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On the multicentric calculus for $n-$tuples of commuting operators

Licentiate Thesis

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Abstract of Licentiate Thesis

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Abstract

One of the central concepts of operator theory is the spectrum of an operator and if one knows that the spectrum is separated then the multicentric calculus is a useful tool introduced by Olavi Nevanlinna in 2011.

This thesis is an attempt of extending the multicentric calculus from single operators to n-tuples of commuting operators, both for holomorphic functions and for non-holomorphic functions.

The multicentric representation of holomorphic functions gives a simple way to generalize the von Neumann result, i.e., the unit disc is a spectral set for contractions in Hilbert spaces. In other words, this calculus provides a way of representing the spectrum of a bounded operator T, by searching for a polynomial p that maps the spectrum to a small disc around origin. Since the von Neumann inequality works for contractions with spectrum in the unit disc, the multicentric representation applies a suitable polynomial p to the operator T so that p(T) becomes a contraction with spectrum in the unit disc and thus the usual holomorphic functional calculus holds. When extending the calculus to n-tuples of commuting operators a constant and some extra conditions are needed for the von Neumann inequality to hold true.

The multicentric calculus without assuming the functions to be analytic provides a way to construct a Banach algebra, depending on the polynomial p, for which a simple functional calculus holds. For a given bounded operator T on a Hilbert space, the polynomial p is such that p(T) is diagonalizable or similar to normal. The operators here are considered to be matrices. In particular, the calculus provides a natural approach to deal with nontrivial Jordan blocks. In the attempt to extend this calculus to n-tuples of commuting matrices, formulas only for cases when n=2 and n=3 are provided because of the complexity and length of a more general formula.

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Preface

In this part I wish to thank all people who contributed to this thesis that was carried out at the Department of Mathematics and Systems Analysis of Aalto University during 2016-2017.

This work would have not been possible without Professor Olavi Nevanlinna, who introduced me to the interesting topic of the thesis. Many thanks go to him for all the advice and guidance which has been most valuable.

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Content

Introduction

One of the central concepts of operator theory is the spectrum of an operator and if one knows that the spectrum is separated then the multicentric calculus is a useful tool to have. This calculus was introduced by Olavi Nevanlinna in 2011 (see [24] and [25]) and this thesis is an attempt of extending the multicentric calculus from single operators to $n-$tuples of commuting operators, both for holomorphic functions and for non-holomorphic functions. We call an $n-$tuple of commuting operators the $n-$tuple of Banach and/or Hilbert space operators $\mathbf{T} = (T_1, T_2, \ldots, T_n)$ satisfying the property

$$T_iT_j = T_jT_i \text{ for all } i, j = 1, 2, \ldots, n.$$

In order to properly understand these concepts we start by introducing notions and properties of tuples of commuting operators, assuming that the reader is familiar with concepts and results for single operators on a Banach and a Hilbert space. Sometimes few definitions or results on single operators are presented so that the transition towards tuples is smoother and the reader can associate easily different concepts. Hence the first chapter is a collection of definitions and spectral properties for $n-$tuples of commuting operators acting on a Banach space and on a Hilbert space.

Notions such as spectrum, point spectrum, approximate point spectrum or surjective spectrum are here extended to tuples of operators and the Taylor spectrum and Fredholm spectrum will be introduced so that in Chapter 2 we can review the Taylor functional calculus for several commuting operators first introduced in 1970 by J. Taylor in [35]. We do this by giving a survey of basic properties of the Taylor spectrum and Taylor functional calculus following [22].

As mentioned before it is well known that one of the central concepts of operator theory is the spectrum of an operator, which is connected with the existence of a functional calculus that provides information about the structure of Banach space operators.

When working with commuting $n-$tuples of Banach space operators this calculus is more involved. There are many possible definitions for a joint spectrum, but the one introduced by J.L. Taylor has the property that there exists a functional calculus for functions analytic on a neighbourhood of this spectrum.

Another important concept presented in Chapter 2 is the von Neumann inequality, which states that giving a contraction T on a Hilbert space H and a holomorphic function φ on the unit disc then

$$\|\varphi(T)\| \leq \sup_{|z|\leq 1} |\varphi(z)|, \quad \forall z \in \mathbb{C}, T \in H.$$

It is well known that the above inequality holds true for a pair of commuting contractions (T_1, T_2), i.e.

$$\|\varphi(T_1, T_2)\| \leq C \sup\{|\varphi(z_1, z_2)|, |z_1| \leq 1, |z_2| \leq 1\},$$

where C is a constant equal to 1, but for three or more commuting contractions the inequality holds true for some constant $C \neq 1$ and some additional conditions.

To get familiar with these concepts the reader should know notions such as positive, completely positive and completely bounded maps, as well as contractions, isometries and dilations, all of which can be found in the book [29] by Vern Paulsen. Here the author said that the key idea behind a dilation is to realize an operator or a mapping into a space of operators as "part" of something simpler on a larger space. The simplest case is the unitary dilation of an isometry. Then one constructs an isometric dilation of a contraction and combining the two dilations one gets the unitary dilation of a contraction.

Many have tried to extend the von Neumann inequality for more than a pair of commuting operators but no success has been achieved without assuming some extra conditions. Useful results for von Neumann's inequality are Ando's theorem, Sz.-Nagy's dilation theorem and Parrot's example which are also presented in this thesis. Proceeding in a chronological way we state a few results due to G. Popescu [30] (1994), which has extended the von Neumann inequality by working on the Hardy space $H^2(D)$ of analytic functions on the unit disc D. In the same year, 1994, B.A. Lotto in [30] extends the inequality for commuting, diagonalizable contractions satisfying some additional conditions and presents in [19] an example to show the need of the extra hypotheses.

In 2006 David Opěla generalizes Andô's theorem and Parrott's example in [27], by stating that any $n-$tuple of contractions that commute according to a graph without a cycle can be dilated to an $n-$tuple of unitaries that commute according to that graph. Conversely, if the graph contains a cycle, he constructed a counterexample. Other generalizations of Andô's result to an $n-$tuple of commuting contractions are due to Gaşpar and Rácz in [16] (1969) who assume only that the $n-$tuple is cyclic commutative, that is,

$$T_1T_2\dots T_n = T_nT_1T_2\dots T_{n-1} = \dots = T_2T_3\dots T_nT_1$$

for a family $T = (T_1, \dots, T_n)$ of linear bounded operators in a Hilbert space H. Gaşpar and Rácz's results are then further generalized by G. Popescu. Instead of starting with $n-$tuples of contractions, one can work with row contractions, that is, with $n-$tuples satisfying $\sum_j T_jT_j^* \le I$. This case has been extensively studied by T. Bhattacharyya in [8] in 2001.

Ambrozie, Engliš, V. Müller in [1] (2007) work with analytic functions on a $D-$space, which is defined as follows: let D be a nonempty open domain in $\mathbb{C}^n$. A Hilbert space $\mathcal{H}$ of functions analytic on D is called a $D-$space if the following conditions are satisfied:

i) $\mathcal{H}$ is invariant under the operators Z_j, $j = 1, \dots, n$, of multiplication by the coordinate functions,

$$(Z_jf)(z) = z_jf(z); \quad f \in \mathcal{H},\ z = (z_1, \dots, z_n) \in D.$$

It follows from the next assumption and the closed graph theorem that the operators Z_j are, in fact, bounded.

ii) For each $z \in D$, the evaluation function $f \to f(z)$ is continuous on $\mathcal{H}$.

By the Riesz representation theorem there is $C_z \in \mathcal{H}$ such that $f(z) = \langle f, C_z\rangle$ for all $f \in \mathcal{H}$. Define the function $C(z,w) := C_{\bar{w}}(z)$, $z \in D, w \in D^*$. (The function $C(z,\bar{w})$ is known as the reproducing kernel of $\mathcal{H}$.)

iii) $C(z,w) \neq 0$ for all $z \in D$ and $w \in D^*$.

Here by D^* we mean the set $\{\bar{z} : z \in D\}$.

The above material is presented in Chapters 1 and 2 and introduces the reader into the theory of tuples of commuting operators and the theory of holomorphic functional calculus.

The following chapter then contains the non-holomorphic functional calculus for commuting operators due to Sandberg [31] (2003) which follows the traditional non-holomorphic fuctional calculus for single operators of Dynkin in [12] (1972), B. Helffer and J. Sjöstrand in "E'quation de Schrödinger avec champ magnétique et e'quation de Harper" (Berlin, 1989) and E. B. Davies in "Spectral Theory and Differential Operators" (Cambridge, 1995).

Note that this calculus and the one that will be presented later in Chapter 5 are using a different approach and therefore they are not related, but more on Chapter 5 is soon discussed.

The multicentric calculus for holomorphic functions presented in Chapter 4 is due to O. Nevanlinna following the papers on the topic [24], [23] and [25], which have results for single operators on a Hilbert space or simply called one variable case. In [24], O. Nevanlinna shows how multicentric representation of functions gives a simple way to generalize the von Neumann result, i.e., the unit disc is a spectral set for contractions in Hilbert spaces. In other words, this calculus provides a way of representing the spectrum of a contraction T, by searching for a polynomial p that maps the spectrum to a smaller disc around origin. Since the von Neumann inequality works for contractions with spectrum in the unit disc, the multicentric representation applies a suitable polynomial p to the operator T so that $p(T)$ becomes a contraction with spectrum in the unit disc and thus the usual holomorphic functional calculus holds.

In [3] one can find a follow up on the papers of O. Nevanlinna [24], [23] and [25]. There we discuss the separation of a compact set, such as the spectrum, into different components by a polynomial lemniscate with few examples and we present applications of the Calculus to the computation and estimation of the Riesz spectral projection.

Knowing that the von Neumann's inequality holds true for 2 commuting operators, further estimates for a pair of commuting operators has been done, and an attempt to extend the calculus to $n-$tuple of commuting operators is worked out. As mentioned earlier, the von Neumann's inequality for more than three contraction needs a constant $C \neq 1$ and some extra conditions. As extra condition one makes use of $k-$homogeneous polynomial in n variables, defined as being a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form

$$P(z_1, \ldots, z_n) = \sum_{\alpha \in \Lambda(k,n)} a_\alpha z^\alpha,$$

where $\Lambda(k, n) := \{\alpha \in \mathbb{N}_0^n : |\alpha| := \alpha_1 + \cdots + \alpha_n = k\}$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $a_\alpha \in \mathbb{C}$.

Due to a recent paper [15] (2015) by Galicer, Muro and Sevilla-Peris, it is known that for this type of polynomials the von Neumann's inequality holds true for a constant. Thus the multicentric calculus can be extended to $n-$tuples.

We finish the thesis with the latest results due to Olavi Nevanlinna [26] (2015) regarding the multicentric calculus without assuming the functions to be analytic. The calculus is worked out for one variable, where the operator is a matrix, and we present it in details in Chapter 5 (although proofs are skipped) so that later one can extend it for more than one variable. When working with $n-$variables the operator is then considered as an $n-$tuple of commuting matrices. Therefore basic notions on commuting matrices are covered.

If one wants to determine which matrices commute with a given set of $n \times n$ matrices two tools appear to be useful, that is, a standard form for a given matrix (Jordan canonical form) and restrictions on the form of the commuting matrices. The latter is a canonical form called the H-form introduced by K.C. O'Meara and C. Vinsonhaler in [28]. We will shortly discuss both this tools and by the end of this thesis an attempt on extending the calculus for n-tuple of commuting operators is worked out.

Chapter 1

Commuting $n-$tuple operators

In this chapter will be presented definitions, properties and results regarding $n-$tuples of commuting operators.

1.1 Basic definitions

Let X be a Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, let $\ker(T)$, $\mathrm{Im}(T)$ and $\sigma(T)$ denote the null space of T, the range of T and the spectrum of T, respectively.

Definition 1.1.1. *An operator* $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{L}(X)^n$ *is called a commuting* $n-$*tuple if* $T_iT_j = T_jT_i$ *for every* $1 \leq i, j \leq n$.

Definition 1.1.2. *We say that two* $n-$*tuples of operators* $\mathbf{R} \in \mathcal{L}(X)^n$ *and* $\mathbf{S} \in \mathcal{L}(X)^n$ *criss-cross commute if*

$$R_iS_jR_k = R_kS_jR_i \quad \textit{and} \quad S_iR_jS_k = S_kR_jS_i$$

for every $1 \leq i, j, k \leq n$.

A simple example of criss-cross commuting tuples of operators is given below, where the operators are considered to be 3×3 matrices. Let $\mathbf{R} = (R_1, R_2)$ and $\mathbf{S} = (S_1, S_2)$ with

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here $R_1R_2 = R_2R_1$ and $S_1S_2 = S_2S_1$ thus $\mathbf{R}$ and $\mathbf{S}$ are commuting pairs. Moreover, R_1S_1, R_2S_2, S_1R_1, and S_2R_2, mutually commute, as well as R_1S_2, R_2S_1, S_1R_2, and S_2R_1, hence $\mathbf{R}$ and $\mathbf{S}$ criss-cross commute.

As a counterexample, let

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $R_1R_2 = R_2R_1$, $S_1S_2 = S_2S_1$, and that R_1S_1, R_2S_2, S_1R_1, S_2R_2, mutually commute, but we do not have criss-cross commutativity since $R_1S_1R_2 \neq R_2S_1R_1$ and $S_1R_1S_2 \neq S_2R_1S_1$. Also we see that $R_1S_2 \neq S_2R_1$ and $R_2S_1 \neq S_1R_2$.

Definition 1.1.3. *Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *and* $\mathbf{S} = (S_1, \ldots, S_n)$ *be commuting* $n-$*tuples. We say that* $\mathbf{R}$ *and* $\mathbf{S}$ *are nearly commuting provided that*

$$R_iS_j = S_jR_i \text{ for every } i \neq j.$$

For example, the commuting pairs $\mathbf{R} = (R_1, R_2)$ and $\mathbf{S} = (S_1, S_2)$ where

$$R_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

are nearly commuting since $R_iS_j = S_jR_1$ for every $i \neq j$ $(i, j = 1, 2)$.

Definition 1.1.4. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting* $n-$*tuple of bounded linear operators acting on a Banach space* X. *For* $x \in X$ $\rho(\mathbf{T}, x)$ *is the set of those points* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ *for which there exist an open set* $U \ni \lambda$ *and* $X-$*valued analytic functions on* $U : f_1, \ldots, f_n$ *such that*

$$\sum_{j=1}^{n} (T_j - z_j) f_j(z) = x, \quad z \in U.$$

The set $\sigma(\mathbf{T}, x) =: \mathbb{C}^n \setminus \rho(\mathbf{T}, x)$ *is called the analytic local spectrum of* $\mathbf{T}$ *at* x.

For $i = 1, \dots, n$ we set

$$\mathbf{SR}_{\{i\}} = (S_1R_1, \dots, R_iS_i, \dots, S_nR_n)$$

(We put in the $i-$th coordinate R_iS_i instead of S_iR_i.)

Proposition 1.1.5. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* $n-$*tuples and let* $x \in X$. *We have:*

i) If $\mathbf{R}$ *and* $\mathbf{S}$ *are nearly commuting, then*

$$\sigma(\mathbf{SR}, S_ix) \subset \sigma(\mathbf{SR}_{\{\mathbf{i}\}}, x) \subset \sigma(\mathbf{SR}, S_ix) \cup [0]^{\{i\}}.$$

ii) If $\mathbf{R}$ *and* $\mathbf{S}$ *are criss-cross commuting, then*

$$\sigma(\mathbf{SR}, S_ix) \subset \sigma(\mathbf{RS}, x) \subset \bigcap_{j=1}^{n} \sigma(\mathbf{SR}, S_jx) \cup [0]^{\{j\}}.$$

The proof of the above proposition can be found in [7].

Now let's review some definitions of spectrum on a Banach space.

Definition 1.1.6. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *Then*

$$\begin{aligned}
\sigma_p(\mathbf{T}) &= \{\lambda \in \mathbb{C}^n : \bigcap_{i=1}^{n} \ker(T_i - \lambda_i) \neq \{0\}\}, \\
\sigma_\pi(\mathbf{T}) &= \{\lambda \in \mathbb{C}^n : \inf\{\sum_{i=1}^{n} \|(T_i - \lambda_i)x\| : x \in X, \|x\| = 1\} = 0\}, \\
\sigma_\delta(\mathbf{T}) &= \{\lambda \in \mathbb{C}^n : (T_1 - \lambda_1)X + \dots + (T_n - \lambda_n)X \neq X\},
\end{aligned}$$

are called the point, approximate point and surjective spectrum of $\mathbf{T}$.

Definition 1.1.7. *Let* T *be an operator in* $\mathcal{L}(X)$. *We define the compression spectrum of* T *as*

$$\sigma_{com} = \{\lambda \in \mathbb{C} : (T - \lambda)X \text{ is not dense in } X\}.$$

Proposition 1.1.8. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *Write* $\mathbf{T}^* = (T_1^*, \dots, T_n^*) \in \mathcal{L}(X^*)$. *Then* $\sigma_\pi(\mathbf{T}^*) = \sigma_\delta(\mathbf{T})$ *and* $\sigma_\delta(\mathbf{T}^*) = \sigma_\pi(\mathbf{T})$.

For the proof of the above result one can check [21].

Now we review some definitions so that we can introduce the notion of Taylor and Fredholm spectrum.

Definition 1.1.9. *Let* $\boldsymbol{e} = \{e_1, e_2, \dots, e_n\}$ *be indeterminates and define* $\Lambda_n[\boldsymbol{e}]$ *to be the exterior algebra on the generators* $e_1, e_2, \dots, e_n$*. This is a linear space over the complex plane* $\mathbb{C}$ *endowed with an anticommutative exterior product* $e_i \wedge e_j = -e_j \wedge e_i$*, for every* $1 \leq i, j \leq n$*. For* $F = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ *with* $i_1 < \cdots < i_p$*, we write* $e_F = e_{i_1} \wedge \cdots \wedge e_{i_p}$*. The exterior algebra over* $\mathbb{C}$ *is then given by*

$$\Lambda_n[\boldsymbol{e}] = \{\sum_F \alpha_F e_F : e_F = e_{i_1} \wedge \cdots \wedge e_{i_p} \text{ and } \alpha_F \in \mathbb{C}\}.$$

We let here $e_\emptyset$ *to be the identity element for the exterior product. If we denote* $\Lambda_n^k[\boldsymbol{e}] = \{\sum_{|F|=k} \alpha_F e_F : \alpha_F \in \mathbb{C}\}$*, where* $|F|$ *is the cardinal of* F*, then clearly* $\dim \Lambda_n^k[\boldsymbol{e}] = \binom{n}{k}$ *for every* $k \leq n$*,* $\Lambda_n^k[\boldsymbol{e}] \wedge \Lambda_n^l[\boldsymbol{e}] = \Lambda_n^{k+l}[\boldsymbol{e}]$ *and* $\Lambda_n[\boldsymbol{e}] = \bigoplus_{k=0}^n \Lambda_n^k[\boldsymbol{e}]$*.*

Definition 1.1.10. *Given a Banach space* X*, the exterior algebra over* X *is defined to be*

$$\Lambda_n[\boldsymbol{e}, X] = \{\sum_F x_F e_F : e_F = e_{i_1} \wedge \cdots \wedge e_{i_p} \text{ and } x_F \in X\}.$$

The subspaces $\Lambda_n^p[\boldsymbol{e}, X] = \{\sum_{|F|=p} x_F e_F : x_F \in X\}$*, for* $p \leq n$ *are given in a similar way. Naturally* $\Lambda_n^0[\boldsymbol{e}, X]$*,* $\Lambda_n^1[\boldsymbol{e}, X]$ *and* $\Lambda_n^n[\boldsymbol{e}, X]$ *can be identified with* X*,* X^n *and* X*, respectively.*

Since no confusion is possible $\Lambda_n^k[X]$ and $\Lambda_n[X]$ can be written instead of $\Lambda_n^k[\mathbf{e}, X]$ and $\Lambda_n[\mathbf{e}, X]$, respectively.

If $T \in \mathcal{L}(X)$, one keeps the same symbol T to denote the operator defined on $\Lambda_n[X]$ by

$$T\left(\sum_F x_F e_F\right) = \sum_F T x_F e_F.$$

Definition 1.1.11. *For* $i \in \{1, 2, \dots, n\}$*, let* $E_i : \Lambda_n[X] \to \Lambda_n[X]$ *be the left multiplication operator by* $e_i : E_i(e_F) = e_i \wedge e_F$*. It is usually called the creation operator. With any commuting* $n-$*tuple* $\mathbf{T} = (T_1, \dots, T_n)$ *we associate the linear mapping defined over* $\Lambda_n[X]$ *by*

$$\delta_{\mathbf{T}} = \sum_{i=1}^n T_i \otimes E_i : \sum_F x_F e_F \to \sum_F \sum_{i=1}^n T_i x_F e_i \wedge e_F.$$

Definition 1.1.12. *Set* $\delta_{\mathbf{T}}^k := \delta_{\mathbf{T}} | \Lambda_n^k[X]$*. We construct a co-chain complex* $K(\mathbf{T})$*, called the Koszul complex associated with* $\mathbf{T}$ *on* X *as follows:*

$$K(\mathbf{T}) : \boldsymbol{0} \xrightarrow{\delta_{\mathbf{T}}^{-1}} \Lambda_n^0[X] \xrightarrow{\delta_{\mathbf{T}}^0} \Lambda_n^1[X] \xrightarrow{\delta_{\mathbf{T}}^1} \cdots \xrightarrow{\delta_{\mathbf{T}}^{n-1}} \Lambda_n^n[X] \xrightarrow{\delta_{\mathbf{T}}^n} \boldsymbol{0}.$$

Definition 1.1.13. *The operator* $\mathbf{T}$ *is said to be non-singular, or Taylor invertible, if* $\ker \delta_{\mathbf{T}}^k = \mathrm{Im}\delta_{\mathbf{T}}^{k-1}$, *for* $k = 1, \ldots, n$, *equivalently* $\ker \delta_{\mathbf{T}} = \mathrm{Im}\delta_{\mathbf{T}}$. *The associated Koszul complex is said to be exact in this case. The Taylor spectrum of* $\mathbf{T}$ *on* X^n *is then the set*

$$\sigma_T(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : K(\mathbf{T} - \lambda) \textit{ is not exact}\}.$$

Definition 1.1.14. *We call cohomology of* $\{\Lambda_n^k[X], \delta_{\mathbf{T}}^k\}$ *the set* $\{H^k(X, \mathbf{T})\}_k$, *where* $H^k(X, \mathbf{T}) = \ker \delta_{\mathbf{T}}^k \setminus \mathrm{Im}\delta_{\mathbf{T}}^{k-1}$.

Notice that the action of $\mathbf{T}$ on X is non-singular if $H^k(X, \mathbf{T}) = 0$ for every k.

Remark 1.1.15. (i) Let $n = 1$. We can identify $\Lambda^0[\mathbf{e}, X]$ and $\Lambda^1[\mathbf{e}, X]$ with X, and so the Koszul complex of a single operator $T_1 \in \mathcal{L}(X)$ becomes

$$0 \to X \xrightarrow{T_1} X \to 0.$$

This complex is exact if and only if T_1 is invertible. Thus for single operators the Taylor spectrum coincides with the ordinary spectrum.

(ii) Let $n = 2$ and let $\mathbf{T} = (T_1, T_2)$ be a commuting pair of operators on X. Then the Koszul complex of $\mathbf{T}$ becomes

$$0 \to X \xrightarrow{\delta_{\mathbf{T}}^0} X \oplus X \xrightarrow{\delta_{\mathbf{T}}^1} X \to 0,$$

where $\delta_{\mathbf{T}}^0$ and $\delta_{\mathbf{T}}^1$ are defined by $\delta_{\mathbf{T}}^0 x = T_1 x \oplus T_2 x$ $(x \in X)$ and $\delta_{\mathbf{T}}^1(x \oplus y) = -T_2 x + T_1 y$ $(x, y \in X)$.

(iii) The most important part of a Koszul complex of an $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ are its ends. The first mapping $\delta_{\mathbf{T}}^0$ can be interpreted as $\delta_{\mathbf{T}}^0 : X \to X^n$ defined by $\delta_{\mathbf{T}}^0 x = \bigoplus_{i=1}^n T_i x$ $(x \in X)$, according to Section 9 in [21]. Thus the Koszul complex of $\mathbf{T}$ is exact at $\Lambda_n^0[\mathbf{e}, X]$ if and only if $0 \neq \sigma_\pi(\mathbf{T})$. Similarly, $\delta_{\mathbf{T}}^1 : X^n \to X$ is defined by $\delta_{\mathbf{T}}^1(x_1 \oplus \cdots \oplus x_n) = \sum_{i=1}^n (-1)^{i-1} T_i x_i$, and so the exactness at $\Lambda_n^n[\mathbf{e}, X]$ means that $0 \neq \sigma_\delta(\mathbf{T})$.

Definition 1.1.16. *A single operator* T *satisfies the SVEP (Single valued extension property) at* $\lambda \in \mathbb{C}$ *if there exists a neighborhood* $\mathcal{U}$ *of* λ *such that the zero function is the only analytic function* f *defined on* $\mathcal{U}$ *satisfying* $(T - \mu)f(\mu) = 0$ *for every* $\mu \in \mathcal{U}$.

