 5. Dispersion diagram of a double wire media with filling ratio $f = 2\pi r_0^2/a^2 = 0.002$ (cubical cell and equal radii). Thin lines, modes of the host medium [singular points of Eq. (28)], thick lines, modes of the 2d WM.

The dispersion equation (35) with substitutions (19), (20), (30), and (34) is the final result for the triple wire media. The explicit equation is cumbersome and cannot be simplified in the general propagation case. However, in the special case when $q_x = 0$, all cross-polarized interaction terms (31)–(34) vanish, and the system (29) splits into two separate sets: the first one is the dispersion equation of the 1d WM $C_{xx} = \alpha_x^{-1}$, the second one is the system (22). The first case corresponds to the extraordinary waves propagating normally to the x wires without interaction with y and z wires. There is no spatial dispersion for those waves (see above). The second case corresponds to the in-plane propagation in 2d WM, which will be studied below. In the present paper we do not consider the general case of the wave propagation in 3d WM.

V. DISPERSION DIAGRAMS AND ISOFREQUENCIES OF A DOUBLE WIRE MEDIUM

The dispersion diagram of a double WM for the in-plane propagation ($q_x = 0$) of the extraordinary waves obtained by numerical solution of (28) is shown in Fig. 5. The chosen parameters of the wire lattice are $a = b = c$, $r_y = r_z$. The filling ratio is $f = 2\pi r_y^2/a^2 = 0.002$.

We use notations $\Gamma = (0,0,0)^T$, $Z = (0,0,\pi/c)^T$, and $L = (0, \pi/b, \pi/c)$ for the central point, the z-bound point, and the corner point of the fundamental Brillouin zone, respectively.

One can notice the significant difference between Fig. 5 and the dispersion diagram of a simple wire medium (see Fig. 3). In Fig. 5 one can see within the interval $L-\Gamma$ two extraordinary modes which do not vanish at low frequencies $k < k_0$ and are not TLM. In simple WM the waves with nonzero longitudinal (with respect to the wires) component of the electric field cannot propagate at low frequencies since the phase shifts between the adjacent wires are small and the reradiation of parallel wires suppresses the wave. In 2d WM it becomes possible due to the electromagnetic interaction of

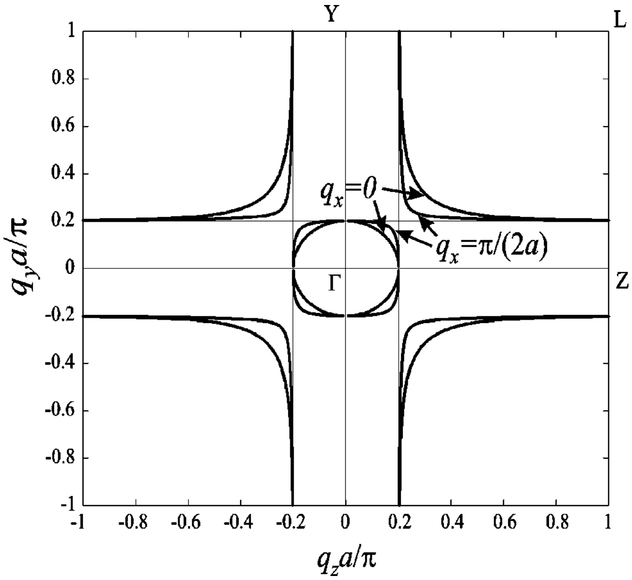

FIG. 6. Isofrequency contours for double wire media at $ka/(2\pi)=0.1$. Two cases $q_x=0$ and $q_x=\pi/(2a)$ are presented.

the two orthogonal wire arrays. This is the result of the cross-polarized interaction of wire arrays. There are terms in C_{yz} that cancel out the terms in C_{yy} and C_{zz} which are responsible for the suppression of the waves propagating obliquely in simple WM at low frequencies.

The horizontal lines $ka/(2\pi)=0.1$ and $ka/(2\pi)=0.3$ in Fig. 5 correspond to isofrequency contours presented in Figs. 6 and 7, respectively. The isofrequency contour located around the L point is very unusual (close to the hyperbolic one). In Fig. 6 one can see that the contours of isofrequencies are rather close to four asymptotes $q_{y,z}=\pm k$. In spite of the rather low frequency as compared to ω_0, the isofrequency contour located around the Γ point ($q=0$) basically differs from the isofrequency of an isotropic dielectric (a circle).