Definition 1.1.17. *Let* $\mathbf{T}$ *be a commuting* $n-$*tuple. We will say that* $\mathbf{T}$ *has the SVEP (Single valued extension property) at* λ *if there exists an open polydisc* D_λ *centred at* λ *such that* $H^k(\mathcal{O}(D_\lambda, X), \mathbf{T}) = 0$ *for* $k = 1, \ldots, n-1$, *where* $\mathcal{O}(D_\lambda, X)$ *denote the Fréchet algebra of analytic functions on* D_λ.

The following two theorems were proved by C. Behinda and E.H. Zerouali in 2011 in [7].

Theorem 1.1.18. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* $n-$*tuples nearly commuting. Then* $\mathbf{T} = (R_1S_1, \dots R_iS_i, \dots, R_nS_n)$ *has SVEP if and only if* $\mathbf{T}'_i = (R_1S_1, \dots, S_iR_i, \dots, R_nS_n)$ *has SVEP for every* $i \in \{1, \dots, n\}$.

In particular, $\mathbf{RS}$ *has SVEP if and only if* $\mathbf{SR}$ *has SVEP.*

Theorem 1.1.19. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* $n-$*tuples satisfying the criss-cross commutativity. Then* $\mathbf{RS}$ *has SVEP if and only if* $\mathbf{SR}$ *has SVEP.*

Theorem 1.1.20. *Let* $\mathbf{T}$ *be a commuting* $n-$*tuple with SVEP at zero. Then* $\mathbf{T}$ *is Taylor invertible if and only if* $T_1(X) + T_2(X) + \cdots + T_n(X) = X$.

Proof. A result of Finch in [13] asserts that for an operator $T \in \mathcal{L}(X)$ with SVEP at zero we have: T is invertible if and only if $T(X) = X$. The above extension of Finch's result for $n-$tuples has been observed in [38].

The condition $T_1(X)+T_2(X)+\dots T_n(X) = X$ implies that $H^n(T) = \{0\}$, and from the proof of lemma 2.1 in [34], it follows that $H^n(\mathbf{T} - \lambda) = \{0\}$ for $\lambda \in D$, with D a polydisc containing zero. We deduce that $H^n(\mathcal{O}(\mathcal{U}(z), X)) = 0$ and hence that $\mathbf{T}$ is Taylor invertible. □

Definition 1.1.21. *The* $n-$*tuple* $\mathbf{T}$ *is said to be Fredholm, if* $\mathrm{Im}\delta_{\mathbf{T}}^{k-1}$ *is closed and* $\ker \delta_{\mathbf{T}}^{k}/\mathrm{Im}\delta_{\mathbf{T}}^{k-1}$ *is finite dimensional for every* $k = 1, \dots, n$, *equivalently* $\mathrm{Im}\delta_{\mathbf{T}}$ *is closed and* $\ker \delta_{\mathbf{T}}/\mathrm{Im}\delta_{\mathbf{T}}$ *is finite dimensional.*

Definition 1.1.22. *The Fredholm (called also the essential) spectrum is*

$$\sigma_{Te}(\mathbf{T}) := \{\lambda \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Fredholm}\}.$$

Definition 1.1.23. *Given a Fredholm* $n-$*tuple* $\mathbf{T}$, *the index of* $\mathbf{T}$ *is defined by the Euler characteristic number of the associated Koszul complex, that is*

$$\mathrm{ind}\ (\mathbf{T}) = \sum_{k=0}^{n} (-1)^k \dim H^k(\mathbf{T}),$$

where $H^k(\mathbf{T}) = \ker \delta_{\mathbf{T}}^{k}/\mathrm{Im}\delta_{\mathbf{T}}^{k-1}$, $k = 1, \dots, n$, *are the associated cohomology groups. A Fredholm operator is said to be Taylor-Weyl if* $\mathrm{ind}\ (\mathbf{T}) = 0$, *and the Taylor-Weyl spectrum is*

$$\sigma_{TW}(\mathbf{T}) := \{\lambda \in \mathbb{C}^n : \mathbf{T} - \lambda \text{ is not Taylor-Weyl}\}.$$

1.2 Spectral properties

In this section are presented results regarding the Taylor and Fredholm spectrum of the operators **RS** and **SR** following [6], therefore complete proofs of the results below can be found in [6].

Theorem 1.2.1. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* $n-$*tuples such that* $R_kS_j = S_jR_k$ *for* $k \neq j$. *Then*

i) $\mathbf{SR}$ *and* $\mathbf{RS}$ *are commuting* $n-$*tuples*

ii) For every $i \in \{1,\dots,n\}$ *and* $\lambda = (\lambda_1,\dots,\lambda_n)$ *such that* $\lambda_i \neq 0$, *we have* $\lambda - \mathbf{SR}$ *is Taylor invertible if and only if* $\lambda - \mathbf{SR}_{\{i\}}$ *is.*

This theorem was first proved by C. Behinda and E.H. Zerouali in year 2007 in article [6] and it uses the following lemma.

Lemma 1.2.2. *Let* $\mathbf{T} = (T_1,\dots,T_n) \in \mathcal{L}(X)^n$ *be a commuting* $n-$*tuple, then*

$$T_j = E_j^* \delta_{\mathbf{T}} + \delta_{\mathbf{T}} E_j^*$$

for every $j = 1,\dots,n$. *In particular,*

$$T_j(\ker(\delta_{\mathbf{T}})) \subset \operatorname{Im}(\delta_{\mathbf{T}}).$$

Let $\mathcal{I}$ be a subset of $\{1,\dots,n\}$. we associate with $\mathcal{I}$ the "partially switched" operator $\mathbf{SR}_{\mathcal{I}}$ of $\mathbf{SR}$ defined as follows:

$$\mathbf{SR}_{\mathcal{I}} := (Q_1,\dots,Q_n),$$

where $Q_i = R_iS_i$ if $i \in \mathcal{I}$, and $Q_i = S_iR_i$ otherwise. Clearly $\mathbf{SR}_{\emptyset} = \mathbf{SR}$ while $\mathbf{SR}_{\{1,\dots,n\}} = \mathbf{RS}$.

Corollary 1.2.3. *Under the assumptions of Theorem 1.2.1, for any* $\mathcal{I}, \mathcal{J} \subset \{1,\dots,n\}$ *and* λ *such that* $\Pi_{i\in\mathcal{I}\cup\mathcal{J}}\lambda_i \neq 0$, *we have*

$$\lambda - \mathbf{SR}_{\mathcal{I}} \quad \text{is Taylor invertible if and only if} \quad \lambda - \mathbf{SR}_{\mathcal{J}} \text{ is.}$$

Denote $[0]^{\mathcal{I}} = \{\lambda = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n : \Pi_{i\in\mathcal{I}}\lambda_i = 0\}$. For convenience, we put $[0] =: [0]^{\{1,\dots,n\}}$ and $[0]^{\emptyset} = \emptyset$.

Theorem 1.2.4. *Under the assumptions of Theorem 1.2.1, for all* $\mathcal{I}, \mathcal{J} \subset \{1,\dots,n\}$, *we have*

$$\sigma_T(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}} = \sigma_T(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}}.$$

In particular,

$$\sigma_T(\mathbf{RS}) \setminus [0] = \sigma_T(\mathbf{SR}) \setminus [0].$$

Corollary 1.2.5. *Under the assumptions of Theorem 1.2.1, we get*

i) If $0 \notin \sigma(R_iS_i) = \sigma(S_iR_i)$ *for every* $i \in \mathcal{I} \subset \{1, \ldots, n\}$, *then*

$$\sigma_T(\mathbf{RS}) \setminus [0]^{\mathcal{I}^c} = \sigma_T(\mathbf{SR}) \setminus [0]^{\mathcal{I}^c},$$

where the set $\mathcal{I}^c$ *is the complement of* $\mathcal{I}$ *in* $\{1, \ldots, n\}$.

ii) If $0 \notin \sigma(R_iS_i) = \sigma(S_iR_i)$ *for every* $i = 1, \ldots, n$, *then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

Theorem 1.2.6. *Let* $\mathbf{R}$ *and* $\mathbf{S}$ *be commuting* $n-$*tuples such that* $R_kS_j = S_jR_k$ *for* $k \neq j$. *Let* $i \in \{1, \ldots, n\}$ *and let* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ *be such that* $\lambda_i \neq 0$. *Then*

$$\lambda - \mathbf{SR}_{\mathcal{I}} \quad \text{is Fredholm if and only if} \quad \lambda - \mathbf{SR}_{\mathcal{J}} \text{ is Fredholm.}$$

Theorem 1.2.7. *Under the assumptions of Theorem 1.2.1, for* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *such that* $\Pi_{i \in \mathcal{I} \cup \mathcal{J}} \lambda_i \neq 0$, *we have*

i) $\lambda - \mathbf{SR}_{\mathcal{I}}$ *is Fredholm if and only if* $\lambda - \mathbf{SR}_{\mathcal{J}}$ *is;*

ii) If $\lambda - \mathbf{SR}_{\mathcal{I}}$ *is Fredholm, then* $\operatorname{ind}(\lambda - \mathbf{SR}_{\mathcal{I}}) = \operatorname{ind}(\lambda - \mathbf{SR}_{\mathcal{J}})$;

iii) In particular, for every λ *such that* $\Pi_{i \in \mathcal{I} \cup \mathcal{J}} \lambda_i \neq 0$, *we have* $\lambda - \mathbf{SR}$ *is Fredholm if and only if* $\lambda - \mathbf{RS}$ *is Fredholm, and in this case,*

$$\operatorname{ind}(\lambda - \mathbf{SR}) = \operatorname{ind}(\lambda - \mathbf{RS}).$$

Corollary 1.2.8. *Under the assumptions of Theorem 1.2.1, we have*

$$\sigma_{Te}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}} = \sigma_{Te}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I} \cup \mathcal{J}}.$$

In particular, we have

$$\sigma_{Te}(\mathbf{SR}) \setminus [0] = \sigma_{Te}(\mathbf{RS}) \setminus [0].$$

Corollary 1.2.9. *Under the assumption of Theorem 1.2.1, we get*

i) If $0 \notin \sigma_e(R_iS_i) = \sigma_e(S_iR_i)$ *for every* $i \in \mathcal{I} \subset \{1, \ldots, n\}$, *then*

$$\sigma_{Te}(\mathbf{RS}) \setminus [0]^{\mathcal{I}^c} = \sigma_{Te}(\mathbf{SR}) \setminus [0]^{\mathcal{I}^c},$$

where the set $\mathcal{I}^c$ *is the complement of* $\mathcal{I}$ *in* $\{1, \ldots, n\}$.

ii) If $0 \notin \sigma_e(R_iS_i) = \sigma_e(S_iR_i)$ *for every* $i = 1, \ldots, n$, *then* $\sigma_{Te}(\mathbf{RS}) = \sigma_{Te}(\mathbf{SR})$.

Corollary 1.2.10. *Under the assumptions of Theorem 1.2.1, we have*

$$\sigma_{TW}(\mathbf{SR}_{\mathcal{I}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}} = \sigma_{TW}(\mathbf{SR}_{\mathcal{J}}) \setminus [0]^{\mathcal{I}\cup\mathcal{J}}$$

and hence

$$\sigma_{TW}(\mathbf{SR}) \setminus [0] = \sigma_{TW}(\mathbf{SR}) \setminus [0].$$

Regarding the Taylor spectrum more developments were done, for example in [9] the following statements were discussed.

If R and S are operators on a Banach space, it is well known that the spectra of the two products RS and SR are very nearly the same:

$$\sigma(RS) \setminus \{0\} = \sigma(SR) \setminus \{0\}. \tag{1.2.1}$$

Necessary and sufficient for full equality is that either of the following two conditions hold:

$$0 \in \sigma(RS) \cap \sigma(SR) \tag{1.2.2}$$

$$0 \notin \sigma(RS) \cup \sigma(SR) \tag{1.2.3}$$

If more generally $\mathbf{R}$ and $\mathbf{S}$ are criss-cross commuting systems of operators, then in particular each of the systems $\mathbf{RS} := (R_1S_1, \ldots, R_nS_n)$ and $\mathbf{SR} := (S_1R_1, \ldots, S_nR_n)$ of products is commutative and the analogue for (1.2.1) is true for the Taylor spectrum.

For a commuting $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ acting on a Banach space X, let $K(\mathbf{T} - \lambda)$ denote the Koszul complex associated with $\mathbf{T} - \lambda$. We define Taylor spectrum $\sigma_T(\mathbf{T})$ and approximate point spectrum $\sigma_\pi(\mathbf{T})$ as follows:

$$\begin{aligned} \sigma_T(\mathbf{T}) &= \{\lambda \in \mathbb{C}^n : K(\mathbf{T} - \lambda) \text{ is not exact}\} \\ \sigma_\pi(\mathbf{T}) &= \{\lambda \in \mathbb{C}^n : \inf_{\|x\|=1} \sum_{i=1}^{n} \|(T_i - \lambda_i)x\| = 0\}. \end{aligned}$$

Given a bounded linear operator T on a Banach space X, we let T^* denote the adjoint of T, acting on X^*, the dual space of X.

Theorem 1.2.11. *Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *and* $\mathbf{S} = (S_1, \ldots, S_n)$ *be criss-cross commuting. If* $\mathbf{0} \in \sigma_\pi(\mathbf{R}) \cap \sigma_\pi(\mathbf{R}^*)$, *then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

For the proof of the above theorem it was used the following theorem.

Theorem 1.2.12. ([34], Theorem 3.6) *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting* $n-$*tuple of operators. Then* $\sigma_T(\mathbf{T}) = \sigma_T(\mathbf{T}^*)$.

Let us set

$$\pi = \{(x, f) \in X \times X^* : \|f\| = f(x) = \|x\| = 1\}.$$

The spacial joint numerical range $V(\mathbf{T})$ and joint numerical range $V(\mathcal{L}(X), \mathbf{T})$ of $\mathbf{T}$ are defined by

$$V(\mathbf{T}) = \{(f(T_1x), \ldots, f(T_nx)) : (x, f) \in \pi\}$$

and

$$V(\mathcal{L}(X), \mathbf{T}) = \{(F(T_1), \ldots, F(T_n)) : F \text{ is a state on } \mathcal{L}(X)\}$$

respectively. The joint spectral radius and joint numerical radius of $\mathbf{T} = (T_1, \ldots, T_n)$ are defined by

$$\begin{aligned} r(\mathbf{T}) &= \sup\{|z| : z \in \sigma_T(\mathbf{T})\} \text{ and} \\ \nu(\mathbf{T}) &= \sup\{|z| : z \in V(\mathbf{T})\} \text{ respectively.} \end{aligned}$$

For an operator $T \in \mathcal{L}(X)$, if $V(T) \subset \mathbb{R}$ then T is called hermitian. An operator $T \in \mathcal{L}(X)$ is called normal if there are hermitian operators H and K such that $T = H + iK$ and $HK = KH$. We denote then the operator $H - iK$ by $\overline{T}$. Then the following are well-known:

i) $\overline{\text{co}}V(T) = V(\mathcal{L}(X), T)$, where $\overline{\text{co}}E$ is the closed convex hull of E.

ii) $\text{co}\sigma(T) \subset \overline{V(T)}$, where $\text{co}E$ and $\overline{E}$ are the convex hull and the closure of E, respectively.

iii) If T is normal, then $\text{co}\sigma(T) = \overline{V(T)} = V(\mathcal{L}(X), T)$. We denote the boundary of E by δE.

Definition 1.2.13. *An $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ is called strongly commuting is, for each $1 \leq j \leq n$, there exist operators R_j and S_j, each with real spectrum, such that $T_j = R_j + iS_j$ and $\mathbf{V} = (R_1, S_1, \ldots, R_n, S_n)$ is a commuting $2n-$tuple.*

Remark 1.2.14. Since the Fuglde theorem holds for Banach space operators, $\mathbf{T} = (T_1, \ldots, T_n)$ is strongly commuting if $\mathbf{T}$ is a commuting $n-$tuple of normal operators.

Theorem 1.2.15. *If $\mathbf{T} = (T_1, \ldots, T_n)$ is a strongly commuting $n-$tuple, then $\mathbf{T}$ has the condition:*

$$\mathbf{0} \in \sigma_T(\mathbf{T}) \Rightarrow \mathbf{0} \in \sigma_\pi(\mathbf{T}) \cap \sigma_\pi(\mathbf{T}^*).$$

For the proof of the above theorem it was used the following theorem that can be found in [10] as Theorem 2.1.

Theorem 1.2.16. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a strongly commuting* $n-$*tuple of operators. Then* $\sigma_T(\mathbf{T}) = \sigma_\pi(\mathbf{T})$.

For a commuting $n-$tuple of operators $\mathbf{T} = (T_1, \ldots, T_n)$, we consider the following two properties:

$$\begin{aligned} \exists a &= (a_1, \ldots, a_n) \in \mathbb{C}^n \text{ and } a \circ \mathbf{T} := a_1T_1 + \cdots + a_nT_n \text{ is} \\ & \quad \text{invertible,} \qquad (1.2.4) \\ \mathbf{0} &= (0, \ldots, 0) \notin \sigma_T(\mathbf{T}). \qquad (1.2.5) \end{aligned}$$

Proposition 1.2.17. *For a commuting* $n-$*tuple of operators* $\mathbf{T} = (T_1, \ldots, T_n)$, *(1.2.4) implies (1.2.5)*

Theorem 1.2.18. *Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *and* $\mathbf{S} = (S_1, \ldots, S_n)$ *be criss-cross commuting* $n-$*tuples. If there exists an invertible operator* T *which is a linear combination of* $R_1, \ldots, R_n$, *then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

Corollary 1.2.19. *Let* $\mathbf{R} = (T, \ldots, T)$ *and* $\mathbf{S} = (S_1, \ldots, S_n)$ *be criss-cross commuting. If* T *is normal, then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

Corollary 1.2.20. *Let* $\mathbf{R} = (R_1, \ldots, R_n)$ *be a commuting* $n-$*tuple of operators which is non-singular, that is,* $\mathbf{0} = (0, \ldots, 0) \notin \sigma(\mathbf{R})$. *Suppose that for* $i = 1, \ldots, n$, *if* $0 \in \sigma(R_i)$, *then* 0 *is an isolated point of* $\sigma(R_i)$. *Let* $\mathbf{S} = (S_1, \ldots, S_n)$ *be an* $n-$*tuple of operators such that* $\mathbf{R}$ *and* $\mathbf{S}$ *criss-cross commute. Then* $\sigma_T(\mathbf{RS}) = \sigma_T(\mathbf{SR})$.

1.3 Operators acting on a Hilbert space

Spectral properties for operators on a Banach space have been presented so far and now we will introduce operators on a Hilbert space, basic definitions and spectral properties for commuting $n-$ tuples. Due to F.-H. Vasilescu in [37] we have the following results.

Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the algebra of all linear continuous operators on H. For convenience we shall recall some definitions.

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a system of commuting operators in $\mathcal{B}(H)$, A any algebra with unit and containing $\mathbf{T}$ in its center and Y an arbitrary $A-$module. Denote by $\Lambda_n^p[Y]$ the set of all antisymmetric functions, defined on the set $\{1, \ldots, n\}^p$, with values in Y. For any $\xi \in \Lambda_n^p[Y]$ let us set

$$(\delta_{\mathbf{T}}^p \xi)(\nu_1, \ldots, \nu_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} T_{\nu_j} \xi(\nu_1, \ldots \hat{\nu_j}, \ldots, \nu_{p+1}),$$

where, the symbol Λ means that the corresponding letter is omitted. It is standard to show that $\delta^p_{\mathbf{T}} : \Lambda^p_n[Y] \to \Lambda^{p+1}_n[Y]$ has the property $\delta^{p+1}_{\mathbf{T}}\delta^p_{\mathbf{T}} = 0$. It is clear also that $\Lambda^p_n[Y] = 0$ for $p > n$ and that $\Lambda^n_n[Y]$ is isomorphic to Y. If we put $\Lambda^0_n[Y] = Y$, $\Lambda^p_n[Y] = 0$ if $p < 0$ and define suitably the maps $\delta^p_{\mathbf{T}}$, then $\{\Lambda^p_n[Y], \delta^p_{\mathbf{T}}\}$ becomes a co-chain complex of $A-$modules. Denote by $\{H^p(Y,\mathbf{T})\}_p$ the cohomology of $\{\Lambda^p_n[Y], \delta^p_{\mathbf{T}}\}$, i.e. $H^p(Y,\mathbf{T}) = \ker \delta^p_{\mathbf{T}}/\mathrm{Im}\delta^{p-1}_{\mathbf{T}}$. We say that the action of $\mathbf{T}$ on Y is non-singular if $H^p(Y,\mathbf{T}) = 0$ for every p. In particular, if Y is equal to H, we denote by $\sigma(\mathbf{T}, H)$ the set of all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $(\lambda - \mathbf{T}) = (\lambda_1, T_1, \ldots, \lambda_n - T_n)$ is singular on H (i.e. it is not non-singular on H). The set $\sigma(\mathbf{T}, H)$ is called the (joint) spectrum of $\mathbf{T}$ on H.

The spaces $\Lambda^p_n[H]$ have a natural Hilbert structure, namely there is the following identification:

$$\Lambda^p_n[H] = \bigoplus_{1 \le i_1 < \cdots < i_p \le n} H.$$

Notice that the adjoint maps $\delta^{p*}_{\mathbf{T}} : \Lambda^{p+1}_n[H] \to \Lambda^p_n[H]$ have the form

$$(\delta^{p*}_{\mathbf{T}}\xi)(\nu_1, \ldots, \nu_p) = \sum_{k=1}^{p} T_k \xi(k, \nu_1, \ldots, \nu_p)$$

for every $\xi \in \Lambda^{p+1}_n[H]$. In particular, by using the maps $\delta^{p*}_{\mathbf{T}}$, one can get easily that $\mathbf{T} = (T_1, \ldots, T_n)$ is non-singular on H if and only if $\mathbf{T}^* = (T^*_1, \ldots, T^*_n)$ is non-singular on H.

Let us set $H^{(n)} = \bigoplus_{k=0}^n \Lambda^k_n[H]$. It is clear that $H^{(n)}$ is a direct sum of 2^n copies of the space H. Denote $\delta_{\mathbf{T}} = \delta^0_{\mathbf{T}} + \oplus \cdots \oplus \delta^n_{\mathbf{T}} : H^{(n)} \to H^{(n)}$.

Theorem 1.3.1. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting system of operators on H. The action of $\mathbf{T}$ is non-singular on H if and only if the operator $\delta_{\mathbf{T}} + \delta^*_{\mathbf{T}}$ is invertible on $H^{(n)}$.*

Corollary 1.3.2. *For any commuting system of operators $\mathbf{T} = (T_1, \ldots, T_n)$ on H we have*

$$\sigma(\mathbf{T}, H) = \mathbb{C}^n \setminus \{\lambda \in \mathbb{C}^n : (\lambda - \mathbf{T})^{-1} \in \mathcal{B}(H^{(n)})\}.$$

Denote by $\alpha(\mathbf{T}) = \delta_{\mathbf{T}} + \delta^*_{\mathbf{T}}$.

Corollary 1.3.3. *If A is any commutative algebra of operators on H then the map*

$$A^n \ni \mathbf{T} \to \alpha(\mathbf{T}) \in \mathcal{B}(H^{(n)})$$

is $\mathbb{R}-$linear.

Denote by $\mathbf{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$, where $1 = 1_H$ is on the $j-$th position. If $\delta_j = \delta_{\mathbf{e}_j}$, we may write for $\mathbf{T} = (T_1, \dots, T_n)$,

$$\alpha(\mathbf{T}) = \sum_{j=1}^{n} (T_j \delta_j + T_j^* \delta_j^*),$$

where $\mathcal{B}(H^{(n)})$ is viewed as a $\mathcal{B}(H)-$module. It is clear that the matrix associated to the operator $\alpha(\mathbf{T})$ on $H^{(n)}$ (considered as a direct sum of 2^n copies of H) does not depend on the space H, moreover, its aspect does not depend on $\mathbf{T}$ either. In particular, we can take $\mathbf{T} = \lambda \in \mathbb{C}^n$.

Proposition 1.3.4. *For any $\lambda \in \mathbb{C}^n$, $\lambda \neq 0$, $\alpha(\lambda)^{-1}$ does exist and $\alpha(\lambda)^{-1} = (|\lambda_1|^2 + \cdots + |\lambda_n|^2)^{-1}\alpha(\lambda)$.*

Corollary 1.3.5. *For any $\lambda \in \mathbb{C}^n$, $\lambda \neq 0$, we have*

$$\begin{aligned} \|\alpha(\lambda)\| &= \|\lambda\| \\ \|\alpha(\lambda)^{-1}\| &= \|\lambda\|^{-1} \end{aligned}$$

where $\|\lambda\|^2 = |\lambda_1|^2 + \dots |\lambda_n|^2$.

Corollary 1.3.6. *The following relations hold,*

$$\delta_j \delta_k^* + \delta_k^* \delta_j = \begin{cases} 1 \text{ if } j = k, \\ 0 \text{ if } j \neq k. \end{cases}$$

Now let us return to an $n-$tuple of commuting operators $\mathbf{T} = (T_1, \dots, T_n)$ on H.

Definition 1.3.7. *The mapping*

$$\mathbb{C}^n \setminus \sigma(\mathbf{T}, H) \ni \lambda \to R(\lambda, \mathbf{T}) = (\alpha(\lambda) - \alpha(\mathbf{T}))^{-1} \in \mathcal{B}(H^{(n)})$$

is called the resolvent of $\mathbf{T}$.

Lemma 1.3.8. *For any commuting $n-$tuple $\mathbf{T} = (T_1, \dots, T_n) \subset \mathcal{B}(H^{(n)})$ the spectrum $\sigma(\mathbf{T}, H)$ is a closed set and the resolvent $R(\lambda, \mathbf{T})$ is an $\mathbb{R}-$analytic function in $\mathbb{C}^n \setminus \sigma(\mathbf{T}, H)$ ($\mathbb{C}^n = \mathbb{R}^{2n}$).*

Lemma 1.3.9. *For any commuting $n-$tuple $\mathbf{T} = (T_1, \dots, T_n) \subset \mathcal{B}(H^{(n)})$ and any $\lambda \in \mathbb{C}^n$ such that $\|\lambda\| > \|\alpha(\mathbf{T})\|$ we have $\lambda \notin \sigma(\mathbf{T}, H)$ and*

$$(\alpha(\lambda) - \alpha(\mathbf{T}))^{-1} = \sum_{k=0}^{\infty} (\alpha(\lambda)^{-1}\alpha(\mathbf{T}))^k \alpha(\lambda)^{-1}, \tag{1.3.1}$$

the series (1.3.1) being absolutely and uniformly convergent on the set $\{\lambda \in \mathbb{C}^n : \|\lambda\| \geq r\}$ with $r > \|\alpha(\mathbf{T})\|$.

Proposition 1.3.10. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a non-singular commuting* $n-$*tuple of operators on* H. *Then both* $\sum_{j=1}^n T_j^* T_j$ *and* $\sum_{j=1}^n T_j T_j^*$ *are invertible.*

Theorem 1.3.11. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a commuting* $n-$*tuple of operators on* $H (\neq 0)$. *Then the spectrum* $\sigma(\mathbf{T}, H)$ *of* $\mathbf{T}$ *is a compact nonempty set in* $\mathbb{C}^n$.

Now, let H be a fixed Hilbert space. For any closed subspace $Y \subset H$, let $Y^{(n)}$ be the Hilbert space defined by $Y^{(n)} = \bigoplus_{k=0}^n \Lambda_n^k[Y]$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n-$tuple of linear operators on H and Y a closed subspace of H, Y reducing $\mathbf{T}$, i.e. $T_j Y \subset Y$ and $T_j^* Y \subset Y$ for any $j = 1, \ldots, n$. Denote by $\mathbf{T}|Y$ the system of restrictions $(T_1|Y, \ldots, T_n|Y)$.

Proposition 1.3.12. *Assume that* $\mathbf{T} = (T_1, \ldots, T_n) \subset \mathcal{B}(H^{(n)})$ *is non-singular on* H *and let* Y *be a closed subspace of* H, Y *reducing* $\mathbf{T}$. *Then* $\mathbf{T}|Y$ *is non-singular if and only if*

$$\alpha(\mathbf{T})^{-1} Y^{(n)} \subset Y^{(n)}.$$

Proposition 1.3.13. *Assume that* $\mathbf{T} = (T_1, \ldots, T_n) \subset \mathcal{B}(H^{(n)})$ *is non-singular on* H *and let* $\{Y_\theta\}_\theta$ *be a family of closed subspaces of* H, Y_θ *reducing* $\mathbf{T}$ *such that* $\mathbf{T}|Y_\theta$ *is non-singular for any* θ. *Then* $\mathbf{T}|Y$ *is non-singular where* $Y = c.l.m.\{Y_\theta\}_\theta$.

For any set $F \in \mathbb{C}^n$ let us denote by ∂F the boundary of F.

Proposition 1.3.14. *Let* Y *be a closed subspace of* H, Y *reducing* $\mathbf{T}$. *Then we have the relation*

$$\partial \sigma(\mathbf{T}, Y) \subset \sigma(\mathbf{T}, H).$$

Corollary 1.3.15. *If* Y *and* Z *are closed subspaces, reducing* $\mathbf{T}$, *such that* $\sigma(\mathbf{T}, Y) \cap \sigma(\mathbf{T}, Z) = \emptyset$, *then* $Y \cap Z = 0$.

For simplicity we will apply the results above to a pair of commuting operators.

Let $\mathbf{T} = (T_1, T_2) \subset \mathcal{B}(H)$ be a pair of commuting operators. Consider the sequence

$$\mathbf{0} \to H \xrightarrow{\delta_T^0} H \oplus H \xrightarrow{\delta_T^1} H \to \mathbf{0} \tag{1.3.2}$$

where $\delta_T^0(x) = T_1 x \oplus T_2 x$ $(x \in H)$ and $\delta_T^1(x_1 \oplus x_2) = T_1 x_2 - T_2 x_1$ $(x_1, x_2 \in H)$. It is clear that $\delta_T^0 \cdot \delta_T^1 = 0$.

We recall that $\mathbf{T}$ is said to be non-singular if the sequence (1.3.2) is exact. The (joint) spectrum $\sigma(\mathbf{T}, H)$ of $\mathbf{T}$ on H is, by definition, the complement of the set of all $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that $\lambda - \mathbf{T} = (\lambda_1 - T_1, \lambda_2 - T_2)$ is non-singular on H.

F-H. Vasilescu has shown in [36] the following results.

Theorem 1.3.16. *Let $\mathbf{T} = (T_1, T_2) \in \mathcal{B}(H)$ be a commuting pair. Then $\mathbf{T}$ is non-singular on H if and only if the operator*

$$\alpha(\mathbf{T}) = \begin{bmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{bmatrix} \tag{1.3.3}$$

is invertible in $H \oplus H$.

Lemma 1.3.17. *If $\mathbf{T} = (T_1, T_2)$ is non-singular on H, then both $T_1T_1^* + T_2T_2^*$ and $T_1^*T_1 + T_2^*T_2$ are invertible on H.*

Corollary 1.3.18. *If $\mathbf{T} = (T_1, T_2) \subset \mathcal{B}(H)$ is a commuting pair, then the spectrum $\sigma(\mathbf{T}, H)$ of $\mathbf{T}$ on H is given by the set*

$$\mathbb{C}^2 \setminus \{\lambda \in \mathbb{C}^2 : (\alpha(\lambda) - \alpha(\mathbf{T}))^{-1} \in \mathcal{B}(H \oplus H)\}.$$

Notice also that $\mathbf{T} = (T_1, T_2) \subset \mathcal{B}(H)$ is non-singular if and only if the matrix

$$\beta(\mathbf{T}) = \begin{bmatrix} T_1 & -T_2^* \\ T_2 & T_1^* \end{bmatrix} = \alpha(T^*)^* \tag{1.3.4}$$

is invertible in $H \oplus H$.

Corollary 1.3.19. *If $\mathbf{T} = (T_1, T_2)$ is non-singular on H, then we have the following commuting relations:*

$$\begin{aligned} T_1^*(T_1T_1^* + T_2T_2^*)^{-1}T_1 + T_2(T_1^*T_1 + T_2^*T_2)^{-1}T_2^* &= 1 \\ T_2^*(T_1T_1^* + T_2T_2^*)^{-1}T_2 + T_1(T_1^*T_1 + T_2^*T_2)^{-1}T_1^* &= 1 \\ T_1^*(T_1T_1^* + T_2T_2^*)^{-1}T_2 - T_2(T_1^*T_1 + T_2^*T_2)^{-1}T_1^* &= 0. \end{aligned} \tag{1.3.5}$$

If H_1, H_2 are Hilbert spaces, then we denote by $H_1 \otimes H_2$ the tensor product of H_1 and H_2 complete for the canonical norm.

Theorem 1.3.20. *Let H_j, $j = 1, 2$ be Hilbert spaces, $T_j \in \mathcal{B}(H_j)$, $H = H_1 \otimes H_2$, $\tilde{T}_1 = T_1 \otimes 1$, $\tilde{T}_2 = 1 \otimes T_2$ and $\tilde{T} = (\tilde{T}_1, \tilde{T}_2) \subset \mathcal{B}(H)$. Then we have*

$$\sigma(\tilde{T}, H) = \sigma(T_1, H_1) \times \sigma(T_2, H_2).$$

Corollary 1.3.21. *With the notations as above we have*

$$\sigma(T_1 \otimes T_2, H) = \sigma(T_1, H_1) \times \sigma(T_2, H_2).$$

Chapter 2

Holomorphic calculus for several variables

2.1 Taylor functional calculus

This section gives a survey of basic properties of the Taylor spectrum, more than the previous section, and Taylor functional calculus, following [22].

One of the central concepts of operator theory is the spectrum of an operator, which is connected with the existence of a functional calculus that provides information about the structure of Banach space operators.

When working with commuting $n-$tuples of Banach space operators this calculus is more involved. There are many possible definitions for a joint spectrum, but the one introduced by J.L. Taylor has the property that there exists a functional calculus for functions analytic on a neighborhood of this spectrum.

We start with the Taylor spectrum and some of its properties, then the split spectrum and essential Taylor spectrum is introduced. We continue with the Taylor functional calculus for the split spectrum and we end this section with the Taylor functional calculus for the Taylor spectrum.

2.1.1 Taylor spectrum

Recall that an $n-$tuple $\mathbf{T} = (T_1, \dots, T_n)$ of commuting operators on a Banach space X is called *Taylor invertible* (or Taylor regular) if $\ker \delta_{\mathbf{T}} = \mathrm{Im}\delta_{\mathbf{T}}$. The *Taylor spectrum* $\sigma_T(\mathbf{T})$ is the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that the $n-$tuple $\mathbf{T} - \lambda = (T_1 - \lambda_1, \dots, T_n - \lambda_n)$ is not Taylor invertible.

Reviewing the definitions 1.1.12, 1.1.13 and remark 1.1.15 may be useful for understanding the following properties of the Taylor spectrum. Also, a more detailed presentation of basic notions may be found in [22]. The properties presented below have complete proves and explanations in [22].

Proposition 2.1.1. *Let $R_1, \dots, R_n, S_1, \dots, S_n$ be commuting operators on a Banach space X satisfying $\sum_{i=1}^n R_i S_i = I$. Then the $n-$tuple $\mathbf{T} = (T_1, \dots, T_n)$ is Taylor invertible.*

Let $\mathcal{A}$ be a unital commutative Banach algebra and $a_1, \dots, a_n \in \mathcal{A}$. The *joint spectrum* is

$$\sigma^{\mathcal{A}}(a_1, \dots, a_n) = \{(f(a_1), \dots, f(a_n)) : f \in \mathcal{M}(\mathcal{A})\},$$

where $\mathcal{M}(\mathcal{A})$ is the set of all multiplicative functionals $f : \mathcal{A} \to \mathbb{C}$ (i.e., the maximal ideal space of $\mathcal{A}$).

Let $a = (a_1, \dots, a_n) \in \mathcal{A}^n$ be a commuting $n-$tuple of elements. The smallest closed unital algebra containing $a_1, \dots, a_n$ is denoted by $\langle a \rangle$. Clearly $\langle a \rangle$ is a unital commutative Banach algebra.

Proposition 2.1.1 implies that

$$\sigma_T(\mathbf{T}) \subset \sigma^{\mathcal{A}}(\mathbf{T}) \tag{2.1.1}$$

for any unital commutative Banach algebra $\mathcal{A} \subset \mathcal{B}(X)$ containing the operators $T_1, \dots, T_n$. In particular, $\sigma_T(\mathbf{T}) \subset \sigma^{\langle \mathbf{T} \rangle}(\mathbf{T})$ for all commuting $n-$tuples $\mathbf{T} \in \mathcal{B}(X)^n$.

The following lemma plays an important role in the study of the Taylor spectrum properties and its proof can be found in [34] as Lemma 2.1.

Lemma 2.1.2. *Let X, Y, Z be Banach spaces, let $T : X \to Y$ and $S : Y \to Z$ be operators satisfying $\mathrm{Im}\ T = \ker S$, and let $\mathrm{Im}\ S$ be closed. Then there exists $\varepsilon > 0$ such that $\mathrm{Im}\ T' = \ker S'$ and $\mathrm{Im}\ S'$ is closed for all pairs of operators $T' : X \to Y$ and $S' : Y \to Z$ satisfying $\|T' - T\| < \varepsilon$, $\|S' - S\| < \varepsilon$ and $S'T' = 0$.*

Corollary 2.1.3. *The set of all commuting Taylor invertible $n-$tuples is relatively open in the set of all commuting $n-$tuples. Consequently, $\sigma_T(\mathbf{T})$ is a compact subset of $\mathbb{C}^n$.*

Moreover, for each $n \in \mathbb{N}$ the mapping $\mathbf{T} \mapsto \sigma_T(\mathbf{T})$ defined on commuting $n-$tuples $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ is upper semi-continuous.

The proof of this corollary can be checked in [22].

An important property of the Taylor spectrum is the projection property, that is, $\sigma_T(T_{i_1}, \dots, T_{i_k}) = P\sigma_T(T_1, \dots, T_n)$ for all $k \leq n,\ 1 \leq i_1 < \cdots < i_k \leq n$, where $P : \mathbb{C}^n \to \mathbb{C}^k$ is the natural projection defined by $P(z_1, \dots, z_n) = (z_{i_1}, \dots, z_{i_k})$.

It is well known that the Taylor spectrum projection property is satisfied both for the *surjective spectrum*

$$\sigma_\delta(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : (T_1 - \lambda_1)X + \cdots + (T_n - \lambda_n)X \neq X\}$$

and the *approximate point spectrum*

$$\sigma_\pi(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : \inf\{\sum_{i=1}^{n} \|(T_i - \lambda_i)x\| : x \in X, \|x\| = 1\} = 0\},$$

see [33]. The proof of projection property for the Taylor spectrum follows [32].

Corollary 2.1.4. *Let $T_1, \dots, T_{n+1} \in \mathcal{B}(X)$ be commuting operators.*

(i) If $(T_1, \dots, T_n)$ is Taylor invertible, then $(T_1, \dots, T_n, T_{n+1})$ is also Taylor invertible.

(ii) If $(T_1, \dots, T_n)$ is Taylor invertible, then there exists $\lambda \in \mathbb{C}$ such that $(T_1, \dots, T_n, T_{n+1} - \lambda)$ is also Taylor invertible.

(iii) Consequently, $\sigma_T(T_1, \dots, T_n) = P\sigma_T(T_1, \dots, T_n, T_{n+1})$, where $P : \mathbb{C}^{n+1} \to \mathbb{C}^n$ is the natural projection onto the first n coordinates.

In particular, since the Taylor spectrum of a single operator $T_1 \in \mathcal{B}(X)$ is equal to the ordinary spectrum, which is non-empty, this corollary implies that $\sigma_T(T_1, \dots, T_n)$ is always non-empty for every commuting $n-$tuple $(T_1, \dots, T_n) \in \mathcal{B}(X)^n$.

Since clearly $\sigma_T(T_{\pi(1)}, \dots, T_{\pi(n)}) = \{(\lambda_{\pi(1)}, \dots, \lambda_{\pi(n)}) : (\lambda_1, \dots, \lambda_n) \in \sigma(T_1, \dots, T_n)\}$ for all permutation $\pi : \{1, \dots, n\} \to \{1, \dots, n\}$, one has the following theorem.

Theorem 2.1.5. (Projection Property of the Taylor Spectrum). *Let* $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ *be a commuting* $n-$*tuple of operators, let* $1 \leq k \leq n$ *and* $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. *Then*

$$\sigma_T(T_{i_1}, \dots, T_{i_k}) = P_{i_1,\dots,i_k}\sigma_T(T_1, \dots, T_n),$$

where $P_{i_1,\dots,i_k} : \mathbb{C}^n \to \mathbb{C}^k$ *is the projection defined by* $P_{i_1,\dots,i_k}(\lambda_1, \dots, \lambda_n) = (\lambda_{i_1}, \dots, \lambda_{i_k})$.

A consequence of the projection property is the spectral mapping property for polynomial mappings.

Theorem 2.1.6. (Spectral Mapping Property). *Let* $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ *be a commuting* $n-$*tuple of operators, let* $k \in \mathbb{N}$ *and let* $p = (p_1, \dots, p_k)$ *be a* $k-$*tuple of polynomials in* n *variables. Let* $p(\mathbf{T}) = (p_1(\mathbf{T}), \dots, p_k(\mathbf{T}))$. *Then*

$$\sigma_T(p(\mathbf{T})) = p(\sigma_T(\mathbf{T})).$$

For the proof see [22].

Remark 2.1.7. Let $\mathcal{A}$ be a unital Banach algebra. By a *spectral system* one means a mapping $\tilde{\sigma}$ that assigns to each commuting tuple $a = (a_1, \dots, a_n) \in (A)^n$ a non-empty compact subset $\tilde{\sigma}(a) \subset \mathbb{C}^n$ such that $\tilde{\sigma}(a) \subset \sigma^{\langle a \rangle}(a)$ and $\tilde{\sigma}$ satisfies the projection property, $\tilde{\sigma}(a_{i_1}, \dots, a_{i_k}) = P_{i_1,\dots,i_k}\tilde{\sigma}(a)$ for all a and $i_1, \dots, i_k$.

A spectral system $\tilde{\sigma}$ is upper semi-continuous if the mapping $(a_1, \dots, a_n) \mapsto \tilde{\sigma}(a_1, \dots, a_n)$ is upper semi-continuous for each n.

Theorem 2.1.5 and Proposition 2.1.1 imply that the Taylor spectrum is an upper semi-continuous spectral system. Further examples of spectral systems are the surjective spectrum and approximate point spectrum.

As in the previous theorem, one can prove that any spectral system satisfies also the spectral mapping property $\tilde{\sigma}(p(\mathbf{T})) = p(\tilde{\sigma}(\mathbf{T}))$ for all $k-$tuples $p = (p_1, \dots, p_k)$ of polynomials in n variables.

The Taylor spectrum has a nice duality property as well.

Theorem 2.1.8. *Let* $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ *be a commuting* $n-$*tuple of operators. Then* $\mathbf{T}$ *is Taylor invertible if and only if* $\mathbf{T}^* = (T_1^*, \dots, T_n^*) \in \mathcal{B}(X^*)^n$ *is Taylor invertible.*

Consequently, $\sigma_T(\mathbf{T}) = \sigma_T(\mathbf{T}^*)$.

The proof of this theorem is based on the following lemma, proved in [32].

Lemma 2.1.9. *Let* X, Y, Z *be Banach spaces, let* $T : X \to Y$ *and* $S : Y \to Z$ *be operators satisfying* $ST = 0$. *the following statements are equivalent:*

(i) $\operatorname{Im} T = \ker S$ *and* $\operatorname{Im} S$ *is closed;*

(ii) $\operatorname{Im} S^* = \ker T^*$ *and* $\operatorname{Im} T^*$ *is closed.*

Remark 2.1.10. If H is a Hilbert space, then it is usual to identify its dual H^* with H. With this convention one has rather

$$\sigma_T(T_1^*, \ldots, T_n^*) = \{(\bar{z}_1, \ldots, \bar{z}_n) : (z_1, \ldots, z_n) \in \sigma_T(T_1, \ldots, T_n)\}$$

for all commuting $n-$tuples $(T_1, \ldots, T_n) \in \mathcal{B}(H)^n$.

2.1.2 Split spectrum and Essential Taylor spectrum

The definitions and results in this section are following [22].

Definition 2.1.11. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *It is said that* $\mathbf{T}$ *is split regular if it is Taylor invertible and the mapping* $\delta_{\mathbf{T}} : \Lambda_n[\mathbf{e}, X]$ *has a generalized inverse, i.e., there exists an operator* $W : \Lambda_n[\mathbf{e}, X] \to \Lambda_n[\mathbf{e}, \mathbf{X}]$ *satisfying* $\delta_{\mathbf{T}} W \delta_{\mathbf{T}} = \delta_{\mathbf{T}}$.

The split spectrum $\sigma_S(\mathbf{T})$ *is the set of all* $\lambda \in \mathbb{C}^n$ *such that the* $n-$*tuple* $\mathbf{T} - \lambda$ *is not split regular.*

The following result is given in [22] to characterize the split regular $n-$tuples of operators and its proof is omitted.

Proposition 2.1.12. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *The following conditions are equivalent:*

(i) $\mathbf{T}$ *is split regular;*

(ii) $\mathbf{T}$ *is Taylor invertible and* $\ker \delta_{\mathbf{T}}^p$ *is a complemented subspace of* $\Lambda_n^p[\mathbf{e}, X]$ *for each* $p = 0, \ldots, n-1$;

(iii) There exists operators $W_1, W_2 : \Lambda_n[\mathbf{e}, X] \to \Lambda_n[\mathbf{e}, X]$ *such that* $W_1 \delta_{\mathbf{T}} + \delta_{\mathbf{T}} W_2 = I_{\Lambda_n[\mathbf{e}, X]}$;

(iv) There exists an operator $V : \Lambda_n[\mathbf{e}, X] \to \Lambda_n[\mathbf{e}, X]$ *such that* $V \delta_{\mathbf{T}} + \delta_{\mathbf{T}} V = I$, $V^2 = 0$ *and* $V \Lambda_n^p[\mathbf{e}, X] \subset \Lambda_n^{p-1}[\mathbf{e}, X]$ $(p = 0, \ldots, n)$. *Equivalently, there are operators* $V_p : \Lambda_n^{p+1}[\mathbf{e}, X] \to \Lambda_n^p[\mathbf{e}, X]$ *(see the diagram below) such that* $V_{p-1} V_p = 0$ *and* $V_p \delta_{\mathbf{T}}^p + \delta_{\mathbf{T}}^{p-1} V_{p-1} = I_{\Lambda_n^p[\mathbf{e}, X]}$ *for every* p *(for* $p = 0$ *and* $p = n$ *this reduces to* $V_0 \delta_{\mathbf{T}}^0 = I_{\Lambda_n^0[\mathbf{e}, X]}$ *and* $\delta_{\mathbf{T}}^{n-1} V_{n-1} = I_{\Lambda_n^n[\mathbf{e}, X]}$, *respectively)*

$$0 \to \Lambda_n^0[\mathbf{e}, X] \underset{V_0}{\overset{\delta_{\mathbf{T}}^0}{\leftrightarrows}} \Lambda_n^1[\mathbf{e}, X] \underset{V_1}{\overset{\delta_{\mathbf{T}}^1}{\leftrightarrows}} \cdots \underset{V_{n-1}}{\overset{\delta_{\mathbf{T}}^{n-1}}{\leftrightarrows}} \Lambda_n^n[\mathbf{e}, X] \to 0.$$

Note that for single operators on a Banach space the split spectrum coincide with the Taylor spectrum and therefore with the regular spectrum. By part (ii) of the above proposition, the split spectrum for $n-$tuples of commuting operators on a Hilbert space coincide with the Taylor spectrum.

Now for $T \in \mathcal{B}(X)$ define the operators $L_T, R_T : \mathcal{B}(X) \to \mathcal{B}(X)$ by $L_T A = TA$ and $R_T A = AT$ $(A \in \mathcal{B}(X))$. For an $n-$tuple $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ write $L_{\mathbf{T}} = (L_{T_1}, \dots, L_{T_n})$ and $R_{\mathbf{T}} = (R_{T_1}, \dots, R_{T_n})$.

The Taylor spectra of $\mathbf{T}, L_{\mathbf{T}}$ and $R_{\mathbf{T}}$ are related in the following way, according to [21].

Theorem 2.1.13. *Let $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$ be a commuting $n-$tuple of operators. Then*

$$\sigma_S(\mathbf{T}) = \sigma_T(L_{\mathbf{T}}) = \sigma_S(L_{\mathbf{T}}) = \sigma_T(R_{\mathbf{T}}) = \sigma_S(R_{\mathbf{T}}).$$

Corollary 2.1.14. *The split spectrum σ_S is an upper semi-continuous spectral system.*

Now we move to the essential Taylor spectrum which is defined below.

Definition 2.1.15. *Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting $n-$tuple of operators on an infinite-dimensional Banach space X. It is said that $\mathbf{T}$ is essentially Taylor regular if* $\dim \ker \delta_{\mathbf{T}} / \operatorname{Im} \delta_{\mathbf{T}} < \infty$. *The essential Taylor spectrum $\sigma_{Te}(\mathbf{T})$ is the set of all $\lambda \in \mathbb{C}^n$ such that $\mathbf{T} - \lambda$ is not essentially Taylor regular.*

The essentially Taylor regular $n-$tuples are an analogy of the Fredholm operators. The following is easy to see.

Proposition 2.1.16. *Let $\mathbf{T} = (T_1, \dots, T_n)$ be an essentially Taylor regular $n-$tuple of operators. Then* $\operatorname{Im} \delta_{\mathbf{T}}$ *is closed.*

If $n = 1$, then T_1 is essentially Taylor regular if and only if T_1 is Fredholm.

2.1.3 Taylor functional calculus for the split spectrum

The existence of the functional calculus for functions analytic on a neighborhood of the Taylor spectrum is the most important property of the Taylor spectrum. Since the construction of the Taylor functional calculus is rather technical, in this section is presented a simpler version for functions analytic on a neighborhood of the split spectrum.

Note that for Hilbert space operators the split spectrum coincide with the Taylor spectrum and so the corresponding functional calculus also coincide. The split functional calculus is also sufficient for the construction of a functional calculus in commutative Banach algebras.