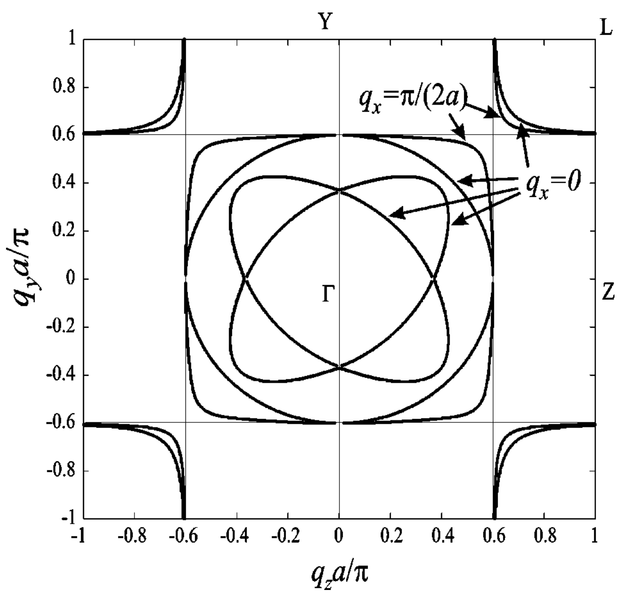

FIG. 7. Isofrequency contours for double wire media at $ka/(2\pi)=0.3$. Two cases $q_x=0$ and $q_x=\pi/(2a)$ are presented.

Only in a special case of the in-plane propagation, the isofrequency centered at the Γ point has nearly circular shape and the phase velocity of this mode coincides with that of the host medium. When $q_x \neq 0$, the shape of this isofrequency becomes superquadric and modes with hyperbolic isofrequency tend to the same asymptotes $q_{y,z}=\pm k$. When $q_x=\pi/a$ the isofrequencies coincide with the asymptotes exactly. This case corresponds to TLM discussed above (which do not transport energy). The plot in Fig. 6 indicates the possibility of the two refracted waves (both extraordinary waves) for the rather large sheer of incidence angles. This effect keeps at the quasistatic limit.

The $2d$ WM with proper orientation of wires with respect to the medium interface can possess the low-frequency negative refraction. It follows from the fact that the angles between the group and the phase velocities for the mode corresponding to the hyperbolic contours in Figs. 6 and 7 can be close to $\pi/2$ (the normal to the isofrequency contour shows the direction of the group velocity vector).

At the frequencies close to the plasma frequency ω_0 and higher two other modes appear with isofrequencies centered at Γ. They are shaped as two crossing ellipses. The modes with isofrequency curves close to $q_{y,z}=\pm k$ are still present. The isofrequency contours for such a case [corresponding to $ka/2\pi=0.3$, $q_x=0$, and $q_x=\pi/(2a)$] are shown in Fig. 7. When q_x increases at fixed frequency, the hyperbolic isofrequency contours in the plane (q_y-q_z) approach the asymptotes in the same way as it happens for lower frequencies. The elliptic contours located around Γ (see Fig. 7) shrink to this point where q_x grows and disappear when q_x becomes greater than k_0.

VI. QUASISTATIC CASE

Let us consider a double wire medium in a quasistatic case where $|\mathbf{q}|a \ll \pi$ and $ka \ll \pi$. Expanding trigonometric functions in the dispersion equation for extraordinary waves (28) into Taylor series and keeping the two first terms in those expansions, we obtain the following equation:

$$(k^2-q_y^2)(k^2-q_z^2)[k^2-k_0^2(r_y,a,c)-q^2][k^2-k_0^2(r_z,a,b)-q^2] = q_y^2 q_z^2 k_0^2(r_y,a,c) k_0^2(r_z,a,b). \qquad (36)$$

Let us consider a special case where $r_y=r_z=r$ and $a=b=c$. In that case (36) could be simplified to the form

$$\sqrt{k^2-q_y^2}\sqrt{k^2-q_z^2}(k^2-k_0^2-q_x^2-q_y^2-q_z^2) \pm q_y q_z k_0^2 = 0, \qquad (37)$$

where $k_0=k_0(r,a,a)$.