Theorem 2.1.17. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *Suppose that* $\mathbf{T}$ *is split regular, i.e.,* $\ker \delta_{\mathbf{T}} = \operatorname{Im} \delta_{\mathbf{T}}$ *and* $\delta_{\mathbf{T}}$ *has a generalized inverse. Then there exists a neighborhood* U *of* 0 *in* $\mathbb{C}^n$ *and an analytic function* $V : U \to \mathcal{B}(\Lambda_n[\mathbf{e}, X])$ *such that* $V(z)\delta_{\mathbf{T}-z} + \delta_{\mathbf{T}-z}V(z) = I_{\Lambda_n[\mathbf{e},X]}$ *for every* $z \in U$.

Moreover, one can assume that $V(z)^2 = 0$ $(z \in U)$ *and*

$$V(z)\Lambda_n^p[\mathbf{e}, X] \subset \Lambda_n^{p-1}[\mathbf{e}, X] \quad (z \in U, p = 0, \ldots, n).$$

For the proof see [22].

Corollary 2.1.18. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be an* $n-$*tuple of commuting operators on a Banach space* X. *Let* $G = \mathbb{C}^n \setminus \sigma_S(\mathbf{T})$. *Then there exists an operator-valued* $C^\infty-$*function* $V : G \to \mathcal{B}(\Lambda_n[\mathbf{e}, X])$ *such that* $\delta_{\mathbf{T}-z}V(z) + V(z)\delta_{\mathbf{T}-z} = I_{\Lambda_n[\mathbf{e},X]}$ *and*

$$V(z)\Lambda_n^p[\mathbf{e}, X] \subset \Lambda_n^{p-1}[\mathbf{e}, X] \quad (z \in G, p = 0, \ldots, n).$$

The following discussion follows [22].

Fix a commuting $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of bounded linear operators on a Banach space X, the set $G = \mathbb{C}^n \setminus \sigma_S(\mathbf{T})$ and a $C^\infty-$function $V : G \to \mathcal{B}(\Lambda_n[\mathbf{e}, X])$ having the properties of Corollary 2.1.18.

For simplicity, for now on, we will write $\Lambda_n[X]$ instead of $\Lambda_n[\mathbf{e}, X]$.

Consider the space $C^\infty(G, \Lambda_n[X])$, which clearly, can be identified with the set $\Lambda_n[C^\infty(G, X)]$.

The function $V : G \to \mathcal{B}(\Lambda_n[X])$ induces naturally the operator (denoted by the same symbol) $V : C^\infty(G, \Lambda_n[X]) \to C^\infty(G, \Lambda_n[X])$ by

$$(Vy)(z) = V(z)y(z) \quad (z \in G, y \in C^\infty(G, \Lambda_n[X])).$$

Similarly, define the operator $\delta_{\mathbf{T}-z}$ (or δ for short if no ambiguity can arise) acting in $C^\infty(G, \Lambda_n[X])$ by

$$(\delta y)(z) = \delta_{\mathbf{T}-z}y(z) \quad (z \in G, y \in C^\infty(G, \Lambda_n[X])).$$

Clearly, $\delta^2 = 0$, $V\delta + \delta V = I_{\Lambda_n[C^\infty(G,X)]}$ and both V and δ are *graded*, i.e.,

$$V\Lambda_n^p[C^\infty(G, X)] \subset \Lambda_n^{p-1}[C^\infty(G, X)] \quad \text{and}$$
$$\delta\Lambda_n^p[C^\infty(G, X)] \subset \Lambda_n^{p+1}[C^\infty(G, X)].$$

Consider now another set of indeterminates $\mathrm{d}\bar{z} = (\mathrm{d}\bar{z}_1, \ldots, \mathrm{d}\bar{z}_n)$ and the space $\Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$. Let $\bar{\partial} : \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the linear mapping defined by

$$\bar{\partial} f e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \mathrm{d}\bar{z} - j_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q} = \sum_{k=1}^{n} \frac{\partial f}{\partial \bar{z}_k} \mathrm{d}\bar{z}_k \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \mathrm{d}\bar{z}_{j_1} \wedge \cdots \wedge \mathrm{d}\bar{z}_{j_q}.$$

Obviously, $\bar{\partial}^2 = 0$.

The operators V and δ can be lifted to $\Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ in the natural way. Clearly, the properties of V and δ are preserved: $\delta^2 = 0$, $V\delta + \delta V = I$ and both V and δ are graded. Note also that $\delta\bar{\partial} = -\bar{\partial}\delta$ and $(\bar{\partial} + \delta)^2 = 0$.

Let $W : \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the mapping defined in the following way: if $\psi \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$, $\psi = \psi_0 + \cdots + \psi_n$, where ψ_j is the part of ψ of degree j in $\mathrm{d}\bar{z}$, then set $W\psi = \eta_0 + \cdots \eta_n$, where

$$\begin{aligned} \eta_0 &= V\psi_0, \\ \eta_1 &= V(\psi_1 - \bar{\partial}\eta_0), \\ &\vdots \\ \eta_n &= V(\psi_n - \bar{\partial}\eta_{n-1}). \end{aligned} \tag{2.1.2}$$

Note that η_j is the part of $W\psi$ of degree j in $\mathrm{d}\bar{z}$.

Lemma 2.1.19. *Let $W : \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ be the mapping defined by (2.1.2). Then:*

(i) $\operatorname{supp} W\psi \subset \operatorname{supp} \psi$ *for all* ψ;

(ii) if G' is an open subset of G and $\psi \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ satisfies $(\bar{\partial} + \delta)\psi = 0$ on G', then $(\bar{\partial} + \delta)W\psi = \psi$ on G';

(iii) $(\bar{\partial} + \delta)W(\bar{\partial} + \delta) = \bar{\partial} + \delta$.

For the proof see [22].

The differential form

$$(2i)^{-n}\mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_n \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n \tag{2.1.3}$$

will be interpreted as the Lebesgue measure in $\mathbb{C}^n = \mathbb{R}^{2n}$.

Let P be the natural projection $P : \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ that annihilates all terms containing at least one of the indeterminates $e_1, \ldots, e_n$ and leaves invariant all the remaining terms.

Let U be a neighborhood of $\sigma_S(\mathbf{T})$. Let f be a function analytic in U. It is possible to find a compact neighborhood Δ of $\sigma_S(\mathbf{T})$ such that $\Delta \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Define $f(\mathbf{T}) : X \to X$ by

$$f(\mathbf{T})x = \frac{-1}{(2\pi i)^n} \int_{\partial\Delta} Pf(z)Wx\mathbf{e} \wedge \mathrm{d}z \quad (x \in X), \tag{2.1.4}$$

where $\mathrm{d}z$ stands for $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and $\mathbf{e} = e_1 \wedge \cdots \wedge e_n$. By the Stokes formula,

$$f(\mathbf{T})x = \frac{-1}{(2\pi i)^n} \int_{\Delta} \bar{\partial}\varphi Pf(z)Wx\mathbf{e} \wedge \mathrm{d}z,$$

where φ is a $C^\infty-$function equal to 0 on a neighborhood of $\sigma_S(\mathbf{T})$ and to 1 on a neighborhood of $\mathbb{C}^n \setminus \Delta$.

On $\mathbb{C}^n \setminus \Delta$ one has

$$\bar{\partial}\varphi PfWx\mathbf{e} = Pf(\bar{\partial}+\delta)Wx\mathbf{e} = Pfx\mathbf{e} = 0.$$

Thus it is possible to write

$$f(\mathbf{T}) = \frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n} \bar{\partial}Pf(z)Wx\mathbf{e}\wedge \mathrm{d}z. \tag{2.1.5}$$

It is clear from the Stokes theorem that the definition of $f(\mathbf{T})x$ does not depend on the choice of the function φ and, by (2.1.5), it is independent of Δ. Moreover, $f(\mathbf{T})$ does not depend on the choice of the mapping W.

Suppose that W_1, W_2 are two operators satisfying

$$(\bar{\partial}+\delta)W_i x\mathbf{e} = x\mathbf{e} \quad (i=1,2).$$

For those z where $\varphi \equiv 1$ one has

$$(\bar{\partial}+\delta)\varphi f(z)(W_1 - W_2)x\mathbf{e} = 0,$$

and so the form $\eta = (\bar{\partial}+\delta)\varphi f(z)(W_1-W_2)x\mathbf{e}$ has a compact support. By the Stokes theorem one has

$$\begin{aligned}
&\textstyle\int_{\mathbb{C}^n} \bar{\partial}\varphi Pf(z)W_1x\mathbf{e}\wedge \mathrm{d}z - \int_{\mathbb{C}^n} \bar{\partial}\varphi Pf(z)W_2x\mathbf{e}\wedge \mathrm{d}z\\
&= \textstyle\int_{\mathbb{C}^n} P\bar{\partial}\varphi f(z)(W_1-W_2)x\mathbf{e}\wedge \mathrm{d}z = \int_{\mathbb{C}^n} P(\bar{\partial}+\delta)\varphi Pf(z)W_1x\mathbf{e}\wedge \mathrm{d}z\\
&= \textstyle\int_{\mathbb{C}^n} P\eta\wedge \mathrm{d}z = \int_{\mathbb{C}^n} P(\bar{\partial}+\delta)W_1\eta\wedge \mathrm{d}z = \int_{\mathbb{C}^n} \bar{\partial}PW_1\eta\wedge \mathrm{d}z = 0.
\end{aligned}$$

In the same way one can show that

$$f(\mathbf{T})x = \frac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n} \bar{\partial}P\psi\wedge \mathrm{d}z \tag{2.1.6}$$

for any form ψ which satisfies $(\bar{\partial}+\delta)\psi = x\mathbf{e}$ on $\mathbb{C}^n \setminus \sigma_S(\mathbf{T})$.

It is possible to express the mapping PW that appears in the definition of the functional calculus more explicitly. By the definition of W,

$$PWx\mathbf{e} = (-1)^{n-1}V_0\bar{\partial}V_1\bar{\partial}\cdots\bar{\partial}V_{n-1}x\mathbf{e}.$$

Note that one can write formulas (2.1.4) and (2.1.5) also globally:

$$\begin{aligned}
f(\mathbf{T}) &= \tfrac{-1}{(2\pi i)^n}\int_{\partial\Delta} Pf(z)WI\mathbf{e}\wedge \mathrm{d}z = \tfrac{-1}{(2\pi i)^n}\int_{\mathbb{C}^n} \bar{\partial}\varphi Pf(z)WI\mathbf{e}\wedge \mathrm{d}z\\
&= \tfrac{(-1)^n}{(2\pi i)^n}\int_{\mathbb{C}^n} \bar{\partial}fV(\bar{\partial}V)^{n-1}I\mathbf{e}\wedge \mathrm{d}z, \qquad (2.1.7)
\end{aligned}$$

where $I = I_X$ is the identity operator on X. The coefficients of forms in (2.1.7) are $\mathcal{B}(X)-$valued $C^\infty-$functions. Therefore $f(\mathbf{T}) \in \mathcal{B}(X)$.

Proposition 2.1.20. *For $n = 1$ the functional calculus defined by (2.1.7) coincides with the classical functional calculus given by the Cauchy formula.*

The proof can be seen in [22].

Remark 2.1.21. If $\mathbf{T} = (T_1, \dots, T_n)$ is a commuting $n-$tuple of Hilbert space operators, then it is possible to choose $V(z) = (\delta_{\mathbf{T}-z} + \delta^*_{\mathbf{T}-z})^{-1}$ (this mapping does not satisfy $V(z)\Lambda_n^p[X] \subset \Lambda_n^{p-1}[X]$, but this property is not essential for the construction). Formula (2.1.7) is then quite explicit.

The split functional calculus for Hilbert space operators was constructed in [38].

2.1.4 Taylor functional calculus

The construction bellow follows [22] which is based on [38].

Let $\mathbf{T} = (T_1, \dots, T_n)$ be an $n-$tuple of commuting operators on a Banach space X. Let $G = \mathbb{C}^n \setminus \sigma_T(\mathbf{T})$. The key is the following theorem stated here without proof (see [38]).

Theorem 2.1.22. *Let $G' \subset G$ be an open subset. Let $\eta \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G', X)]$ satisfy $(\bar{\partial} + \delta)\eta = 0$. Then there exists $\psi \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G', X)]$ such that $(\bar{\partial} + \delta)\psi = \eta$.*

Moreover, it is possible to find ψ such that its support is contained in any given neighborhood of $\operatorname{supp} \eta$.

Corollary 2.1.23. *Let $x \in X$. Then there exists $\psi_x \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ such that $(\bar{\partial} + \delta)\psi_x = x\mathbf{e}$.*

Let f be a function analytic on a neighborhood of $\sigma_T(\mathbf{T})$. As in (2.1.6), the form ψ_x can be used to define the vector $f(\mathbf{T})x \in X$. However, this definition of $f(\mathbf{T})$ is local, defined for each $x \in X$ separately, and it is not clear at the first glance that $f(\mathbf{T})$ defined in this way is continuous and linear.

For functions f analytic on a neighborhood of $\sigma_S(\mathbf{T})$ it was possible to find a mapping W acting on $\Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(\mathbb{C}^n \setminus \sigma_S(\mathbf{T}), \mathcal{B}(X))]$ such that $Wx\mathbf{e}$ served as ψ_x. Thus all considerations were done in the Banach space $\mathcal{B}(X)$.

For functions f analytic on a neighborhood of $\sigma_T(\mathbf{T})$ it is no longer possible. To simplify the situation, it is possible to consider the Banach space $\mathcal{H}(X)$ of all bounded homogeneous mappings $\varphi : X \to X$, i.e., the mappings satisfying $\varphi(\lambda x) = \lambda x$ $(\lambda \in \mathbb{C}, x \in X)$ and $\|\varphi\| := \sup\{\|\varphi(x)\| : x \in X, \|x\| \leq 1\} < \infty$ (no additivity is assumed).

For $i = 1, \dots, n$ let $L'_{T_i} : \mathcal{H}(X)$ be defined by $L'_{T_i}\varphi = T_i\varphi$ $(\varphi \in \mathcal{H}(X))$. Let $L'_{\mathbf{T}} = (L'_{T_1}, \dots, L'_{T_n})$. Clearly $L'_{\mathbf{T}}$ is a commuting $n-$tuple of bounded linear operators acting on the Banach space $\mathcal{H}(X)$.

Moreover, it is possible to show that $\sigma_T(L'_{\mathbf{T}}) = \sigma_T(\mathbf{T})$. Thus one has:

Corollary 2.1.24. *There is a form* $W_T \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, \mathcal{H}(X))]$ *such that* $(\bar{\partial} + \delta_{L'_{\mathbf{T}-\lambda}})W_T(\lambda) = I\mathbf{e}$, *where* I *is the identity operator on* X.

The form W_T can be also considered to be a mapping $W_T : X \to \Lambda_n^{n-1}[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$. Then $(\bar{\partial} + \delta_{\mathbf{T}-\lambda})W_T(\lambda)x = x\mathbf{e}$ for all $x \in X$.

The definition of the Taylor functional calculus is analogue to the definition of the split functional calculus.

Recall that P is the projection $P : \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)] \to \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ that annihilates all terms containing at least one of the indeterminates $e_1, \dots, e_n$ and leaves invariant all the remaining terms.

Let U be a neighborhood of $\sigma_T(\mathbf{T})$ and let f be a function analytic on U. It is possible to find a compact neighborhood Δ of $\sigma_T(\mathbf{T})$ such that $\Delta \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Define $f(\mathbf{T}) : X \to X$ by

$$f(\mathbf{T}) = \frac{-1}{(2\pi i)^n} \int_{\partial\Delta} P f W_T \wedge \mathrm{d}z. \tag{2.1.8}$$

By Stokes formula,

$$f(\mathbf{T}) = \frac{-1}{(2\pi i)^n} \int_{\Delta} \bar{\partial}\varphi P f W_T \wedge \mathrm{d}z,$$

where φ is a $C^\infty-$function equal to 0 on a neighborhood of $\sigma_T(\mathbf{T})$ and to 1 on a neighborhood of $\mathbb{C}^n \setminus \Delta$.

On $\mathbb{C}^N \setminus \Delta$ one has $\bar{\partial}\varphi P f W_T = P f(\bar{\partial} + \delta)W_T = P f I\mathbf{e} = 0$. Thus it is possible to write

$$f(\mathbf{T}) = \frac{-1}{(2\pi i)^n} \int_{\mathbb{C}^n} \bar{\partial}\varphi P f W_T \wedge \mathrm{d}z. \tag{2.1.9}$$

It is clear from the Stokes theorem that the definition of $f(\mathbf{T})$ does not depend on the choice of the function φ and, by (2.1.9), it is independent of Δ.

It is possible to show that $f(\mathbf{T})$ does not depend on the choice of the form W_T. The following simple lemma will be used frequently.

Proposition 2.1.25. *Let* $\eta \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G, X)]$ *be a differential form with compact support disjoint with* $\sigma_T(\mathbf{T})$ *such that* $(\bar{\partial} + \delta)\eta = 0$. *Then*

$$\int_{\mathbb{C}^n} P\eta \wedge \mathrm{d}z.$$

Proof. Using theorem 2.1.22 and Stokes theorem the result follows immediately. For complete proof see [22]. □

Let $x \in X$ and let $\psi_1, \psi_2 \in \Lambda_n[\mathbf{e}, \mathrm{d}\bar{z}, C^\infty(G,X)]$ satisfy $(\bar{\partial}+\delta)\psi_1 = x\mathbf{e} = (\bar{\partial}+\delta)\psi_2$. Let φ be a $C^\infty-$function equal to 0 on a neighborhood of $\sigma_T(\mathbf{T})$ and to 1 on a neighborhood of $\mathbb{C}^n \setminus U$. Then

$$\int \bar{\partial}\varphi P f \psi_1 \wedge \mathrm{d}z - \int \bar{\partial}\varphi P f \psi_2 \wedge \mathrm{d}z = \int P(\delta+\bar{\partial})\varphi f(\psi_1 - \psi_2) \wedge \mathrm{d}z.$$

On $\mathbb{C}^n \setminus \Delta$ one has $\varphi \equiv 1$, and so $(\delta+\bar{\partial})\varphi f(\psi_1-\psi_2) = f(\delta+\bar{\partial})(\psi_1-\psi_2) = 0$. Thus the form $(\delta+\bar{\partial})\varphi f(\psi_1-\psi_2)$ has a compact support disjoint with $\sigma_T(\mathbf{T})$. By proposition 2.1.25, $\int P(\delta+\bar{\partial})\varphi f(\psi_1-\psi_2)\wedge \mathrm{d}z = 0$.

In particular, the definition of $f(\mathbf{T})$ does not depend on the choice of W_T.

Note that for the definition of $f(\mathbf{T})x$ one can use any form ψ satisfying $(\delta_{\mathbf{T}-z}+\bar{\partial})\psi = x\mathbf{e}$ on a neighborhood of supp φ. This implies that for functions analytic on a neighborhood of $\sigma_S(\mathbf{T})$ the Taylor functional calculus coincides with the split functional calculus introduced in the previous section. By proposition 2.1.20, for $n = 1$ the Taylor functional calculus coincide with the usual functional calculus for single operators.

Lemma 2.1.26. $f(\mathbf{T}) \in \mathcal{B}(X)$.

Proof. See [22]. □

The next result is the first step to show the multiplicativity of the Taylor functional calculus and for its proof, as well as for all the following results proofs, see [22].

Proposition 2.1.27. *Let f be a function analytic on a neighborhood of $\sigma_T(\mathbf{T})$, $1 \le j \le n$ and $g(z) = z_j f(z)$. Then $g(\mathbf{T}) = T_j f(\mathbf{T})$.*

This proposition implies that the definition of the Taylor functional calculus for polynomials coincides with the usual definition.

Proposition 2.1.28. *Let $\mathbf{T} = (T_1, \dots, T_n) \in \mathcal{B}(X)^n$, $\mathbf{S} = (S_1, \dots, S_m) \in \mathcal{B}(X)^m$. Suppose that $(\mathbf{T}, \mathbf{S}) = (T_1, \dots, T_n, S_1, \dots, S_m)$ is a commuting $(n+m)-$tuple and let f and g be functions analytic on a neighborhood of $\sigma_T(\mathbf{T})$ and $\sigma_T(\mathbf{S})$, respectively. Let h be defined by $h(z,w) = f(z)\cdot g(w)$. Then $h(\mathbf{T},\mathbf{S}) = f(\mathbf{T})\cdot g(\mathbf{S})$.*

The following lemma is used in the proof of the next theorem.

Lemma 2.1.29. *Let K be a compact subset of $\mathbb{C}^n$ and let f be a function analytic on an open neighborhood of K. Then there are functions h_j ($j = 1, \dots, n$) analytic on a neighborhood of the set $D = \{(z,z) : z \in K\}$ such that*

$$f(z) - f(w) = \sum_{j=1}^{n} (z_j - w_j)\cdot h_j(z,w).$$

Now denote by H_K the algebra of all functions analytic on a neighborhood of a compact set $K \subset \mathbb{C}^n$ (more precisely, the algebra of all germs of functions analytic on a neighborhood of K).

Theorem 2.1.30. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an $n-$tuple of commuting operators on X. Then*

(i) the mapping $f \mapsto f(\mathbf{T})$ is linear and multiplicative, i.e., the Taylor functional calculus is a homomorphism from $H_{\sigma_T(\mathbf{T})}$ to $\mathcal{B}(X)$;

(ii) if p is a polynomial, $p(z) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$, then $p(\mathbf{T}) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha \mathbf{T}^\alpha$;

(iii) if $f_n \to f$ uniformly on a compact neighborhood of $\sigma_T(\mathbf{T})$, then $f_n(\mathbf{T}) \to f(\mathbf{T})$ in the norm topology;

(iv) $f(\mathbf{T}) \in (\mathbf{T})''$ for each $f \in H_{\sigma_T(\mathbf{T})}$, where $(\mathbf{T})''$ denotes the bicommutant of the set $\{T_1, \ldots, T_n\}$.

It follows from the general theory that the Taylor spectrum satisfies the spectral mapping property for all polynomials (and consequently, for all functions that can be approximated by polynomials uniformly on a neighborhood of the Taylor spectrum). In fact, the spectral mapping property is true for all analytic functions.

The next lemma shows that each operators T_j behaves as the zero on the quotient $\ker \delta_{\mathbf{T}}/\mathrm{Im}\ \delta_{\mathbf{T}}$.

Lemma 2.1.31. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n-$tuple of operators acting on a Banach space X. Let $j \in \{1, \ldots, n\}$. Then $T_j \ker \delta_{\mathbf{T}} \subset \mathrm{Im}\ \delta_{\mathbf{T}}$.*

It is natural to expect that $f(\mathbf{T})$ behaves as $f(0)$ on the quotient space $\ker \delta_{\mathbf{T}}/\mathrm{Im}\ \delta_{\mathbf{T}}$. However, there is a technical difficulty because in general $\mathrm{Im}\ \delta_{\mathbf{T}}$ is not close, and so the quotient $\ker \delta_{\mathbf{T}}/\mathrm{Im}\ \delta_{\mathbf{T}}$ is not a Banach space. Therefore the proof is a little bit more complicated.

Lemma 2.1.32. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n-$tuple of operators on X, let $c = (c_1, \ldots, c_n) \in \sigma_T(\mathbf{T})$ and let f be a function analytic on a neighborhood of $\sigma_T(\mathbf{T})$. Consider the indeterminates $t = (t_1, \ldots, t_n)$ and the operator $\delta_{\mathbf{T}-c,t} : \Lambda_n[t, X] \to \Lambda_n[t, X]$ defined by $\delta_{\mathbf{T}-c,t}\psi = \sum_{j=1}^n (T_j - c_j) t_j \wedge \psi$ for all $\psi \in \Lambda_n[t, X]$. Let $\eta \in \ker \delta_{\mathbf{T}-c,t}$. Then $(f(\mathbf{T}) - f(c))\eta \in \delta_{\mathbf{T}-c,t}\Lambda_n[t, X]$.*

Proposition 2.1.33. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n-$tuple of operators on X, let $c = (c_1, \ldots, c_n) \in \sigma_T(\mathbf{T})$ and let f be a function analytic on a neighborhood of $\sigma_T(\mathbf{T})$. Then the $(n+1)-$tuple $(T_1 - c_1, \ldots, T_n - c_n, f(\mathbf{T}))$ is Taylor invertible if and only if $f(c) \neq 0$.*

Lemma 2.1.34. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* $n-$*tuple of operators on* X *and let* f *be a function analytic on a neighborhood of* $\sigma_T(\mathbf{T})$*. Denote by* $\mathcal{A}$ *the commutative algebra generated by* $T_1, \dots T_n$ *and* $f(\mathbf{T})$*. Let* φ *be a multiplicative functional on* $\mathcal{A}$ *such that* $\varphi(\mathbf{S}) \in \sigma_T(\mathbf{S})$ *for all tuples* $\mathbf{S} = (S_1, \dots, S_m)$ *of operators in* $\mathcal{A}$*. Then* $\varphi f(\mathbf{T}) = f(\varphi(\mathbf{T}))$*.*

Corollary 2.1.35. (Spectral mapping property). *Let* $\tilde{\sigma}$ *be a spectral system on* $\mathcal{B}(X)$ *which is contained in the Taylor spectrum. Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* $n-$*tuple of operators on* X *and let* $f = (f_1, \dots, f_m)$ *be an* $m-$*tuple of functions analytic on a neighborhood if* $\sigma_T(\mathbf{T})$*. Then* $\tilde{\sigma}(f(\mathbf{T})) = f(\tilde{\sigma}(\mathbf{T}))$*.*

In particular, $\sigma_T(f(\mathbf{T})) = f(\sigma_T(\mathbf{T}))$*. Similarly,* $\sigma_{\pi k}(f(\mathbf{T})) = f(\sigma_{\pi k}(\mathbf{T}))$ *and* $\sigma_{\delta k}(f(\mathbf{T})) = f(\sigma_{\delta k}(\mathbf{T}))$ *for all* $k = 0, \dots, n$*.*

Corollary 2.1.36. *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* $n-$*tuple of operators on* X*. Suppose that* $\sigma_T(\mathbf{T}) \subset U_1 \cup U_2$*, where* U_1, U_2 *are open disjoint sets. Then there exists closed subspaces* $X_1, X_2 \subset X$ *invariant for* $T_1, \dots, T_n$ *such that* $X = X_1 \oplus X_2$ *and* $\sigma_T(T_1|X_j, \dots, T_n|X_j) \subset U_j$ *for* $j = 1, 2$*.*

The following theorem was proved by M. Putinar in 1982.

Theorem 2.1.37. (Superposition Principle). *Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be a commuting* $n-$*tuple of operators on* X*, let* $f = (f_1, \dots, f_m)$ *be an* $m-$*tuple of functions analytic on a neighborhood of* $\sigma_T(\mathbf{T})$*, let* $\mathbf{S} = f(\mathbf{T})$*, let* g *be a function analytic on a neighborhood of* $\sigma_T(\mathbf{S})$*, and let* $h(z) = g(f_1(z), \dots, f_m(z))$*. Then* $h(\mathbf{T}) = g(\mathbf{S})$*.*

As a corollary of the Taylor functional calculus it is possible to obtain the properties of the functional calculus in commutative Banach algebras.

Theorem 2.1.38. *Let* $\mathcal{A}$ *be a commutative Banach algebra. To each finite family* $a = (a_1, \dots, a_n)$ *of elements of* $\mathcal{A}$ *and each function* $f \in H_{\sigma(a)}$ *it is possible to assign an element* $f(a) \in \mathcal{A}$ *such that the following conditions are satisfied:*

(i) if $f(z_1, \dots, z_n) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ *is a polynomial in* n *indeterminates, then* $f(a_1, \dots, a_n) = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n}$*;*

(ii) the mapping $f \mapsto f(a_1, \dots, a_n)$ *is an algebra homomorphism from the algebra* $H_{\sigma(a_1, \dots, a_n)}$ *to* $\mathcal{A}$*;*

(iii) if U *is a neighborhood of* $\sigma(x_1, \dots, x_n)$*,* f*,* f_k $(k \in \mathbb{N})$ *are analytic in* U *and* f_k *converge to* f *uniformly on* U*, then*

$$f_k(a_1, \dots a_n) \to f(a_1, \dots, a_n);$$

(iv) *if* $\varphi \in \mathcal{M}(\mathcal{A})$ *and* $f \in H_{\sigma(a_1,\ldots,a_n)}$*, then*

$$\varphi(f(a_1,\ldots,a_n)) = f(\varphi(a_1),\ldots,\varphi(a_n));$$

(v) $\tilde{\sigma}(f(a_1,\ldots,a_n)) = f(\tilde{\sigma}(a_1,\ldots,a_n))$ *for each compact-valued spectral system in* $\mathcal{A}$;

(vi) *if* $a_1,\ldots,a_m \in \mathcal{A}$, $n < m$, $f \in H_{\sigma(a_1,\ldots,a_n)}$ *and* $\tilde{f} \in H_{\sigma(a_1,\ldots,a_m)}$ *satisfy* $\tilde{f}(z_1,\ldots,z_m) = f(z_1,\ldots,z_n)$ *for all* $z_1,\ldots,z_m$ *in a neighborhood of* $\sigma(a_1,\ldots,a_m)$*, then*

$$\tilde{f}(a_1,\ldots,a_m) = f(a_1,\ldots,a_n);$$

(vii) *If* $f_1,\ldots,f_m \in H_{\sigma(a)}$, $b_i = f_i(a)$, $g \in H_{\sigma(b_1,\ldots,b_m)}$ *and* $h \in H_{\sigma(a)}$ *is defined by* $h(z) = g(f_1(z),\ldots,f_m(z))$*, then* $h(a) = g(b)$.

2.2 Von Neumann's inequality

We expect the reader to be familiar with positive, completely positive and completely bounded maps as well as contractions, isometries and dilations notions. For these and the above definitions and results, one can check the book [29] by Vern Paulsen which said that the key idea behind a dilation is to realize an operator or a mapping into a space of operators as "part" of something simpler on a larger space.

Following [29], the simplest case is the *unitary dilation of an isometry.* Let V be an isometry on a Hilbert space H, and let $P = I_H - VV^*$ be the projection onto the orthocomplement of the range of V. If we define U on $H \oplus H = K$ via

$$U = \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix},$$

then it is easy checked that $U^*U = UU^* = I_K$, so that U is a unitary on K. Moreover if we identify H with $H \oplus 0$, then

$$V^n = P_H U^n|_H \text{ for all } n \geq 0.$$

Thus any isometry V can be realized as a restriction of some unitary to one of its subspaces in a manner that also respects the powers of both operators.

In a similar way one can construct an *isometric dilation of a contraction.* Let T be an operator on H, $\|T\| \leq 1$, and let $D_T = (I - T^*T)^{1/2}$. Observe that $\|Th\|^2 + \|D_T h\|^2 = \langle T^*Th, h\rangle + \langle D_T^2 h, h\rangle = \|h\|^2$, for any $h \in H$.

Set

$$\ell^2 = \{(h_1, h_2, \dots) : \ h_n \in H \text{ for all } n, \ \sum_{n=1}^{\infty} \|h_n\|^2 < +\infty\}.$$

This is a Hilbert space with $\|(h_1, h_2, \dots)\|^2 = \sum_{n=1}^{\infty} \|h_n\|^2$, and inner product $\langle (h_1, h_2, \dots), (k_1, k_2, \dots)\rangle = \sum_{n=1}^{\infty} \langle h_n, k_n\rangle$.

Define $V : \ell^2(H) \to \ell^2(H)$ by $V((h_1, h_2, \dots)) = (Th_1, D_T h_1, h_2, \dots)$. Since $\|V((h_1, h_2, \dots))\|^2 = \|Th_1\|^2 + \|D_T h_1\|^2 + \|h_2\|^2 + \cdots = \|(h_1, h_2, \dots)\|^2$, V is an isometry on $\ell^2(H)$. If we identify H with $H \oplus 0 \oplus \dots$ then it is clear that $T^n = P_H V^n|_H$ for all $n \geq 0$.

Therefore we can combine these and get the unitary dilation of a contraction.

Theorem 2.2.1. (Sz.-Nagy's dilation theorem). *Let T be a contraction operator on a Hilbert space H. Then there is a Hilbert space K containing H as a subspace and a unitary operator U on K such that*

$$T^n = P_H U^n|_H.$$

The proof can be found in [29].

Definition 2.2.2. *Whenever Y is an operator on a Hilbert space K, H is a subspace of K, and $X = P_H Y|_H$, we call X a compression of Y.*

Theorem 2.2.1 was used by Sz.-Nagy to prove von Neumann's inequality, stated in the following theorem. Its proof can be checked in V. Paulsen's book [29].

Theorem 2.2.3. (von Neumann's inequality). *Let T be a contraction on a Hilbert space. Then for any polynomial p,*

$$\|p(T)\| \leq \sup\{|p(z) : |z| \leq 1|\}.$$

The following is a slightly refined version of Theorem 2.2.1.

Theorem 2.2.4. (Sz.-Nagy's dilation theorem). *Let $T \in \mathcal{B}(H)$ with $\|T\| \leq 1$. Then there exists a Hilbert space K containing H as a subspace and a unitary U on K with the property that K is the smallest closed reducing subspace for U containing H such that*

$$T^n = P_H U^n|_H, \quad \textit{for all nonnegative integers } n.$$

Moreover, if (U', K') is another pair with the above property, then there is a unitary $V : K \to K'$ such that $Vh = h$ for $h \in H$ and $VUV^* = U'$.

The proof can be check in [29] and the techniques used to prove this theorem can be used to prove a far more general result. Let $X \subseteq \mathbb{C}$ be a compact set, and let $\mathcal{R}(X)$ be the algebra of all rational functions of X. An operator T is said to be a *normal* $\partial X-$*dilation* if there is a Hilbert space K containing H as a subspace and a normal operator N on K with $\sigma(N) \subseteq \partial X$ such that

$$r(T) = P_H r(N)|_H$$

for all $r \in \mathcal{R}(X)$. We shall call N a *minimal normal* $\partial X-$*dilation of* T, provided that K is the smallest reducing subspace for N that contains H.

Now we turn to families of commuting contractions on a Hilbert space H. There is an analogue of Sz.-Nagy unitary dilation theorem due to Ando for a pair of commuting contractions, and consequently there is a two-variable analogue of von Neumann's inequality, both being presented below after a generalization for sets of commuting isometries, following [29].

Theorem 2.2.5. (Dilation theorem) *Let $\{V_1, V_2, \ldots, V_n\}$ be a set of commuting isometries on a Hilbert space H. Then there is a Hilbert space K containing H and a set of commuting unitaries $\{U_1, U_2, \ldots, U_n\}$ on K such that*

$$V_1^{m_1} \cdots V_n^{m_n} = P_H U_1^{m1} \cdots U_n^{m_n}|_H$$

for all sets $\{m_1, \ldots, m_n\}$ of nonnegative integers.

Proof. Let U_1 on K_1 be the minimal unitary dilation of V_1 given by Theorem 2.2.4. Recall that the span of $\{U_1^n H : \ n \in \mathbb{Z}\}$ is dense in K_1.

For $i \neq 1$ we claim that there is a well-defined isometry $W_i : K_1 \to K_1$ given by the formula

$$W_i \left(\sum_{n=-N}^{+N} U_1^n h_n \right) = \sum_{n=-N}^{+N} U_1^n V_i h_n.$$

To see this note that

$$\begin{aligned} \| \sum_{n=-N}^{+N} U_1^n V_i h_n \|^2 &= \sum_{n \geq m} \langle U_1^{n-m} V_i h_n, V_i h_m \rangle + \sum_{n<m} \langle V_i, h_n, U_1^{m-n} V_i h_m \rangle \\ &= \sum_{n \geq m} \langle V_1^{n-m} V_i h_n, V_i h_m \rangle + \sum_{n<m} \langle V_i h_n, V_1^{m-n} V_i h_m \rangle \\ &= \sum_{n \geq m} \langle V_i V_1^{n-m} h_n, V_i h_m \rangle + \sum_{n<m} \langle V_i h_n, V_i V_1^{m-n} h_m \rangle \\ &= \sum_{n \geq m} \langle V_1^{n-m} h_n, h_m \rangle + \sum_{n<m} \langle h_n, V_1^{m-n} h_m \rangle \end{aligned}$$

$$= \sum_{n \geq m} \langle U_1^{n-m} h_n, h_m \rangle + \sum_{n<m} \langle h_n, U_1^{m-n} h_m \rangle$$

$$= \| \sum_{n=-N}^{+N} U_1^n h_n \|^2.$$

This equality of norms proves that W_i is well defined and an isometry. Note that if V_i is unitary, then W_i is onto a dense subspace of K_1 and hence is also unitary.

It is easy to see that $\{U_1, W_2, \ldots, W_n\}$ commute and that

$$V_1^{m_1} \cdots V_n^{m_n} = P_H U_1^{m_1} W_2^{m_2} \cdots W_n^{m_n}|_H.$$

Now continue by next taking the unitary dilation of W_2 on K_2 and extending $U_1, W_3, \ldots, W_n$ to be isometries on K_2. Since U_1 is unitary, its extension will also be a unitary on K_2. Thus, after n such dilations and extensions we shall obtain an $n-$tuple of unitaries on a space K_n with the desired properties.

□

Corollary 2.2.6. *Let $\{V_1, V_2, \ldots, V_n\}$ be a set of commuting isometries on a Hilbert space H and $p_{i,j}$, $i, j = 1, \ldots m$, be polynomials in n variables. Then*

$$\|(p_{i,j}(V_1, \ldots V_n))\|_{\mathcal{B}(H^{(m)})} \leq \sup\{\|(p_{i,j}(z_1, \ldots, z_n))\|_{M_m} : |z_k| \leq 1, 1 \leq k \leq n\}.$$

Corollary 2.2.7. *Let $\{V_1, V_2, \ldots, V_n\}$ be a set of commuting isometries on a Hilbert space H. Then $(V_j^* V_i) \geq (V_i V_j^*) \geq 0$.*

Definition 2.2.8. *Let G be an abelian group. We call $\mathcal{P} \subseteq G$ a spanning cone provided:*

i) $0 \in \mathcal{P}$

ii) *if $g_1, g_2 \in \mathcal{P}$, then $g_1 + g_2 \in \mathcal{P}$*

iii) *if $g \in G$, then there exists $g_1, g_2 \in \mathcal{P}$ such that $g = g_1 - g_2$.*

Definition 2.2.9. *We call $\rho : \mathcal{P} \to \mathcal{B}(H)$ a semigroup homomorphism if $\rho(0) = I$ and $\rho(g_1 + g_2) = \rho(g_1)\rho(g_2)$.*

Theorem 2.2.10. *Let G be an abelian group with spanning cone $\mathcal{P}$, and let $\rho_{\mathcal{P}} \to \mathcal{B}(H)$ be a semigroup homomorphism such that $\rho(g)$ is an isometry for every $g \in \mathcal{P}$. Then there exists a Hilbert space K containing H and a unitary representation $\pi : G \to \mathcal{B}(K)$ such that $\rho(g) = P_H \pi(g)|_H$ for every $g \in \mathcal{P}$.*

Theorem 2.2.11. (Ando's dilation theorem) *Let T_1 and T_2 be commuting contractions on a Hilbert space H. Then there exists a Hilbert space K containing H as a subspace, and commuting unitaries U_1, U_2 on K, such that*

$$T_1^n T_2^m = P_H U_1^n U_2^m|_H$$

for all nonnegative integers n, m.

Corollary 2.2.12. *Let T_1 and T_2 be commuting contractions on a Hilbert space H, and let $p_{i,j}$, $i, j = 1, \dots m$, be polynomials in two variables. Then*

$$\|(p_{i,j}(T_1, T_2))\|_{\mathcal{B}(H^{(m)})} \leq \sup\{\|(p_{i,j}(z_1, z_2))\|_{M_m} : |z_1| \leq 1, |z_2| \leq 1\}.$$

We shall refer to the result above as the *two-variable von Neumann's inequality.* Ando's construction cannot be generalized to more than two commuting contractions, otherwise one could prove an $n-$variable von Neumann inequality, but the analogue of von Neumann's inequality fails for three or more commuting contractions. There are several such counterexamples in the literature. The following is perhaps the simplest.

Example 2.2.13. (Kaijser-Varopoulos). Consider the following operators on $\mathbb{C}^5$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} & 0 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \end{pmatrix}.$$

It is easy to check that $\|A_i\| \leq 1$ and that $A_i A_j = A_j A_i$, for $1 \leq i, j \leq 3$. If one considers the polynomial

$$p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3,$$

then $\|p\|_\infty = \sup\{|p(z_1, z_2, z_3)| : |z_i| \leq 1\} = 5$, as the little calculus shows. However,

$$p(A_1, A_2, A_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3\sqrt{3} & 0 & 0 & 0 & 0 \end{pmatrix},$$

and since $3\sqrt{3} \geq 5$, the analogue of von Neumann's inequality fails.

Next we give some applications of Ando's theorem, begining with some re-formulations due to Sz.-Nagy and Foiaş in *Harmonic Analysis of Operators on Hilbert Space,* published by the American Elsevier in 1970.

Theorem 2.2.14. (Commutant lifting theorem) *Let T be a contraction on a Hilbert space H and let (U, K) be the minimal unitary dilation of T. If R commutes with T, then there exists on operator S commuting with U such that $\|R\| = \|S\|$ and $RT^n = P_H SU^n|_H$ for $n \geq 0$.*

The following is an equivalent re-formulation of Ando's theorem. Given $T_i \in \mathcal{B}(H_i)$, $i = 1, 2$, and $A \in \mathcal{B}(H_1, H_2)$, we say that A *intertwines* T_1 and T_2 provided that $AT_1 = T_2A$. Observe that this is equivalent to $\begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$ commuting with $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$.

Theorem 2.2.15. (Intertwining dilation theorem) *Let T_i, $i = 1, 2$, be contraction operators on Hilbert spaces H_i with minimal unitary dilations (U_i, K_i). If A intertwines T_1 and T_2, then there exists R intertwining U_1 and U_2 such that $\|A\| = \|R\|$ and*

$$AT_1^n = T_2^n A = P_{H_2} RU_1^n|_{H_1} = P_{H_2} U_2^n R|_{H_1}$$

for all $n \geq 0$.

Definition 2.2.16. *A set of operators $\{T_i\}$ is said to doubly commute if $T_iT_j = T_jT_i$ and $T_i^*T_j = T_jT_i^*$ for $i \neq j$.*

This is equivalent to requiring that the C^*-algebra generated by each of these operators commutes with the C^*-algebra generated by any of the other operator, but does not require that each of these C^*-algebras be commutative.

The simplest example of tuples of doubly commuting operators is the tuple of diagonal matrices and the simplest example of $n-$tuples of doubly commuting isometries is the tuple of multiplication operators $(M_{z_1}, \ldots, M_{z_n})$ by the coordinate functions on the Hardy space $H^2(\mathbb{D}^n)$ over the polydisc $\mathbb{D}^n$ $(n \geq 2)$.

Theorem 2.2.17. (Sz.-Nagy-Foiaş) *Let $\{T_i\}_{i=1}^n$ be a doubly commuting family of contractions on a Hilbert space H. Then there exists a Hilbert space K containing H as a subspace, and a doubly commuting family of unitary operators $\{U_i\}_{i=1}^n$ on K, such that*

$$T_1(k_1)\cdots T_n(k_n) = P_h U_1^{k_1} \cdots U_n^{k_n}|_H, \text{ where } T(k) = \begin{cases} T^k, & k \geq 0 \\ T^{*-k}, & k < 0. \end{cases}$$

Moreover, if K is the smallest reducing subspace for the family $\{U_i\}_{i=1}^n$ containing H, then $\{U_i\}_{i=1}^n$ is unique up to unitary equivalence. That is, if $\{U_i'\}_{i=1}^n$ and K' are another such set and space, then there is a unitary $W : K \to K'$ leaving H fixed such that $WU_iW^* = U_i'$, $i = 1, \ldots, n$.

2.2.1 Von Neumann's inequality on Fock-spaces

In 1994, Gelu Popescu [30] has extended the von Neumann inequality in the following way.

Let $H^2(D)$ be the Hardy space of analytic functions on the unit disk D, i.e.,

$$H^2(D) = \{u(\lambda) = \sum_{k=0}^{\infty} \lambda^k a_k : a_k \in \mathbb{C}, \|u\|^2_{H^2(D)} = \sum_{k=0}^{\infty} |a_k|^2 < \infty\}.$$

J. von Neumann's well-known inequality on Hilbert space operators asserts that if T is a contraction on a complex Hilbert space H (i.e., $\|T\| \leq 1$) and p is an analytic polynomial in $H^2(D)$, then the operator $p(T)$ satisfies the inequality

$$\|p(T)\| \leq \sup_{|\lambda|\leq 1} |p(\lambda)| = \sup_{q\in(\mathcal{P}_+)_1} \|pq\|_{H^2(D)}, \tag{2.2.1}$$

where $(\mathcal{P}_+)_1$ stands for the unit ball of $\mathcal{P}_+ \subset H^2(D)$ and $\mathcal{P}_+$ denote the set of all analytic polynomials in $H^2(D)$.

For a natural number n let $\mathcal{B}(H)^n$ denote the set of $n-$tuples $\mathbf{T} = (T_1, \ldots, T_n)$ of elements from $\mathcal{B}(H)$ (i.e. the algebra of all bounded operators on the Hilbert space H). We define a Banach space norm on $\mathcal{B}(H)^n$ asking that the injective map

$$\pi : \mathcal{B}(H)^n \to \mathbb{M}_n(\mathcal{B}(H))$$

given by

$$\pi(\mathbf{T})_{1j} = T_j \text{ for } 1 \leq j \leq n \text{ and } \pi(\mathbf{T})_{ij} = 0 \text{ for } i > 1,$$

be an isometry. The norm gives $\mathcal{B}(H)^n$ the product topology, and for each $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{B}(H)^n$ we have

$$\|\mathbf{T}\| = \|\pi(\mathbf{T})\| = \| \sum_{i=1}^{n} T_i T_i^* \|^{\frac{1}{2}}.$$

Let $(\mathcal{B}(H)^n)_1$ denote the unit ball of $\mathcal{B}(H)^n$, i.e.,

$$(\mathcal{B}(H)^n)_1 = \{(T_1, \ldots, T_n) \in \mathcal{B}(H)^n : \sum_{i=1}^{n} T_i T_i^* \leq I_H\}.$$

Let's consider the full Fock-space

$$\mathcal{F}(H_n) = \mathbb{C}I \oplus \bigoplus_{m \geq 1} H_n^{\otimes m}, \tag{2.2.2}$$

where H_n is an $n-$dimensional complex Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. We shall denote by $\mathcal{P}$ the set of all $p \in \mathcal{F}(H_n)$ of the form

$$p = a_0 + \sum_{\substack{1 \leq i_1, \ldots, i_k \leq n \\ 1 \leq k \leq m}} a_{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbb{N}, \tag{2.2.3}$$

where $a_0, a_{i_1 \ldots i_k} \in \mathbb{C}$ and the sum contains only a finite number of summands.

The set $\mathcal{P}$ may be viewed as the algebra of the polynomials in n noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}$ as multiplication.

Let $p(T_1, \ldots, T_n)$ stand for the operator acting on H, given by

$$p(T_1, \ldots, T_n) = a_0 I_H + \sum a_{i_1 \ldots i_k} T_{i_1} \ldots T_{i_k}. \tag{2.2.4}$$

The von Neumann inequality for $(\mathcal{B}(H)^n)_1$ asserts that if $(T_1, \ldots, T_n) \in (\mathcal{B}(H)^n)_1$ and $p \in \mathcal{P}$, then

$$\|p(T_1, \ldots, T_n)\| \leq \sup_{q \in \mathcal{P}_1} \|p \otimes q\|_{\mathcal{F}(H_n)}, \tag{2.2.5}$$

where

$$\mathcal{P}_1 = \{p \in \mathcal{P} : \|p\|_{\mathcal{F}(H_n)} \leq 1\}.$$

2.2.2 Commuting, diagonalizable contractions

B.A Lotto in [18] extends von Neumann inequality for commuting, diagonalizable contractions satisfying some additional conditions, therefore this section follows [18] and presents the main results.

We say that a subspace M of $\mathbb{C}^N$ is a reducing subspace of the $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of operators on $\mathbb{C}^N$ if both M and $M^\perp$ are invariant under every T_j.

Theorem 2.2.18. *Suppose that $\mathbf{T} = (T_1, \ldots, T_n)$ is an $n-$tuple of commuting, diagonalizable contractions on $\mathbb{C}^N$ that has no nontrivial reducing subspace. If there exists a diagonalizable contraction X on $\mathbb{C}^N$ that commutes with every T_j such that $I - X^*X$ has rank 1, then von Neumann's inequality holds for $\mathbf{T}$.*

The hypotheses that $\mathbf{T}$ has no nontrivial reducing subspace is not really a restriction, as shown by the following proposition.

Proposition 2.2.19. *Suppose that M is a reducing subspace for $\mathbf{T}$. Then von Neumann's inequality holds for $\mathbf{T}$ if and only if holds for both $\mathbf{T}|M$ and $\mathbf{T}|M^\perp$.*

Fix an $n-$tuple $\mathbf{T} = (T_1, \ldots, T_n)$ of commuting, diagonalizable contractions on $\mathbb{C}^N$. Since any set of commuting, diagonalizable operators is simultaneously diagonalizable, there is a basis $V = \{v_1, \ldots, v_N\}$ of $\mathbb{C}^N$ consisting entirely of eigenvectors for every T_j. For $w \in \mathbb{C}^N$, let D_w denote the operator defined by $D_w v_j = w_j v_j$ and on the rest on $\mathbb{C}^N$ by linearity. Note that each T_j is of the form D_w for an appropriate choice of w and that $p(\mathbf{T})$ is also of this form for any polynomial p in n variables.

To prove von Neumann's inequality for $\mathbf{T}$, it is enough (by scaling) to show that if p is a polynomial with $\|p\|_\infty \leq 1$, then $p(\mathbf{T})$ is a contraction.

Lemma 2.2.20. *Let $w \in \mathbb{C}^N$. Then D_w is a contraction if and only if the matrix*

$$((1 - w_j \bar{w}_k)\langle v_j, v_k\rangle)_{j,k=1}^N \tag{2.2.6}$$

is positive semidefinite.