We can express q_x from (37) in the form

$$q_x^2 = k^2 - k_0^2 \pm \frac{q_y q_z k_0^2}{\sqrt{k^2-q_y^2}\sqrt{k^2-q_z^2}} - q_t^2. \qquad (38)$$

In Ref. [4] the following approximate dispersion equation was introduced under the conditions $k \approx k_0$ and $q_t=\sqrt{q_x^2+q_y^2} \ll k_0$ (in our notations):

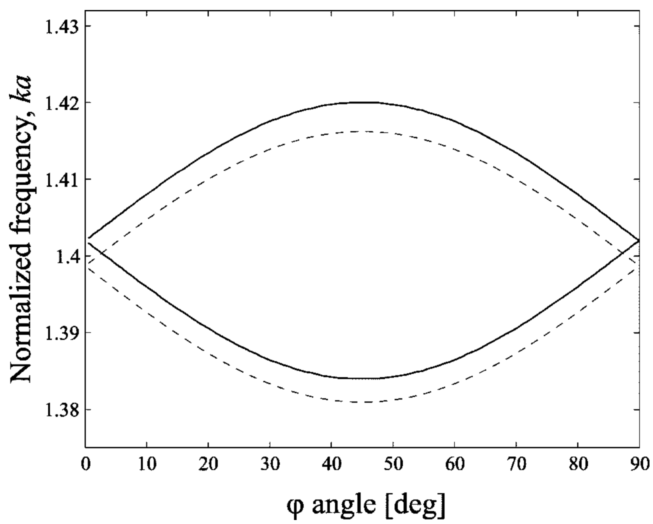

FIG. 8. Dependence of the normalized wave number ka on φ angle near the "plasma" resonance of the wire medium with $r=a/100$, $q_t=0.1\pi/a$ (dashed line). Comparison with the exact result (solid line corresponds to the numerical data from Fig. 6 [4]).

$$q_x^2 \approx k^2 - k_0^2 \pm q_y q_z - q_t^2. \qquad (39)$$

The birefringence of the dispersion branches near the plasma frequency corresponds to the two signs on the right-hand sides of (38) and (39). The difference between Eqs. (38) and (39) is not significant if $k\approx k_0$ and $q_t=\sqrt{q_x^2+q_y^2}\ll k_0$.

In Ref. [4] the propagation of waves at the frequencies close to ω_0 in triple wire medium has been numerically studied for the case where $q_x=0$ (in-plane propagation). In that case the presence of x wires does not influence the propagation characteristics, and our dispersion equation (37) is applicable. We compare the solution of (37) with results from Ref. [4] in order to validate our theory.

In Ref. [4] one chose the following parameters: $r=a/100$ (it corresponds to $k_0a\approx 1.4$), $q_t=0.1\pi/a$. One calculates the medium dispersion for $q_x=0$ at the frequencies $k\approx k_0$. In Fig. 6 of this work one shows the dependence of the normalized eigenfrequency ka on the angle φ. The angle φ is indicated in Fig. 4 ($q_y=q_t \sin\varphi$ and $q_z=q_t\cos\varphi$). The plot ka vs φ shown in Fig. 8 represents the comparison of (37) with the numerical data from Ref. [4]. The upper dispersion branch corresponds to the case where before q_yq_z in (37) there is a plus sign, and the lower branch corresponds to the case where there is a minus sign. This plot illustrates the effect of the dispersion branch birefringence near the "plasma frequency" of wire medium (see ellipses in Fig. 7). Equation (37) shows that this effect holds also for $q_x\neq 0$. Figure 8 verifies that the quasistatic equation (37) is correct even at rather high frequencies (slightly higher than ω_0) outside the initial approximation $ka\ll\pi$.