Lemma 2.2.21. *Let $w \in \mathbb{C}^N$ and suppose that D_w is a contraction. Then either D_w is a scalar multiple of the identity or $|w_j| < 1$ for $1 \leq j \leq N$.*

It follows from Lemma 2.2.21 that if D_w is a contraction, not a scalar multiple of the identity, then it is unitarily equivalent to its Sz.-Nagy-Foiaş model, which is described below.

Let d denote the rank of the operator $I - D_w^* D_w$, and let H_d^2 denote the Hardy space of $\mathbb{C}^d-$valued functions that are holomorphic in D and have square-summable Taylor coefficients. Then D_w is unitarily equivalent to the restriction of the backward shift operator S^* to an invariant subspace K. (The backward shift is the adjoint of the shift operator S of "multiplication by z" on H_d^2.) Under this unitary equivalence, the basis V of eigenvectors is mapped to a basis of K consisting of eigenvectors for S^*. These are of the form $c_j k_{\bar{w}_j}$, where $0 \neq c_j \in \mathbb{C}^d$ and $k_{\bar{w}_j}(z) = (1 - w_j z)^{-1}$ is the reproducing kernel function at $\bar{w}_j$ for scalar-valued H^2. We note that the model operator (and hence D_w) possesses an H^∞ functional calculus for which von Neumann's inequality is valid, that is, $\|f(S^*|K)\| \leq \|f\|_\infty$ for all $f \in H^\infty$.

Theorem 2.2.22. *Suppose that there exists $z \in \mathbb{C}^N$ such that D_z is a contraction and $I - D_z^* D_z$ has rank 1. The following are then equivalent for any $w \in \mathbb{C}^N$:*

- *i) D_w is a contraction.*
- *ii) The matrix $((1 - w_j \bar{w}_k)/(1 - z_j \bar{z}_k))_{j,k=1}^N$ is positive semidefinite.*
- *iii) There is a function $f \in H^\infty$ with $\|f\|_\infty \leq 1$ such that $f(z_j) = w_j$.*
- *iv) There is a function $f \in H^\infty$ with $\|f\|_\infty \leq 1$ such that $f(D_z) = D_w$.*

Corollary 2.2.23. *Suppose that there exists $z \in \mathbb{C}^N$ such that D_z is a contraction and $I - D_z^* D_z$ has rank 1. Then von Neumann's inequality holds for any $n-$tuple of contractions of the form $(D_{w_1}, \ldots, D_{w_n})$.*

Theorem 2.2.24. *If* $\mathbf{T}$ *is any $n-$tuple of commuting, diagonalizable operators on a two-dimensional space, then von Neumann's inequality holds for* $\mathbf{T}$.

Now suppose that $\mathbf{T}$ is an $n-$tuple of commuting, diagonalizable operators on a three-dimensional space. If $\mathbf{T}$ has a nontrivial reducing subspace, the von Neumann's inequality holds for $\mathbf{T}$ by Proposition 2.2.19 and the preceding result for two dimensional. In the contrary case it is not always possible to apply Theorem 2.2.18. For the next theorem, assume (without loss of generality) that the eigenvectors v_1, v_2 and v_3 have unit norm.

Theorem 2.2.25. *The hypotheses of Theorem 2.2.18 can be satisfied if and only if*

$$|\alpha\beta|^2 + |\alpha\gamma|^2 + |\beta\gamma|^2 = |\alpha\beta\gamma|^2 + 2\mathrm{Re}(\alpha\bar{\beta}\gamma), \tag{2.2.7}$$

where $\alpha = \langle v_1, v_2 \rangle$, $\beta = \langle v_1, v_3 \rangle$ and $\gamma = \langle v_2, v_3 \rangle$.

So far no one has been able to determine whether von Neumann's inequality holds for an arbitrary $n-$tuple of commuting, diagonalizable contractions on a three-dimensional space. This question is important in light of the following result.

Theorem 2.2.26. *If von Neumann's inequality holds for all $n-$tuples of commuting, diagonalizable contractions on a three-dimensional space, then it holds for all $n-$tuples of commuting contractions on a three-dimensional space.*

Theorem 2.2.26 follows from the following lemma, which implies that a counterexample to von Neumann's inequality could be perturbed to a counterexample consisting of diagonalizables.

Lemma 2.2.27. *Any $n-$tuple of commuting operators on $\mathbb{C}^3$ can be perturbed to commuting diagonalizables.*

Proof. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be an $n-$tuple of commuting operators on $\mathbb{C}^3$. If any T_j has two distinct eigenvalues, then the matrices of $T_1, \ldots, T_n$ with respect to a basis that realizes the Jordan form of T_j are all block diagonal. One can then perturb each $n-$tuples of blocks to commuting diagonalizables using, if necessary, the two-dimensional result mentioned above as Theorem 2.2.24. One thus obtain a perturbation of $\mathbf{T}$ consisting of commuting diagonalizables.

One may therefore assume that every T_j has a single eigenvalue. After subtracting a constant multiple of the identity from T_j (which does not affect the possibility of perturbation), one may assume that every T_j is nilpotent.

Now the only matrices that commute with

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

are polynomials of this matrix. Hence, if any T_j has this as its Jordan form, then $T_k = p_k(T_j)$ for some polynomial p_k. If one now perturbs T_j to a diagonalizable T_j' and then replaces T_k by $p_k(T_j')$, the resulting $n-$tuple will be the desired perturbation consisting of diagonalizables.

The other possibility is that $T_j^2 = 0$ for all j. Now the only nilpotent matrices of order two that commute with

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the one of the form

$$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & a' & 0 \\ 0 & 0 & 0 \\ 0 & b' & 0 \end{pmatrix}.$$

Since these matrices do not commute with each other if $bb' \neq 0$, one must have that every T_j is of the same form, either the one on the left or the one on the right. In either case, every T_j is a polynomial in two fixed, commuting matrices. As any two commuting matrices can be perturbed to commuting diagonalizables, one can replace T_j by its polynomial in these perturbations as above to get the desired perturbation of commuting diagonalizables. The lemma is now proved. □

In [18], B.A. Lotto showed that von Neumann's inequality

$$\|p(\mathbf{T})\| \leq \|p\|_\infty \tag{2.2.8}$$

holds for all polynomials p in n variables, where $\mathbf{T}$ is an $n-$tuple of commuting, diagonalizable contractions on $\mathbb{C}^N$ that satisfies some additional hypotheses. Here $\|p(\mathbf{T})\|$ denotes the operator norm of $p(\mathbf{T})$ and $\|p\|_\infty$ denotes the supremum norm of p over the unit polydisc of $\mathbb{C}^N$. In [19] it is presented an example to show that the extra hypotheses cannot be removed. The example is based on an example due to Kaijser and Varopoulos (see, N.Th. Varopoulos, *On an inequality of von Neumann and an application of the metric theory of tensor products to operator theory*, J. Funct. Anal. 16 (1974), 83-100) that shows that (2.2.8) can fail with $n = 3$ and $N = 5$. Therefore the following theorem can be found in [19].

Theorem 2.2.28. *There are three commuting, diagonalizable contractions T_1, T_2 and T_3 on $\mathbb{C}^5$ and a polynomial p in three variables such that $\|p(T_1, T_2, T_3)\| > \|p\|_\infty$.*

2.2.3 Analytic functions

Ambrozie, Engliš, V. Müller in 2007 in [1] have proved the following theorem and extended the von Neumann's inequality as presented below.

Theorem 2.2.29.

i) Let T be a Hilbert space contraction with spectrum contained in the open unit disc. Then T is unitarily equivalent to a restriction of the backward shift of infinite multiplicity to an invariant subspace.

ii) More generally, a Hilbert space contraction T is unitarily equivalent to a restriction of the backward shift of infinite multiplicity to an invariant subspace if and only if $T^n \to 0$ in the strong operator topology.

Let D be a nonempty open domain in $\mathbb{C}^n$. Set $D^* = \{\bar{z} : z \in D\}$. For $f : D \to \mathbb{C}$, set $\tilde{f}(w) := \overline{f(\bar{w})}$, $w \in D^*$.

Definition 2.2.30. *Let D be a nonempty open domain in $\mathbb{C}^n$. A Hilbert space $\mathcal{H}$ of functions analytic on D is called a $D-$space if the following conditions are satisfied:*

i) $\mathcal{H}$ is invariant under the operators Z_j, $j = 1, \ldots, n$, of multiplication by the coordinate functions,

$$(Z_j f)(z) = z_j f(z); \quad f \in \mathcal{H},\ z = (z_1, \ldots, z_n) \in D.$$

It follows from the next assumption and the closed graph theorem that the operators Z_j are, in fact, bounded.

ii) For each $z \in D$, the evaluation function $f \to f(z)$ is continuous on $\mathcal{H}$.

By the Riesz representation theorem there is $C_z \in \mathcal{H}$ such that $f(z) = \langle f, C_z \rangle$ for all $f \in \mathcal{H}$. Define the function $C(z, w) := C_{\bar{w}}(z)$, $z \in D, w \in D^*$. (The function $C(z, \bar{w})$ is known as the reproducing kernel of $\mathcal{H}$.)

iii) $C(z, w) \neq 0$ for all $z \in D$ and $w \in D^*$.

Lemma 2.2.31. *Let $\mathcal{H}$ be a $D-$space and $\{\psi_k\}$ an orthonormal basis in $\mathcal{H}$. Then*

$$C(z, w) = \sum_{l=1}^{\infty} \psi_k(z) \tilde{\psi}_k(w)$$

where the series converges uniformly and absolutely on each compact subset of $D \times D^*$.

Let H be a Hilbert space. Denote by $\mathcal{H} \otimes H$ the completed Hilbertian tensor product. Consider the multiplication operators M_{z_j} on $\mathcal{H} \otimes H$ defined by

$$M_{z_j} = Z_j \otimes I_H, \quad j = 1, \ldots, n.$$

and write

$$M_z = (M_{z_1}, \ldots, M_{z_n}) \in \mathcal{B}(\mathcal{H} \otimes H)^n.$$

The basic prototype of a $D-$space $\mathcal{H}$ is the Hardy space H^2 on the unit disc. In this case $C(z, w) = (1 - zw)^{-1}$ and M_z^* is a backward shift of infinite multiplicity. Theorem 2.2.29 can thus be restated as saying that if an operator T on H satisfies $\frac{1}{C}(T,T^*) = I - TT^* \geq 0$ and (i) $\sigma(T) \subset D$ or (ii) $T^{*n} \to 0$, then T^* is unitarily equivalent to the restriction of M_z^* to an invariant subspace.

Let H be a Hilbert space. Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting tuple of operators. Denote by $\sigma(\mathbf{T})$ the Taylor spectrum of $\mathbf{T}$, and let

$$M_{\mathbf{T}} = (L_{T_1^*}, \dots, L_{T_n^*}, R_{T_1}, \dots, R_{T_n}).$$

Here $L_A(X) = AX$ and $R_A(X) = XA$ are the left and right multiplication operators by A on $\mathcal{B}(H)$. Let f, g be analytic functions in a neighborhood of $\sigma(\mathbf{T})$ and h be analytic in a neighborhood of $\sigma(\mathbf{T}^*)$. Then we have the equalities

$$L_{f(\mathbf{T})} = f(L_{\mathbf{T}}) \tag{2.2.9}$$
$$R_{h(\mathbf{T}^*)} = h(R_{\mathbf{T}^*}) \tag{2.2.10}$$
$$g(\mathbf{T}^*) = \tilde{g}(\mathbf{T}^*). \tag{2.2.11}$$

Let f be a function analytic on a neighborhood of $\sigma(M_{\mathbf{T}})$. Define $f(\mathbf{T}, \mathbf{T}^*) \in \mathcal{B}(H)$ by $f(\mathbf{T}, \mathbf{T}^*) := F(M_{\mathbf{T}})(I)$.

Lemma 2.2.32. *Let $\mathbf{T}$ be a tuple on a Hilbert space. Let $f = f(z)$, $g = g(z,w)$ and $h = h(w)$ be analytic in neighborhoods of $\sigma(\mathbf{T})$, $\sigma(M_{\mathbf{T}})$ and $\sigma(\mathbf{T}^*)$, respectively. Set $F(z,w) := f(z)g(w,z)h(w)$. Then*

$$F(\mathbf{T}, \mathbf{T}^*) = f(\mathbf{T})g(\mathbf{T}, \mathbf{T}^*)h(\mathbf{T}^*).$$

Let now $D \in \mathbb{C}^n$ be a domain and $\mathcal{H}$ a $D-$space. Let $\mathbf{T}$ be a $n-$tuple of operators on a Hilbert space H such that $\sigma(\mathbf{T}) \subset D$.

Define the linear map $C_{\mathbf{T}} : H \to \mathcal{H} \otimes H$ by

$$C_{\mathbf{T}} h := \int_{\partial\Delta} C_{\bar{w}} \otimes k_{\mathbf{T}^*}(w) h \quad (h \in H), \tag{2.2.12}$$

where Δ is a bounded open domain with smooth boundary such that $\sigma(\mathbf{T}^*) \subset \Delta$ and $\overline{\Delta} \subset D^*$. This definition is motivated by the formal identities

$$C_{\mathbf{T}}(z) = C(z, \mathbf{T}^*) = \int_{\partial\Delta} C(z,w) k_{\mathbf{T}^*}(w) = \int_{\partial\Delta} C_{\bar{w}}(z) k_{\mathbf{T}^*}(w).$$

Lemma 2.2.33. *We have*

$$\langle C_{\mathbf{T}} h, f \otimes h' \rangle = \langle h, f(\mathbf{T}) h' \rangle$$

for all $h, h' \in H$ and $f \in \mathcal{H}$. In particular, C_T does not depend on the choice of Δ. Moreover, $C_{\mathbf{T}} : H \to \mathcal{H} \otimes H$ is a bounded operator.

Let $\mathcal{H}$ be a $D-$space. Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Set $D_{\mathbf{T}} = \frac{1}{C}(\mathbf{T}, \mathbf{T}^*)^{1/2}$ and define the mapping $V : H \to \mathcal{H} \otimes H$ by $V := (I_{\mathcal{H}} \otimes D_{\mathbf{T}})C_{\mathbf{T}}$.

By Lemma 2.2.33, it is easy to see that $V^* : \mathcal{H} \otimes H \to H$ is defined by the formula $V^*(f \otimes h) = f(\mathbf{T})D_{\mathbf{T}}h$ $(f \in \mathcal{H},\ h \in H)$.

Theorem 2.2.34. *Let $\mathcal{H}$ be a $D-$space. Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Then the mapping $V : H \to \mathcal{H} \otimes H$ defined by $V := (I_{\mathcal{H}} \otimes D_{\mathbf{T}})C_{\mathbf{T}}$ satisfies the equality*

$$VT_j^* = M_{z_j}^* V \quad (j = 1, \ldots, n).$$

Theorem 2.2.35. *Let $\mathcal{H}$ be a $D-$space. Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Then the mapping $V : H \to \mathcal{H} \otimes H$ defined by $V := (I_{\mathcal{H}} \otimes D_{\mathbf{T}})C_{\mathbf{T}}$ is an isometry.*

Corollary 2.2.36. *Let $\mathcal{H}$ be a $D-$space. Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Then $\mathbf{T}^*$ is unitarily equivalent to a restriction of M_z^* to an invariant subspace.*

Corollary 2.2.37. *Let $D \subset \mathbb{C}^n$ be a bounded domain, let $\mathcal{H}$ be a $D-$space and suppose that $\mathcal{H}$ is a subspace of $L^2(\phi)$ where ϕ is a nonnegative finite Borel measure with* supp $\phi \subset \overline{D}$. *Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Then $\mathbf{T}$ has a normal dilation $\mathbf{N}$. More precisely, there are a Hilbert space $K \supset H$ and a commuting $n-$tuple $\mathbf{N}$ of normal operators on K such that $\sigma(\mathbf{N}) \subset \overline{D}$ and*

$$p(\mathbf{T}) = P_H p(\mathbf{N})|H$$

for all polynomials p is n variables.

Corollary 2.2.38. (von Neumann inequality) *Let $\mathcal{H}$ be a $D-$space satisfying the conditions of the preceding corollary. Let $\mathbf{T}$ be an $n-$tuple of operators on H such that $\sigma(\mathbf{T}) \subset D$ and $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$. Then*

$$\|p(\mathbf{T})\| \leq \sup_{z \in D} |p(z)|$$

for all polynomials p in n variables. Moreover, if D is polynomially convex, then the above von Neumann inequality is true for all functions analytic on D.

From now on are studied models for $n-$tuples of operators which need not satisfy $\sigma(\mathbf{T}) \subset D$. Stronger assumptions on $\mathcal{H}$ are therefore needed.

Let $\mathcal{H}$ be a $D-$space such that:

i) $\mathcal{H}$ contains the constant functions (hence, also all polynomials) and the polynomials are dense in $\mathcal{H}$,

ii) $\frac{1}{C}$ is a polynomial.

The monomials z^α ($\alpha \in \mathbb{Z}_+^n$) are then a (non-orthogonal) basis for $\mathcal{H}$. Arranging them in some order, by the Gram-Schmidt orthogonalization we can find (and fix from now on) an orthonormal basis $\{\psi_k\}$ consisting of polynomials and such that, conversely, any polynomial is a finite linear combination of ψ_k.

For $m \geq 0$ set

$$f_m(z,w) = \sum_{k=0}^{\infty} \psi_k(z) \frac{1}{C}(z,w) \tilde{\psi_k}(w).$$

By Lemma 2.2.31, the series converges and $f_0(z,w) = 1$. Note that

$$f_m(z,w) = 1 - \sum_{k=1}^{m-1} \psi_k(z) \frac{1}{C}(z,w) \tilde{\psi_k}(w)$$

is a polynomial for each m. In particular, $f_m(\mathbf{T}, \mathbf{T}^*)$ makes sense for any operator tuple $\mathbf{T}$.

Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting $n-$tuple of operators satisfying $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$ and $\sup_m \|f_m(\mathbf{T}, \mathbf{T}^*)\| < \infty$. Define $V : H \to \mathcal{H} \otimes H$ by

$$Vh = \sum \psi_k \otimes D_{\mathbf{T}} \psi_k(\mathbf{T}^*)h. \tag{2.2.13}$$

Proposition 2.2.39. *Let $D, \mathcal{H}, \mathbf{T}$ be as above. Let $h \in H$. Then $I = f_0(\mathbf{T}, \mathbf{T}^*) \geq f_1(\mathbf{T}, \mathbf{T}^*) \geq f_2(\mathbf{T}, \mathbf{T}^*) \geq \ldots$ and we have $\|Vh\|^2 = \|h\|^2 - \lim_m \langle f_m(\mathbf{T}, \mathbf{T}^*)h, h\rangle$.*

Lemma 2.2.40. *For all $g \in H$ and any polynomial $f \in \mathcal{H}$,*

$$V^*(f \otimes g) = f(\mathbf{T}) D_{\mathbf{T}} g.$$

Proposition 2.2.41. *Let $\mathcal{H}$ be a $D-$space. Suppose that $\frac{1}{C}$ is a polynomial and the polynomials are (contained and) dense in $\mathcal{H}$. Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a $n-$tuple of commuting operators satisfying $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$ and $\lim_m \langle f_m(\mathbf{T}, \mathbf{T}^*)h, h\rangle = 0$ for each $h \in H$. Then $\mathbf{T}^*$ is unitarily equivalent to the restriction of M_z^* to an invariant subspace.*

Proposition 2.2.42. *Assume that C^{-1} is a polynomial and that $\mathcal{H}$ contains the constant functions. Then*

$$\frac{1}{C}(Z, Z^*) = \|1\|^2_{\mathcal{H}} P,$$

where P is the orthogonal projection onto the constant functions in $\mathcal{H}$.

Lemma 2.2.43. *Let $\mathcal{H}$ be a $D-$space such that $\frac{1}{C}$ is a polynomial and the polynomials are dense in $\mathcal{H}$. Let f_m be defined as above. Then $f_m(Z, Z^*)$ is the orthogonal projection onto $\vee\{\psi_k, k \geq m\}$. In particular, $f_m(Z, Z^*) \geq 0$ and $\lim_{m\to\infty} f_m(Z, Z^*)h = 0$ for each $h \in H$.*

Corollary 2.2.44. *Let $\mathcal{H}$ be a $D-$space such that the polynomials are dense in $\mathcal{H}$ and $\frac{1}{C}$ is a polynomial. Let $\mathbf{T}$ be a commuting $n-$tuple of operators on a Hilbert space H. The following statements are equivalent:*

i) $\mathbf{T}^*$ *is unitarily equivalent to the restriction M_z^* to an invariant subspace;*

ii) $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$ *and* $\lim_m f_m(\mathbf{T}, \mathbf{T}^*)h = 0$ $(h \in H)$.

Corollary 2.2.45. (dilation and a von Neumann inequality) *Let $\mathcal{H}$ be a $D-$space on a bounded domain $D \subset \mathbb{C}^n$. Suppose that $\mathcal{H}$ is the closure of the polynomials in $L^2(\phi)$ where ϕ is a finite nonnegative Borel measure with* $\operatorname{supp} \phi \subset \overline{D}$, *and that $\frac{1}{C}(z, w)$ is a polynomial. Let $\mathbf{T}$ be an $n-$tuple of operators on a Hilbert space H such that $\frac{1}{C}(\mathbf{T}, \mathbf{T}^*) \geq 0$ and $f_m(\mathbf{T}, \mathbf{T}^*) \to 0$ in the strong operator topology. Then the multiplications $\mathbf{N} = (N_1, \dots, N_n)$ by the coordinate functions on $L^2(\phi) \otimes H$ are a normal dilation of $\mathbf{T}$, and for any polynomial p*

$$\|p(\mathbf{T})\| \leq \sup_{z\in D} |p(z)|.$$

Moreover, if D is polynomially convex, then $\sigma(\mathbf{T}) \in \overline{D}$. To see this, denote by σ_π the approximate point spectrum. Clearly $\sigma_\pi(M_z) \subset \sigma_\pi(\mathbf{N}) = \sigma(\mathbf{N}) \subset \overleftarrow{D}$, so $\sigma(M_z) \subset \overline{D}$. Further $\sigma_\pi(\mathbf{T}^*) \subset \sigma(M_z^*) \subset \overline{D}^*$ so $\sigma(\mathbf{T}^*) \subset \overline{D}^*$ and $\sigma(\mathbf{T}) \subset \overline{D}$.

2.2.4 Contractions that commute according to a graph

David Opěla in [27] generalizes Andô's theorem and Parrott's example as follows. He states that any $n-$tuple of contractions that commute according to a graph without a cycle can be dilated to an $n-$tuple of unitaries that commute according to that graph. Conversely, if the graph contains a cycle, he constructed a counterexample.

Now we review some definitions and concepts that help understanding Opěla's work presented below.

Definition 2.2.46. *If H and K are two Hilbert spaces, $T \in \mathcal{B}(H)$, $W \in \mathcal{B}(K)$ operators, we say that W is an extension of T, if T is the restriction of W to H, i.e., if $Tx = Wx$ for all $x \in H$.*

Definition 2.2.47. *With the same notation, we say that W is a dilation of T, or that T is a compression of W, if $T^n = PW^n|H$, for all $n \geq 0$, where P is the orthogonal projection from K onto H.*

By a result of Sarason, we can say equivalently, that W has the following structure

$$W = \begin{pmatrix} T & 0 & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad \begin{matrix} \dots & H \\ \dots & \tilde{K} \ominus H, \\ \dots & K \ominus \tilde{K} \end{matrix} \tag{2.2.14}$$

that is, H is the orthogonal difference of two invariant subspaces for W.

Definition 2.2.48. *$A \in \mathcal{B}(H)$ is a co-isometry if A^* is an isometry, that is $AA^* = I$.*

The first result of the theory, due to Sz.-Nagy, asserts that every contraction $T \in \mathcal{B}(H)$ has a co-isometric extension. In fact, there is a minimal co-isometric extension $W_0 \in \mathcal{B}(K_0)$ of T that is characterized by

$$K_0 = \overline{\text{span}}\{(W_0^*)^n H;\ n \in \mathbb{N}\}.$$

Any two minimal co-isometric extensions W_0 and $\tilde{W}_0$ are unitarily equivalent via a unitary that restricts to the identity on H - we denote this by $W_0 \cong \tilde{W}_0$. For every co-isometric extension W of T we have $W \cong W_0 \oplus \tilde{W}$, where $\tilde{W}$ is another co-isometry.

If T is already a co-isometry, then any of its co-isometric extensions is the direct sum of T and another co-isometry. A minimal co-isometric extension of an isometry is a unitary.

If T is a contraction, denote by V any of the co-isometric extensions of the contraction T^*. Let U be a minimal co-isometric extension of the isometry V^*, then one easily checks (by Sarason's characterization) that U is a dilation of T.

Then Sz.-Nagy dilation theorem follows - any contraction has a unitary dilation. Given $T \in \mathcal{B}(H)$ there is a minimal unitary dilation U_0, unique up to a unitary that restricts to I_H, such that any unitary dilation U of T has the form $U \cong U_0 \oplus U_1$.

By the Wold decomposition, any co-isometry is the orthogonal sum of (a finite or infinite number of) copies of the unilateral backward shift and a unitary.