Now, let us turn to the consideration of the effective permittivity of 2d WM. In the dyadic form the tensor of effective relative permittivity of arbitrary anisotropic dielectric media can be written as $\bar{\bar{\varepsilon}}=\varepsilon_{xx}\mathbf{x}_0\mathbf{x}_0+\varepsilon_{yy}\mathbf{y}_0\mathbf{y}_0+\varepsilon_{zz}\mathbf{z}_0\mathbf{z}_0$. The dispersion equation of anisotropic dielectric (following from Maxwell's equation and from the definition of relative permittivity in terms of $\mathbf{D}=\varepsilon_0\bar{\bar{\varepsilon}}\cdot\mathbf{E}$) corresponds to the zero-value determinant of the following system: $(\bar{\bar{\varepsilon}}k^2+\mathbf{qq}-q^2\bar{\bar{I}})\mathbf{E}=0$:

$$(\varepsilon_{xx}k^2-q_y^2-q_z^2)E_x+q_xq_yE_y+q_xq_zE_z=0,$$

$$q_xq_yE_x+(\varepsilon_{yy}k^2-q_x^2-q_z^2)E_y+q_yq_zE_z=0,$$

$$q_xq_zE_x+q_yq_zE_y+(\varepsilon_{zz}k^2-q_x^2-q_y^2)E_z=0. \qquad (40)$$

This system helps to find the polarization of eigenmodes when $\bar{\bar{\varepsilon}}$ is known. The study of eigenmode polarization will be considered in our next paper.

The dispersion equation has the following form:

$$(q_y^2+q_z^2-k^2\varepsilon_{xx})(q_x^2+q_z^2-k^2\varepsilon_{yy})(q_x^2+q_y^2-k^2\varepsilon_{zz})$$

$$-(q_y^2+q_z^2-k^2\varepsilon_{xx})q_y^2q_z^2-(q_x^2+q_z^2-k^2\varepsilon_{yy})q_x^2q_z^2$$

$$-(q_x^2+q_y^2-k^2\varepsilon_{zz})q_x^2q_y^2-2q_x^2q_y^2q_z^2=0. \qquad (41)$$

In Ref. [4] the following expressions have been heuristically introduced for components of the permittivity of 3d WM:

$$\varepsilon_{xx}=1-\frac{k_0^2(r_x,b,c)}{k^2-q_x^2},$$

$$\varepsilon_{yy}=1-\frac{k_0^2(r_y,a,c)}{k^2-q_y^2},\quad \varepsilon_{zz}=1-\frac{k_0^2(r_z,a,b)}{k^2-q_z^2}. \qquad (42)$$

The effects of the low-frequency spatial dispersion are deemed to be described by terms $q_{x,y,z}$ in the denominators of the components of $\bar{\bar{\varepsilon}}$ (see also in Ref. [3]).

It has been noted in Ref. [4] that the expressions (42) and the dispersion equation (37) fit perfectly with the results of numerical simulations for $\omega\approx\omega_0$. Following (42), the components of $\bar{\bar{\varepsilon}}$ for triple WM are the permittivities of the three orthogonal simple wire media stretched along the Cartesian axes. We have assumed that the same rule holds for 2d WM. In the case of 2d wire medium there are no x-directed wires and $\varepsilon_{xx}=1$. We have analytically verified that (36) exactly coincides with (41) if the effective permittivity of a double WM takes the following form:

$$\bar{\bar{\varepsilon}}_{\text{double}}=\mathbf{x}_0\mathbf{x}_0+\varepsilon_{yy}\mathbf{y}_0\mathbf{y}_0+\varepsilon_{zz}\mathbf{z}_0\mathbf{z}_0, \qquad (43)$$

where ε_{yy} and ε_{zz} are given by the relations (42). It should be noted that formula (43) has been obtained (very recently) for 2d WM by other authors [9] as a result of a very complicated analytical-numerical approach. From (42) it follows, that at every point of the central isofrequency contour in Fig. 6 (where $k<k_0$ and $q_{y,z}<k$) both components of the permittivity tensor ε_{yy} and ε_{zz} are negative. The propagation of such a wave ($I\neq 0$ for it and the electric field can contain y and z components) is the spatial dispersion effect.