Andô's theorem states that given a pair of commuting contractions, we can extend each of them to a co-isometry such that the two co-isometries commute. As in the single operator case, if the contractions are isometries, the co-isometries can be constructed to be unitaries. The dilation version asserts that for any pair of commuting contractions A, B, we can find commuting unitaries U, V such that

$$PU^m V^n|H = A^m B^n, \text{ for all } m, n \geq 0.$$

Note that the equality above implies that U (resp. V) is a dilation of A (resp. B) by taking $n = 0$ ($m = 0$, respectively). However, not every pair of dilations satisfies this property. This more restrictive relation is more desirable, since it implies that the map $U \mapsto A, V \mapsto B$ extends to an algebra homomorphism between the operator algebras generated by U, V and A, B, respectively.

The commutant lifting theorem of Foiaş and Sz.-Nagy asserts that given a pair of commuting contractions and a co-isometric extension (or a unitary dilation) of one of them, one can extend (or dilate) the other one to a contraction that commutes with the given co-isometric extension (unitary dilation, respectively).

Andô's theorem cannot be generalised to three (or more) commuting contractions. The first counterexample was constructed by S. Parrott. There are some sufficient conditions on when an $n-$tuple of commuting contractions can be dilated to an $n-$tuple of commuting unitaries, e.g., if the operators doubly commute, see the below theorem (Theorem 12.10 in V. Paulsen's book [29]).

Theorem 2.2.49. *(Theorem 12.10 in [29]) Let $\{T_j\}_{j=1}^n$ be a doubly commuting family of contractions on a Hilbert space H. Then there exists a Hilbert space K containing H as a subspace, and a doubly commuting family of unitary operators $\{U_i\}_{i=1}^n$ on K, such that*

$$T_1(k_1)\cdots T_n(k_n) = P_H U_1^{k_1}\cdots U_n^{k_n}|H, \quad \textit{where } T(k) = \begin{cases} T^k, & k \geq 0 \\ T^{*-k}, & k < 0. \end{cases}$$

Moreover, if K is the smallest reducing subspace for the family $\{U_i\}_{i=1}^n$ containing H, then $\{U_i\}_{i=1}^n$ is unique up to unitary equivalence. That is, if $\{U_i'\}_{i=1}^n$ and K' are another such set and space, then there is a unitary $W : K \to K'$ leaving H fixed such that $WU_iW^* = U_i'$, $i = 1, \ldots, n$.

There are generalizations of Andô's result to an $n-$tuple of contractions, for example, Gaşpar and Rácz in [16] assume only that the $n-$tuple is cyclic commutative. Their work is briefly described below.

Definition 2.2.50. *We say that a family* $\mathbf{T} = (T_1, \ldots, T_n)$ *of linear bounded operators in a Hilbert space* H*, is cyclic commutative if*

$$T_1T_2\ldots T_n = T_nT_1T_2\ldots T_{n-1} = \cdots = T_2T_3\ldots T_nT_1$$

Theorem 2.2.51. *Let* $\mathbf{T} = (T_1, \ldots, T_n)$ *be a cyclic commutative family of contractions in a Hilbert space* H*. There exists a cyclic commutative family* $\mathbf{V} = (V_1, \ldots V_n)$ *of isometries in a Hilbert space* $K \supset H$*, with the property that*

$$PV_{i_1}^{m_1}\cdots V_{i_n}^{m_n}|H = T_{i_1}^{m_1}\cdots T_{i_n}^{m_n}, \quad (m_j \geq 0, j = 1, 2, \ldots, n)$$

where $(i_1, i_2, \ldots, i_n)$ *is an arbitrary permutation of* $(1, 2, \ldots, n)$*, and* P *is the orthogonal projection of* K *onto* H*.*

Proposition 2.2.52. *Every cyclic commutative family* $\mathbf{V} = (V_1, \ldots V_n)$ *of isometries in a Hilbert space* K *can be extended to a cyclic commutative family* $\mathbf{U} = (U_1, \ldots, U_n)$ *of unitary operators in a Hilbert space* $K_1 \supset K$*.*

Theorem 2.2.53. *For every cyclic commutative family* $\mathbf{T} = (T_1, \ldots, T_n)$ *of contractions in a Hilbert space* H*, there exists a cyclic commutative family* $\mathbf{U} = (U_1, \ldots, U_n)$ *of unitary operators in a Hilbert space* $K \supset H$*, such that*

$$PU_{i_1}^{m_1}\cdots U_{i_n}^{m_n}|H = T_{i_1}^{m_1}\cdots T_{i_n}^{m_n}, \quad (m_j \geq 0, j = 1, 2, \ldots, n)$$

for every permutation $(i_1, i_2, \ldots, i_n)$ *of* $(1, 2, \ldots, n)$*.*

These results were further generalized by G. Popescu. Instead of starting with $n-$tuples of contractions, one can work with row contractions, that is, with $n-$tuples satisfying $\sum_j T_jT_j^* \leq I$. This case has been extensively studied by T. Bhattacharyya in [8]. Main results from [8] are presented below starting with reviewing definitions and results for single contractions and then continue with tuples of commuting contractions.

All the Hilbert spaces in T. Bhattacharyya's work are over the complex field and are separable. Given two Hilbert spaces H and K, the notations $K \supset H$ and $H \subset K$ will mean that H is a closed subspace of K or that H is isometrically embedded into K, i.e., there is a linear isometry V mapping H into K. In the latter case, one shall identify H with the closed subspace VH of K. Any bounded operator T on H is then identified with the bounded operator VTV^* on VH.

Definition 2.2.54. *Let* $H \subset K$ *be two Hilbert spaces. Suppose* T *and* V *are bounded operators on* H *and* K *respectively. Then* V *is called a dilation of* T *if*

$$T^nh = P_HV^nh$$

for all $h \in H$ and all nonnegative integers n where P_H is the projection of K onto H. A dilation V of T is called minimal if $\overline{span}\{V^n h : h \in H, n = 0, 1, 2, \dots\} = K$. An isometric (respectively unitary) dilation of T is a dilation V which is an isometry (respectively unitary).

Definition 2.2.55. *An element T of $\mathcal{B}(H)$, algebra of all bounded operators of H, is called a contraction is $\|T\| \leq 1$.*

Definition 2.2.56. *Two dilations V_1 and V_2 on the Hilbert spaces K_1 and K_2 respectively, of the same operator T on H are called unitarily equivalent is there is a unitary $U : K_1 \to K_2$ such that $UV_1U^* = V_2$.*

Theorem 2.2.57. *For every contraction T on a Hilbert space H, there is a minimal isometric dilation which is unique up to a unitary equivalence.*

Definition 2.2.58. *An isometry V on a Hilbert space K is called a unilateral shift if there is a subspace L of K satisfying*

i) $V^n L \perp L$ *for all* $n = 1, 2, \dots$ *and*

ii) $L \oplus VL \oplus V^2 L \cdots = K$.

The subspace L is called the generating subspace for V and $\dim L$ *is called the multiplicity of V.*

Definition 2.2.59. *Let $H \subset K$ be two Hilbert spaces. Suppose V and U are bounded operators on H and K respectively, such that*

$$U^n h = V^n h, \quad \textit{for all } h \in H.$$

Then U is called an extension of V. A unitary extension is an extension which is also a unitary operator.

An extension U of a bounded operator V is also a dilation of V because $P_H U^n h = P_H V^n h = V^n h$, for all $h \in H$.

Remark 2.2.60. Let H be a Hilbert space and $T \in \mathcal{B}(H)$. Suppose V is a dilation of T on a Hilbert space $K_1 \supset H$ and U is an extension of V on a Hilbert space $K_2 \supset K_1$. Then U is a dilation of T. Indeed,

$$\begin{aligned} P_H U^n h &= P_H V^n h \text{ because } U \text{ is an extension of } V \text{ and } h \in H \subset K_1 \\ &= T^n h \text{ because } V \text{ is a dilation of } T. \end{aligned}$$

Definition 2.2.61. *A unitary operator U on a Hilbert space K is called a bilateral shift is there is a subspace L of K satisfying* *i)* $U^nL \perp L$ *for all* $n \neq 0$ *and*

ii) $\bigoplus_{n=-\infty}^{\infty} U^nL = K$.

The subspace L *is called a generating subspace for* U *and* $\dim L$ *is called the multiplicity of* U.

Lemma 2.2.62. *A unilateral shift* V *on* K *always has an extension to a bilateral shift. Moreover, the extension preserves multiplicity.*

Corollary 2.2.63. *An isometry* V *on* H *always has an extension to a unitary.*

Theorem 2.2.64. *For every contraction* T *on a Hilbert space* H, *there is a minimal unitary dilation which is unique up to unitary equivalence.*

Theorem 2.2.65. (von Neumann's inequality) *For every polynomial* $p(z) = a_0 + a_1 z + \cdots + a_m z^m$, *let*

$$\|p\| = \sup\{|p(z)| : |z| \leq 1\}.$$

If T *is a contraction and* p *is a polynomial, then*

$$\|p(T)\| \leq \|p\|.$$

After reviewing these notions of single contractions, T. Bhattacharyya [8] continues with definitions and results on tuples of commuting operators, as follows.

Definition 2.2.66. *Let* $H \subset K$ *be two Hilbert spaces. Suppose* $\mathbf{T} = (T_1, \ldots, T_n)$ *and* $\mathbf{V} = (V_1, \ldots, V_n)$ *are tuples of bounded operators acting on* H *and* K *respectively, i.e.,* $T_i \in \mathcal{B}(H)$ *and* $V_i \in \mathcal{B}(K)$. *The operator tuple* $\mathbf{V}$ *is called a dilation of the operator tuple* $\mathbf{T}$ *if*

$$T_{i_1}T_{i_2}\ldots T_{i_k} = P_H V_{i_1}V_{i_2}\ldots V_{i_k}$$

for all $h \in H$, $k \geq 0$, *and all* $1 \leq i_1, i_2, \ldots, i_k \leq n$.

If V_i *are isometries with orthogonal ranges, i.e.,* $V_i^* V_j = \delta_{ij}$ *for* $1 \leq i, j \leq n$, *then* $\mathbf{V}$ *is called an isometric dilation. A dilation* $\mathbf{V}$ *of* $\mathbf{T}$ *is called minimal if* $\overline{span}\{V_{i_1}V_{i_2}\ldots V_{i_k}h : h \in H, k \geq 0 \text{ and } 1 \leq i_1, i_2, \ldots, i_k \leq n\} = K$.

Theorem 2.2.67. *For a pair* $\mathbf{T} = (T_1, T_2)$ *of commuting contractions on a Hilbert space* H, *there is a commuting isometric dilation* $\mathbf{V} = (V_1, V_2)$.

Theorem 2.2.68. *Let V_1 and V_2 be two commuting isometries on a Hilbert space H. Then there is a Hilbert space K and two commuting unitaries U_1 and U_2 on K such that*

$$U_1 h = V_1 h \text{ and } U_2 h = V_2 h \text{ for all } h \in H.$$

In other words, two commuting isometries can be extended to two commuting unitaries.

Theorem 2.2.69. *Given two commuting contractions T_1 and T_2 on a Hilbert space H, there is a Hilbert space $K \supset H$ and two commuting unitaries U_1 and U_2 on K such that*

$$T_1^m T_2^n h = P_H U_1^m U_2^n h, \text{ for all } h \in H \text{ and } n, m \geq 0. \tag{2.2.15}$$

Corollary 2.2.70. (von Neumann's inequality) *Let T_1 and T_2 be two commuting contractions acting on a Hilbert space H. Suppose $p(z_1, z_2)$ is any polynomial in two variables. Then*

$$\|p(T_1, T_2)\| \leq \sup\{|p(z_1, z_2)| : |z_1| \leq 1, |z_2| \leq 1\}. \tag{2.2.16}$$

Definition 2.2.71. *Let H be a Hilbert space and let $\mathbf{T} = (T_1, \ldots, T_n)$ be a tuple of bounded operators acting on H. Then $\mathbf{T}$ is called a contractive tuple if $\sum_{i=1}^n T_i T_i^* \leq \mathbf{1}_H$. The tuple is called commuting if $T_i T_j = T_j T_i$ for all $i, j = 1, 2, \ldots, n$. The positive operator $(\mathbf{1}_H - \sum_{i=1}^n T_i T_i^*)^{1/2}$ and the closure of its range are respectively called the Defect operator of $\mathbf{T}$ and the Defect space of $\mathbf{T}$ and are denoted by $D_{\mathbf{T}}$ and $\mathcal{D}_{\mathbf{T}}$.*

Contractivity of a tuple is equivalent to demanding that for all $h_1, \ldots, h_n \in H$,

$$\|T_1 h_1 + T_2 h_2 + \cdots + T_n h_n\|^2 \leq \|h_1\|^2 + \|h_2\|^2 + \cdots + \|h_n\|^2.$$

A prototype of a commuting contractive tuple is the so-called $n-$shift which we shall simply call the shift since n is fixed.

Suppose we have a single linear contraction T on a Hilbert space H. Consider the usual Toeplitz algebra $\mathcal{T}$, i.e., the unital C^*-algebra generated by the unilateral shift S. Then there is a unique unital completely positive map φ on $\mathcal{T}$ which maps S to T and moreover any 'sesqui-polynomial' $\sum a_{k,l} S^k (S_*)^l$ to $\sum a_{k,l} T^k (T^*)^l$. Actually this is a way of looking at the Sz.-Nagy dilation of contractions. Indeed if we consider the minimal Stinesring representation π of φ, we see that $\pi(S)$ is nothing but the minimal isometric dilation of T.

Given a Hilbert space L and $k = 0, 1, 2, \ldots,$ we write $L^{\otimes_s k}$ for the symmetric tensor product of k copies of L. The space $L^{\otimes_s 0}$ is defined as the one dimensional vector space $\mathbb{C}$ with its usual inner product. For $k \geq 2$, $L^{\otimes_s k}$ is the subspace of the full tensor product $L^{\otimes k}$ consisting of all vectors fixed under the natural representation of the permutation group σ_k,

$$L^{\otimes_s k} = \{\xi \in L^{\otimes k} : U_\pi \xi = \xi, \pi \in \sigma_k\},$$

U_π denoting the isomorphism of $L^{\otimes k}$ defined on elementary tensors by

$$U_\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\pi^{-1}(1)} \otimes x_{\pi^{-1}(2)} \otimes \cdots \otimes x_{\pi^{-1}(n)}, \quad x_i \in L. \tag{2.2.17}$$

Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis on $\mathbb{C}^n$. Then an orthonormal basis for the full tensor product space $L^{\otimes k}$ is $\{e_{i_1}, e_{i_2}, \dots, e_{i_k} : 1 \leq i_1, \dots, i_k \leq n\}$. The full Fock space over L and the symmetric Fock space over L are respectively

$$\Gamma(L) = \mathbb{C} \oplus L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes k} \oplus \cdots$$

and

$$\Gamma_s(L) = \mathbb{C} \oplus L \oplus L^{\otimes_s 2} \oplus \cdots \oplus L^{\otimes_s k} \oplus \cdots .$$

In both the Fock space, the one dimensional subspace $\mathbb{C} \oplus \{0\} \oplus \{0\} \cdots$ is called the vacuum space. The unit norm element $(1, 0, 1, \dots)$ in this space is called the vacuum vector and is denoted by ω. The projection on to the vacuum space is denoted by E_0. Define the creation operator tuple $\mathbf{V} = (V_1, V_n, \dots, V_n)$ on $\Gamma(\mathbb{C}^n)$ by

$$V_i \xi = e_i \otimes \xi \text{ for } i = 1, 2, \dots, n \text{ and } \xi \in \Gamma(\mathbb{C}^n).$$

It is easy to see that V_i are isometries with orthogonal ranges. Denoting by P_+ the orthogonal projection onto the subspace $\Gamma_s(L)$ of $\Gamma(L)$, define the tuple of bounded operators $\mathbf{S} = (S_1, S_2, \dots, S_n)$ on $\Gamma_s(L)$ by

$$S_i \xi = P_+(e_i \otimes \xi) \text{ for } i = 1, 2, \dots, n \text{ and } \xi \in \Gamma(\mathbb{C}^n).$$

Since V_i are isometries, the S_i are contractions. The projection P_+ acts on the full tensor product space $L^{\otimes k}$ by the following action on the orthonormal basis:

$$P_+(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}) = \frac{1}{k!} \sum e_{\pi(i_1)} \otimes e_{\pi(i_2)} \otimes \cdots \otimes e_{\pi(i_k)}$$

where π varies over the permutation group σ_k. Using this it is easy to see that S forms a commuting tuple. The operator tuple $\mathbf{S}$ is called the *commuting $n-$shift*. For contractivity of $\mathbf{S}$, we start with the following lemma.

Lemma 2.2.72. $\mathbf{1}_{\Gamma(\mathbb{C}^n)} - \sum V_i V_i^*$ *is the one-dimensional projection onto the vacuum space.*

This lemma immediately gives the contractivity property for $\mathbf{V}$ and $\mathbf{S}$:

Corollary 2.2.73. $\mathbf{V}$ *and* $\mathbf{S}$ *are contractive tuples.*

Before we proceed further, it will be helpful to list some properties of the shift which we will need later.

Lemma 2.2.74. $\sum_{i=1}^{n} S_i^* S_i$ *is an invertible operator on* $\Gamma_s(\mathbb{C}^n)$.

The C^*−subalgebra of $\mathcal{B}(\Gamma_s(\mathbb{C}^n))$ generated by $S_1, \ldots, S_n$ will be denoted by $\mathcal{T}_n$ and called the Toeplitz C^*−algebra. The Toeplitz C^*−algebra is unital. We do not have to, a priori, include $\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}$ in the C^*−algebra $\mathcal{T}_n$ because the operator $\sum S_i^* S_i$ is invertible in $\mathcal{B}(\Gamma_s(\mathbb{C}^n))$. Since C^*−algebras are inverse closed, $(\sum S_i^* S_i)^{-1}$ is in the C^*−algebra generated by $S_1, \ldots, S_n$ and hence $\mathcal{T}_n$ is unital. The subalgebra of $\mathcal{T}_n$ consisting of polynomials in $S_1, \ldots, S_n$ and $\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}$ will be denoted by $\mathcal{A}$. The following two lemmas give more information about $\mathcal{T}_n$.

Lemma 2.2.75. $\mathcal{T}_n = \overline{span} \mathcal{A}\mathcal{A}^*$.

Lemma 2.2.76. *All compact operators are in* $\mathcal{T}_n$.

Now we proceed towards developing the model and dilation for a given commuting contractive tuple $\mathbf{T}$ on a Hilbert space H, as in [8]. By an operator space we shall mean a vector subspace of $\mathcal{B}(L)$ where L is a Hilbert space. Given an operator space $\mathcal{E}$ and an algebra $\mathcal{A} \subseteq \mathcal{E}$, a completely positive map φ, from $\mathcal{E}$ to $\mathcal{B}(H)$ for some Hilbert space H, is called an $\mathcal{A}$−morphism if

$$\varphi(AX) = \varphi(A)\varphi(X), \text{ for any } A \in \mathcal{A} \text{ and } X, AX \in \mathcal{E}.$$

Every unital $\mathcal{A}$−morphism $\varphi : \mathcal{T}_n \to \mathcal{B}(H)$ for some Hilbert space H gives rise to a commuting contractive tuple $(T_1, \ldots, T_n)$ on H by way of $T_i = \varphi(S_i)$, $i = 1, 2, \ldots, n$. Indeed, $\sum T_i T_i^* = \varphi(\sum S_i S_i^*) \leq \varphi(\mathbf{1}_{\Gamma_s(\mathbb{C}^n)}) = \mathbf{1}_H$ and $T_i T_j = \varphi(S_i)\varphi(S_j) = \varphi(S_i S_j) = \varphi(S_j S_i) = \varphi(S_j)\varphi(S_i) = T_j T_i$ for all $1 \leq i, j \leq n$. Given any commuting contractive tuple $\mathbf{T}$ acting on H, the aim is to produce an $\mathcal{A}$−morphism from the C^*−algebra $\mathcal{T}_n$ to $\mathcal{B}(H)$. This will be achieved by the help of the following theorem. The $\mathcal{A}$−morphism is the key element in finding dilation and proving von Neumann inequality. In fact, it could be thought of as the model for $\mathbf{T}$. We shall associate a completely positive map $P_{\mathbf{T}}$ with $\mathbf{T}$ which acts on $\mathcal{B}(H)$ by

$$P_{\mathbf{T}}(X) = \sum_{i=1}^{n} T_i X T_i^*.$$

Since $\mathbf{T}$ is a contractive tuple, the completely positive map $P_{\mathbf{T}}$ is contractive and hence $\mathbf{1}_H \geq P_{\mathbf{T}}(\mathbf{1}_H) \geq P_{\mathbf{T}}^2(\mathbf{1}_H) \geq \cdots$. This decreasing sequence of positive contractions converges strongly and A_∞ will denote the positive contraction which is the strong limit:

$$A_\infty = \lim_{m\to\infty} P_{\mathbf{T}}^m(\mathbf{1}_H).$$

The commuting contractive tuple $\mathbf{T}$ will be called pure if $A_\infty = 0$.

Theorem 2.2.77. *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be a commuting contractive tuple of operators on a Hilbert space H. Then there is a unique bounded operator $R : \Gamma_s(\mathbb{C}^n) \otimes \mathcal{D}_{\mathbf{T}} \to H$ satisfying $R(\omega \otimes \xi) = D_{\mathbf{T}}\xi$ and*

$$R(e^{\underline{k}} \otimes \xi) = \mathbf{T}^{\underline{k}} D_{\mathbf{T}}\xi \tag{2.2.18}$$

for every multi-index $\underline{k}$ with $|\underline{k}| = 1, 2, \ldots$. In general, $\|R\| \leq 1$, and if $(T_1, \ldots, T_n)$ is a pure tuple, then R is a co-isometry: $RR^* = \mathbf{1}_H$.

Given a commuting contractive tuple, the following theorem constructs an $\mathcal{A}-$morphism from the Toeplitz C^*-algebra into the unital C^*-algebra generated by $T_1, T_2, \ldots, T_n$. Note that 2.2.18 implies that $R(S^{\underline{k}} \otimes \mathbf{1}_{\mathcal{D}_{\mathbf{T}}}) = \mathbf{T}^{\underline{k}} R$.

Theorem 2.2.78. *For every commuting contractive tuple $\mathbf{T} = (T_1, \ldots, T_n)$ acting on a Hilbert space H there is a unique unital $\mathcal{A}-$morphism*

$$\varphi : \mathcal{T}_n \to \mathcal{B}(H)$$

such that $\varphi(S_i) = T_i$, $i = 1, \ldots, n$.

Corollary 2.2.79. (von Neumann's inequality) *Let $\mathbf{T} = (T_1, \ldots, T_n)$ be any commuting contractive tuple acting on a Hilbert space H and $\mathbf{S} = (S_1, \ldots, S_n)$ be the shift. Then for any polynomial p in $n-$variables,*

$$\|p(T_1, \ldots, T_n)\| \leq \|p(S_1, \ldots S_n)\|.$$

Let M be a Hilbert space of dimension m (which is a nonnegative integer or ∞). We mean by $m \cdot \mathbf{S}$ the operator tuple $(S_1 \otimes \mathbf{1}_M, \ldots, S_n \otimes \mathbf{1}_M)$ acting on $\Gamma_s(\mathbb{C}^n) \otimes M$.

Given a Hilbert space N, and a representation β of $\mathcal{T}_n$ on N, the operator tuple

$$\mathbf{A} \stackrel{\text{def}}{=} m \cdot \mathbf{S} \oplus \beta(\mathbf{S})$$

is clearly a commuting contractive tuple on $\hat{H} \stackrel{\text{def}}{=} (\Gamma_s(\mathbb{C}^n) \otimes M) \oplus N$.

Let H be a subspace of $\hat{H}$ such that $A_i^* H \subseteq H$ for all $i = 1, \ldots, n$. Recall that such subspaces are called *co-invariant* with respect to the tuple $\mathbf{A}$.

Consider the compression $\mathbf{T}$ of $\mathbf{A}$ to H as follows,

$$T_i \stackrel{\text{def}}{=} P_K A_i | H.$$

This $\mathbf{T}$ is clearly a commuting contractive tuple on H and moreover for any polynomial $p(z_1, \ldots, z_n)$, $p(\mathbf{T})$ is the compression of $p(\mathbf{A})$ due to the co-invariance of H with respect to $\mathbf{A}$. It is proved in [8] that every commuting contractive tuple has such a representation with β sending all compact operators to zero, see the following theorem.

Theorem 2.2.80. (Dilation) *Let $\mathbf{T}$ be any commuting contractive tuple acting on a Hilbert space H and* rank $D_{\mathbf{T}} = m$ *(which is a nonnegative integer or ∞). Then there is a separable Hilbert space M of dimension m, another separable Hilbert space N with commuting tuple of operators $\mathbf{Z} = (Z_1, \ldots, Z_n)$ acting on it, satisfying $Z_1 Z_1^* + \cdots + Z_n Z_n^* = \mathbf{1}_N$ such that:*

i) H is contained in $\hat{H} \stackrel{def}{=} (\Gamma_s(\mathbb{C}^n) \otimes M) \oplus N$ as a subspace and it is co-invariant under $\mathbf{A} \stackrel{def}{=} m \cdot \mathbf{S} \oplus \mathbf{Z}$.

ii) $\mathbf{T}$ is the compression of $\mathbf{A}$ to H, that is $\mathbf{T}^{\underline{k}} = P_H \mathbf{A}^{\underline{k}} | H$ for every multi-index $\underline{k}$.

iii) $\hat{H} = \overline{span}\{\mathbf{A}^{\underline{k}} h : h \in H$ and $\underline{k}$ is any multi-index$\}$.