We have also proved that the whole system (42) holds in our model of a triple wire media. The quasistatic analog of (35) has the form

$$(k^2 - q_x^2)(k^2 - q_y^2)(k^2 - q_z^2)[k^2 - k_0^2(r_x,b,c) - q^2]$$
$$\times[k^2 - k_0^2(r_y,a,c) - q^2][k^2 - k_0^2(r_z,a,b) - q^2]$$
$$- (k^2 - q_x^2)[k^2 - k_0^2(r_x,b,c) - q^2]q_y^2 q_z^2 k_0^2(r_y,a,c)k_0^2(r_z,a,b)$$
$$- (k^2 - q_y^2)[k^2 - k_0^2(r_y,a,c) - q^2]q_x^2 q_z^2 k_0^2(r_x,b,c)k_0^2(r_z,a,b)$$
$$- (k^2 - q_z^2)[k^2 - k_0^2(r_z,a,b) - q^2]q_x^2 q_y^2 k_0^2(r_x,b,c)k_0^2(r_y,a,c)$$
$$+ 2q_x^2 q_y^2 q_z^2 k_0^2(r_x,b,c)k_0^2(r_y,a,c)k_0^2(r_z,a,b) = 0. \quad (44)$$

It coincides with the dispersion equation (41) if the permittivity takes the form [see relations (42)]:

$$\bar{\bar{\varepsilon}}_{\rm triple} = \varepsilon_{xx}\mathbf{x}_0\mathbf{x}_0 + \varepsilon_{yy}\mathbf{y}_0\mathbf{y}_0 + \varepsilon_{zz}\mathbf{z}_0\mathbf{z}_0. \quad (45)$$

We have thus verified that dielectric permittivities for 2d and 3d wire media in the form (43) and (45) suggested in Ref. [4,9] fit successfully in our theory. We have analytically verified, that the quasistatic analogs of dispersion equations (23) and (35) in the form (37) and (44) coincide with the dispersion equations of anisotropic dielectrics (43) and (45).

VII. CONCLUSION

In the present paper we have generalized a recently developed analytical theory of a simple wire medium to the case of double wire media and obtained some results for triple WM. We have validated our theory by comparison with Ref. [4] and proved that the effective permittivity of 2d and 3d WM introduced in Refs. [4,9] fits our dispersion equations fairly well.

We have theoretically revealed the effects of low-frequency spatial dispersion for 2d WM, such as the following.

(i) Propagation of z-polarized TLM along y wires is not suppressed by the presence of z wires (the same is correct for the y-polarized TLM propagating along z).

(ii) There are TLM which can exist in both y and z arrays simultaneously. These modes do not transport energy, since the directions of the currents in wires are alternating along the x axis.

(iii) There are two propagating modes at low frequencies $\omega < \omega_0$ which are not TLM and not ordinary waves. One mode has nonzero electric field component in the plane (y-z) whereas both y and z components of the permittivity tensor are negative. For the other one the isofrequency contour is nearly hyperbolic.

(iv) Near the plasma frequency the two other waves appear with crossing isofrequency contours.

The materials under consideration could find various applications due to the properties discussed in this paper. We would like to note especially such applications as creation of a low frequency superprism and design of materials with negative refraction. Those properties of double WM will be discussed in a future paper.

Finally, let us discuss the problem of the homogenization of WM. Equation (41) relates three unknown components of $\bar{\bar{\varepsilon}}$, three components of the wave vector $\mathbf{q}$ and the frequency (or wave number k). The components of $\mathbf{q}$ are related through dispersion equation with k. It is clear that the problem of $\bar{\bar{\varepsilon}}$ has no unique solution in this formulation. The same concerns 2d WM. Though (42) fits our dispersion equations, this result is heuristic and the permittivity has been introduced and not derived. Is it reasonable to try to find other possible expressions for $\bar{\bar{\varepsilon}}$?

It is well known that the effective material parameters of spatially dispersive media have meaning other than those of continuous media. The effective susceptibility of such media in the presence of a point source depends on the source position and has nothing to do with the effective medium susceptibility for plane waves. The usual boundary conditions are not valid on the medium interface. So, the material parameters are not very helpful in solving the boundary problem for media with spatial dispersion. The goal of the homogenization of WM is modest: to describe the low-frequency propagating properties of an infinite medium in terms of those parameters. Therefore, all we need is to introduce the permittivity which would (1) describe all effects we can reveal solving the correct (quasistatic) dispersion equation and (2) allow one to find the polarization of eigenmodes correctly. For both 2d and 3d WM the permittivity (42) comprises all the dispersion properties at low frequencies ($ka < 1$). As to eigenwaves polarization, the result (42) requires further study, which will be also presented in a future paper.

ACKNOWLEDGMENT

The authors are grateful to Mario Silveirinha for very fruitful discussion.

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