Thus any commuting contractive tuple has a minimal commuting dilation. Moreover, this dilation is unique up to unitary equivalence.

Coming back to David Opěla, he derives a different generalization of Andô's theorem, namely, he assumes that only some of the $\binom{n}{2}$ pairs commute. His results on the matter are presented below, following [27].

Definition 2.2.81. *A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a set (the elements of $V(G)$ are called vertices of G) and $E(G)$ is a set of unordered pairs of distinct vertices - these are called edges.*

A graph G' is a subgraph of a graph G, if $V(G') \subset V(G)$ and $E(G') \subset E(G)$.

A cycle of length n is the graph G with $V(G) = \{v_1, \ldots v_n\}$ and $E(G) = \{(v_j, v_{j+1}) : 1 \leq j \leq n-1\} \cup \{(v_1, v_n)\}$.

A graph G is connected, if for every two vertices v, w there exists a sequence of vertices $\{v_k\}_{k=0}^m \subset V(G)$ with $v_0 = v$, $v_m = w$ and such that $(v_j, v_{j+1}) \in E(G)$ for $j = 0, \ldots, m-1$.

A graph is a-cyclic, if it does not contain any cycle as a subgraph, and a connected a-cyclic graph is called a tree. Every tree has a vertex that lies on exactly one edge.

Definition 2.2.82. *Let $T_1, T_2, \dots, T_n \in \mathcal{B}(H)$ be an $n-$tuple of operators, and let G be a graph on the vertices $\{1, 2, \dots n\}$. We say that the operators $T_1, T_2, \dots, T_n$ commute according to G, if $T_iT_j = T_jT_i$ whenever (i, j) is an edge of G.*

Lemma 2.2.83. *Let $A, B \in \mathcal{B}(H)$ be commuting contractions and let $\tilde{X} \in \mathcal{B}(\tilde{K})$ be a co-isometric extension of A. Then there exists a Hilbert space K containing $\tilde{K}$ and commuting co-isometries $X, Y \in \mathcal{K}$ such that X extends $\tilde{X}$ and Y extends B. Moreover, if A, B are isometries and $\tilde{X}$ is a unitary, we can construct X and Y to be unitary.*

Lemma 2.2.84. *Let G be a graph without a cycle on vertices $\{1, 2, \dots n\}$ and let $T_1, \dots T_n \in \mathcal{B}(H)$ be an $n-$tuple of contractions that commute according to G. Then there exist a Hilbert space K containing H and an $n-$tuple of co-isometries $V_1, \dots V_n \in \mathcal{B}(K)$ that commute according to G and such that V_j extends T_j, for $j = 1, 2, \dots n$. Moreover, if the T_j's are all isometries, the V_j's can be chosen to be unitaries.*

The main result in [27] is thus the following theorem that is stated here without proof.

Theorem 2.2.85. *Let G be an a-cyclic graph on n vertices $\{1, 2, \dots, n\}$. Then for any $n-$tuple of contractions $T_1 \dots, T_n$ on a Hilbert space H that commute according to G, there exists an $n-$tuple of unitaries $U_1, \dots, U_n$ on a Hilbert space K that commute according to G and such that*

$$PU_{j_1} \dots U_{j_k}|H = T_{j_1} \dots T_{j_k}, \tag{2.2.19}$$

for all $k \in \mathbb{N}$, $j_i \in \{1, 2, \dots, n\}$, $1 \leq i \leq k$. Here $P : K \to H$ is the orthogonal projection.

Conversely, if G contains a cycle, there exists an $n-$tuple of contractions that commute according to G with no $n-$tuple of unitaries dilating them that also commute according to G.

Remark 2.2.86. The above theorem holds for infinite graphs.

Chapter 3

Nonholomorphic calculus for several variables

This chapter is based on [31] and provides a general scheme to extent Taylor's holomorphic functional calculus for several commuting operators (see chapter 2) to classes of non-holomorphic functions. We start by recalling some notions, definitions and results about commuting operators, as well as a brief description of the holomorphic functional calculus and we end this chapter by presenting the non-holomorphic functional calculus due to Sandberg [31].

3.1 Basic notions

Let X, Y be two Banach spaces and denote by $\mathcal{L}(X, Y)$ the Banach space of all continuous linear operators from X to Y and let $\mathcal{L}(X) = \mathcal{L}(X, X)$. We denote by I the identity operator of $\mathcal{L}(X)$. For a subset $A \subset \mathcal{L}(X)$, let A'' be the commutant, i.e. the Banach algebra of all operators in $\mathcal{L}(X)$ which commute with every operator in $\{R \in \mathcal{L}(X) : TR = RT \text{ for all } T \in A\}$.

Let $T \in \mathcal{L}(X)$, and recall that the spectrum on T is the set

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

If f is a holomorphic function in a neighborhood of the spectrum $\sigma(T)$ then the operator $f(T)$ is defined by the integral

$$f(T) = \frac{1}{2\pi i} \int_{\partial D} f(z)(z - T)^{-1} \mathrm{d}z, \tag{3.1.1}$$

where D is an appropriate neighborhood of $\sigma(T)$. This expression defines a continuous algebra homomorphism

$$f \mapsto f(T) : \mathcal{O}(\sigma(T)) \to (T)'',$$

such that $1(T) = I$ and $z(T) = T$, called the Riesz functional calculus. S. Sandberg in [31] extends this algebra homomorphism to functions not necessarily holomorphic in a neighborhood of the spectrum. Following Dynkin [12] he defines $f(T)$ by

$$f(T) = -\frac{1}{2\pi i} \int \bar{\partial} f(z) \wedge (z - T)^{-1} \mathrm{d}z \tag{3.1.2}$$

for all $f \in S_T$, where

$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} \mathrm{d}\bar{z}$$

and S_T is the set

$$S_T = \{f \in C_c^1(\mathbb{C}) : \|f\|_T := \|\bar{\partial} f(z) \wedge (z - T)^{-1} \mathrm{d}z\|_\infty < \infty\}.$$

It is clear that $f(T)$ is a bounded linear operator on X which commutes with every operator that commutes with T, i.e., $f(T) \in (T)''$. By Stokes theorem the definition of $f(T)$ only depends on the behaviour of f near the spectrum. Suppose that D is an open set such that it contains the spectrum, $\sigma(T) \subset D$, and that $f \in \mathcal{O}(D)$. Then if $\phi \in C_c^1(D)$ is equal to 1 in a neighborhood of $\sigma(T)$, one has that $\phi f \in S_T$ and $\phi f(T)$ defined by (3.1.2) agrees with $f(T)$ defined by (3.1.1).

The proof that $f \mapsto f(T)$ is an algebra homomorphism and that the spectral mapping theorem holds for functions in S_T was done by Dynkin [12] and one can check them there, but they can also be found in [31]. We will just state the results as follows.

Theorem 3.1.1. *The mapping*

$$f \mapsto f(T) : S_T \to (T)'',$$

where $T \in \mathcal{L}(X)$, is a continuous algebra homomorphism that continuously extends the holomorphic functional calculus. Moreover, if $f \in S_T$ then $\sigma(f(T)) = f(\sigma(T))$.

Furthermore, one has a rule of composition for the functional calculus.

Theorem 3.1.2. (Rule of composition) *If $g \in S_T$ and f is a holomorphic function in a neighborhood of $\sigma(T)$, then $\phi(f \circ g) \in S_T$ and $f(g(T)) = \phi(f \circ g)(T)$, if $\phi \in C_c^1(\mathbb{C})$ is equal to 1 in a neighborhood of $\sigma(T)$.*

Further results regarding this functional calculus can be found in [12].

Now we review notions on tuples of commuting operators. Suppose $\mathbf{T} = (T_1, \ldots, T_n) \in \mathcal{L}(X)^n$ is a commuting tuple of operators, that is $T_iT_j = T_jT_i$ for all i and j.

Let $\mathbf{e} = \{e_1, e_2, \ldots, e_n\}$ be indeterminates and define $\Lambda_n[\mathbf{e}]$ to be the exterior algebra on the generators $e_1, e_2, \ldots, e_n$. This is a linear space over the complex plane $\mathbb{C}$ endowed with an anticommutative exterior product $e_i \wedge e_j = -e_j \wedge e_i$, for every $1 \leq i, j \leq n$. For $F = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_p$, we write $e_F = e_{i_1} \wedge \cdots \wedge e_{i_p}$. The exterior algebra over $\mathbb{C}$ is then given by

$$\Lambda_n[\mathbf{e}] = \{\sum_F \alpha_F e_F : e_F = e_{i_1} \wedge \cdots \wedge e_{i_p} \text{ and } \alpha_F \in \mathbb{C}\}.$$

We let here $e_\emptyset$ to be the identity element for the exterior product. If we denote $\Lambda_n^k[\mathbf{e}] = \{\sum_{|F|=k} \alpha_F e_F : \alpha_F \in \mathbb{C}\}$, where $|F|$ is the cardinal of F, then clearly $\dim \Lambda_n^k[\mathbf{e}] = \binom{n}{k}$ for every $k \leq n$, $\Lambda_n^k[\mathbf{e}] \wedge \Lambda_n^l[\mathbf{e}] = \Lambda_n^{k+l}[\mathbf{e}]$ and $\Lambda_n[\mathbf{e}] = \bigoplus_{k=0}^n \Lambda_n^k[\mathbf{e}]$.

Definition 3.1.3. *Given a Banach space X, the exterior algebra over X is defined to be*

$$\Lambda_n[\boldsymbol{e}, X] = \{\sum_F x_F e_F : e_F = e_{i_1} \wedge \cdots \wedge e_{i_p} \text{ and } x_F \in X\}.$$

The subspaces $\Lambda_n^p[\boldsymbol{e}, X] = \{\sum_{|F|=p} x_F e_F : x_F \in X\}$, for $p \leq n$ are given in a similar way. Naturally $\Lambda_n^0[\boldsymbol{e}, X]$, $\Lambda_n^1[\boldsymbol{e}, X]$ and $\Lambda_n^n[\boldsymbol{e}, X]$ can be identified with X, X^n and X, respectively.

Since no confusion is possible $\Lambda_n^k[X]$ and $\Lambda_n[X]$ can be written instead of $\Lambda_n^k[\mathbf{e}, X]$ and $\Lambda_n[\mathbf{e}, X]$, respectively.

If $T \in \mathcal{L}(X)$, one keeps the same symbol T to denote the operator defined on $\Lambda_n[X]$ by

$$T\left(\sum_F x_F e_F\right) = \sum_F T x_F e_F.$$

For $i \in \{1, 2, \dots, n\}$, let $E_i : \Lambda_n[X] \to \Lambda_n[X]$ be the left multiplication operator by e_i : $E_i(e_F) = e_i \wedge e_F$. It is usually called the creation operator. With any commuting $n-$tuple $\mathbf{T} = (T_1, \dots, T_n)$ we associate the linear mapping defined over $\Lambda_n[X]$ by

$$\delta_{\mathbf{T}} = \sum_{i=1}^n T_i \otimes E_i : \sum_F x_F e_F \to \sum_F \sum_{i=1}^n T_i x_F e_i \wedge e_F.$$

Set $\delta_{\mathbf{T}}^k := \delta_{\mathbf{T}} | \Lambda_n^k[X]$. We construct a co-chain complex $K(\mathbf{T})$, called the Koszul complex associated with $\mathbf{T}$ on X as follows:

$$K(\mathbf{T}) : \mathbf{0} \xrightarrow{\delta_{\mathbf{T}}^{-1}} \Lambda_n^0[X] \xrightarrow{\delta_{\mathbf{T}}^0} \Lambda_n^1[X] \xrightarrow{\delta_{\mathbf{T}}^1} \cdots \xrightarrow{\delta_{\mathbf{T}}^{n-1}} \Lambda_n^n[X] \xrightarrow{\delta_{\mathbf{T}}^n} \mathbf{0}.$$

Definition 3.1.4. *The operator* $\mathbf{T}$ *is said to be non-singular, or Taylor invertible, if* $\ker \delta_{\mathbf{T}}^k = \mathrm{Im}\ \delta_{\mathbf{T}}^{k-1}$, *for* $k = 1, \dots, n$, *equivalently* $\ker \delta_{\mathbf{T}} = \mathrm{Im}\ \delta_{\mathbf{T}}$. *The associated Koszul complex is said to be exact in this case. The Taylor spectrum of* $\mathbf{T}$ *on* X^n *is then the set*

$$\sigma_T(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : K(\lambda - \mathbf{T}) \textit{ is not exact}\}.$$

We call cohomology of $\{\Lambda_n^k[X], \delta_{\mathbf{T}}^k\}$ the set $\{H^k(X, \mathbf{T})\}_k$, where $H^k(X, \mathbf{T}) = \ker \delta_{\mathbf{T}}^k \backslash \mathrm{Im}\ \delta_{\mathbf{T}}^{k-1}$. Notice that the action of $\mathbf{T}$ on X is non-singular if $H^k(X, \mathbf{T}) = 0$ for every k.

One also defines the split spectrum

$$\sigma_S(\mathbf{T}) = \{\lambda \in \mathbb{C}^n : K(\lambda - \mathbf{T}) \text{ is not split}\},$$

where split means that for every integer k there are operators R and S such that $I = \delta_{\mathbf{T}}^{k+1} R + S \delta_{\mathbf{T}}^k$. If X is a Hilbert space or $n = 1$ then $\sigma_T(\mathbf{T}) = \sigma_S(\mathbf{T})$. In general one has $\sigma_T(\mathbf{T}) \subset \sigma_S(\mathbf{T})$.

Consider operators parametrized by a variable $z \in \mathbb{C}^n$, such that $z \mapsto z - \mathbf{T}$. In this case the boundary map $\delta_{\mathbf{T}}^k$ depends on z and we henceforth suppress the index k and write $\delta_{\mathbf{T}}^k$ as $\delta_{z-\mathbf{T}}$ for every k. Also set $e_i = \mathrm{d}z_i$.

Now suppose that $T \in \mathcal{L}(X, Y)$ has closed range and let $k(T)$ be the norm of the inverse of T considered as a map from $X / \ker(T)$ to $\mathrm{Im}\ T$. The next lemma implies that if T_0 is a non-singular tuple then T is non-singular if $\|T_0 - T\|$ is small enough.

Lemma 3.1.5. *Suppose that* X, Y, Z *are Banach spaces,* $T_0 \in \mathcal{L}(X, Y)$, $R_0 \in \mathcal{L}(Y, Z)$, $\mathrm{Im}\ R_0$ *closed and* $\ker R_0 = \mathrm{Im}\ T_0$, *that is*

$$X \xrightarrow{T_0} Y \xrightarrow{R_0} Z$$

is exact. Let r *be a number such that* $r > \max\{k(T_0), k(R_0)\}$. *If* $T \in \mathcal{L}(X, Y)$, $R \in \mathcal{L}(Y, Z)$, $\mathrm{Im}\ T \subset \ker R$ *and* $\|T - T_0\|, \|R - R_0\| < 1/6r$ *then* $\mathrm{Im}\ T = \ker R$ *and* $k(T) \leq 4r$.

Hence $\sigma_T(\mathbf{T})$ is closed. Furthermore, the spectrum has the projection property.

Theorem 3.1.6. *If* $\mathbf{T} \in \mathcal{L}(X)^n$ *and* $\mathbf{T}' = (\mathbf{T}, T_{n+1}) \in \mathcal{L}(X)^{n+1}$ *are commuting and* $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ *is defined by* $\pi(z, z_{n+1}) = z$ *then* $\pi(\sigma_T(\mathbf{T}')) = \sigma_T(\mathbf{T})$.

It follows that

$$\sigma_T(\mathbf{T}) \subset \sigma(T_1) \times \cdots \times \sigma(T_n)$$

and hence $\sigma_T(\mathbf{T})$ is bounded. Thus $\sigma_T(\mathbf{T})$ is a compact subset of $\mathbb{C}^n$. Conversely, any compact set K in $\mathbb{C}^n$ can arise as the spectrum of a commuting tuple of operators. This one sees by letting the operators T_i to be multiplication by z_i on the Banach space $C(K)$ of continuous functions on $K \subset \mathbb{C}^n$.

3.2 Another holomorphic functional calculus

In this section is shortly presented Taylor's holomorphic functional calculus with the Cauchy-Franttapie-Leray formulas, which was introduced by Andersson in 1997 in [2]. For detailed proofs on the results in this section, one can also check [31].

The purpose is to generalize Theorem 3.1.1 to the case of several commuting operators. Suppose that E is a set such that there is a smooth function s such that $\delta_{z-\mathbf{T}} s = e$ outside E. In that case one can use the integral representation (3.2.9) to extend the holomorphic functional calculus.

The main difficulty is to show the multiplication property; for this, the resolvent identity, that is

$$(w-z)(z-T)^{-1}(w-T)^{-1} = (z-T)^{-1} - (w-T)^{-1}, \text{ where } z, w \in \mathbb{C}, \quad (3.2.1)$$

will be generalized to several commuting operators.

Let X be a Banach space, $\mathbf{T} \in \mathcal{L}(X)^n$ be a tuple of commuting operators on X and $z \in \mathbb{C}^n$ be a variable. If the complex $K(z - \mathbf{T})$ is exact for every z in an open set U then there is a smooth solution u in U to the equation $\delta_{z-\mathbf{T}} u = f$ if f is a closed and smooth $X-$valued form in U.

Sandberg in [31] constructs the resolvent on $\mathbb{C}^n \setminus \sigma_T(\mathbf{T})$. The following discussion is based on [31]. Remark that

$$\delta_{z-\mathbf{T}}\bar{\partial}\sum_k f_k \mathrm{d}z_k = -2\pi i \sum_{k,l}(z_k - T_k)\frac{\partial f_k}{\partial \bar{z}_l}\mathrm{d}\bar{z}_l = -\bar{\partial}\delta_{z-\mathbf{T}}\sum_k f_k \mathrm{d}z_k,$$

and therefore $\delta_{z-\mathbf{T}}\bar{\partial} = -\bar{\partial}\delta_{z-\mathbf{T}}$ for $1-$forms and hence for all forms since $\delta_{z-\mathbf{T}}$ and $\bar{\partial}$ are anti-derivations. Suppose that $K(z-\mathbf{T})$ is exact and $x \in X$. Then one can define a sequence u_i in $\mathbb{C}^n \setminus \sigma_T(\mathbf{T})$ by

$$\delta_{z-\mathbf{T}}u_1 = x, \quad \delta_{z-\mathbf{T}}u_{i+1} = \bar{\partial}u_i, \tag{3.2.2}$$

since $\bar{\partial}$ and $\delta_{z-\mathbf{T}}$ anti-commute. If this sequence starts with $x = 0$ then there is a form w_n such that $u_n = \bar{\partial}w_n$, this follows from the fact that one successively can find w_i such that

$$w_1 = 0, \quad \delta_{z-\mathbf{T}}w_{i+1} = \bar{\partial}w_i - u_i. \tag{3.2.3}$$

Thus if one has two sequences u_i and u_i' as in (3.2.2) then the difference $u_n - u_n'$ is exact. Hence u_n defines a Dolbeault cohomology class $\omega_{z-\mathbf{T}}x$ of bidegree $(n, n-1)$, which is called a resolvent cohomology class.

Suppose one has two cohomology classes $\omega_{z-\mathbf{T}}x$ and $\omega'_{w-\mathbf{R}}x$, where $z, w \in \mathbb{C}^n$, $\mathbf{T}, \mathbf{R} \in \mathcal{L}(X)^n$, corresponding to sequences u_i and v_i, respectively. Then one defines the $X-$valued cohomology class $\omega_{z-\mathbf{T}} \wedge \omega_{w-\mathbf{R}}x$ as the class of c_{2n}, where c_i solve

$$c_1 = 0, \quad \delta_{z-\mathbf{T},w-\mathbf{R}}c_{i+1} = \bar{\partial}c_i + v_i - u_i. \tag{3.2.4}$$

To see that this is well defined cohomology class, let u_i', v_i' and c_i' be other choices of sequences. Let w_i^u and w_i^v be the sequences given by (3.2.3) for the sequences $u_i - u_i'$ and $v_i - v_i'$ respectively. Then we obtain

$$c_1 - c_1' + w_1^v - w_1^u = 0$$

and

$$\delta_{z-\mathbf{T},w-\mathbf{R}}(c_{i+1} - c_{i+1}' + w_{i+1}^v - w_{i+1}^u) = \bar{\partial}(c_i - c_i' + w_i^v - w_i^u).$$

Hence, by (3.2.3) again, there exists a sequence w_i^c such that $c_{2n} - c_{2n}' = \bar{\partial}w_{2n}^c$.

Now suppose that one instead has operator valued forms, u_i, such that

$$\delta_{z-\mathbf{T}}u_1 = I, \quad \delta_{z-\mathbf{T}}u_{i+1} = \bar{\partial}u_i, \tag{3.2.5}$$

so that u_n represents the operator valued cohomology class $\omega_{z-\mathbf{T}}$. Then one has that $\omega_{z-\mathbf{T}} \wedge \omega_{w-\mathbf{R}}x$ is the class of $u_n \wedge v_n$, where v_i is an $X-$valued sequence defining $\omega_{w-\mathbf{R}}x$. This follows from the fact

$$\delta z - \mathbf{T}(u_1 \wedge v_n) = v_n, \quad \delta_{z-\mathbf{T}}(u_{i+1} \wedge v_n) = \bar{\partial}(u_i \wedge v_n)$$

and the following proposition.

Proposition 3.2.1. *If v_i is a sequence defining $\omega_{w-\mathbf{R}}x$ and*

$$\delta_{z-\mathbf{T}}f_1 = v_n, \quad \delta_{z-\mathbf{T}}f_{i+1} = \bar{\partial} f_i,$$

then f_n represents $\omega_{z-\mathbf{T}} \wedge \omega_{w-\mathbf{R}}x$.

Complete proof of the above proposition can be seen in [31].

In one variable, there is only one possible representative for $\omega_{z-T}x, T \in \mathcal{L}(X)$,

$$\omega_{z-T}x = \frac{1}{2\pi i}(z-T)^{-1}\mathrm{d}zx,$$

and one has that ω_{z-T} is operator valued. The key part in the proof of the holomorphic functional calculus in one variable is the resolvent identity (3.2.1), which Sandberg reformulates as

$$\omega_{z-T} \wedge \omega_{w-T} + \omega_{w-T} \wedge \omega_{z-w} + \omega_{w-z} \wedge \omega_{z-T} = 0.$$

He then generalizes this equality to several commuting operators as follows. Let $\Delta = \{(z,w) \in \mathbb{C}^{2n} : z = w\}$ be the diagonal.

Lemma 3.2.2. *For every $x \in X$, we have the equality*

$$\omega_{z-\mathbf{T}} \wedge \omega_{w-\mathbf{T}}x + \omega_{w-\mathbf{T}} \wedge \omega_{z-w}x + \omega_{w-z} \wedge \omega_{z-\mathbf{T}}x = 0, \tag{3.2.6}$$

on $((\mathbb{C}^n \setminus \sigma_T(\mathbf{T})) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma_T(\mathbf{T}))) \setminus \Delta$.

For the proof it is used the following sequence

$$m_k = \frac{1}{(2\pi i)^k} \frac{\partial |z-w|^2}{|z-w|^2} \wedge \left(\bar{\partial} \frac{\partial |z-w|^2}{|z-w|^2} \right)^{k-1}. \tag{3.2.7}$$

For details check the proof of Lemma 2.2 in [31].

Sandberg continues by choosing representatives $\tilde{\omega}_{z-\mathbf{T}}x$, $\tilde{\omega}_{w-\mathbf{T}}x$ and $\tilde{\omega}_{z-\mathbf{T}} \wedge \tilde{\omega}_{w-\mathbf{T}}x$ for $\omega_{z-\mathbf{T}}x$, $\omega_{w-\mathbf{T}}x$ and $\omega_{z-\mathbf{T}} \wedge \omega_{z-\mathbf{T}}x$, respectively on $(\mathbb{C}^n \setminus \sigma_T(\mathbf{T})) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma_T(\mathbf{T}))$. Let $\tilde{\omega}_{z-\mathbf{T}} = m_n$. Then (3.2.6) says that the form

$$\tilde{\omega}_{z-\mathbf{T}} \wedge \tilde{\omega}_{w-\mathbf{T}}x + \tilde{\omega}_{w-\mathbf{T}} \wedge \tilde{\omega}_{z-w}x + \tilde{\omega}_{w-z} \wedge \tilde{\omega}_{z-\mathbf{T}}x, \tag{3.2.8}$$

defined on $((\mathbb{C}^n \setminus \sigma_T(\mathbf{T})) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma_T(\mathbf{T}))) \setminus \Delta$, is exact. Suppose that (3.2.8) is exact on $(\mathbb{C}^n \setminus \sigma_T(\mathbf{T})) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma_T(\mathbf{T}))$. One has $[\Delta] = \bar{\partial}\tilde{\omega}_{z-w}$, where $[\Delta]$ denotes the current of integration over Δ. If one applies $\bar{\partial}$ to (3.2.8), interpreted as a current, one gets

$$0 = -\tilde{\omega}_{w-\mathbf{T}}x \wedge [\Delta] + [\Delta] \wedge \tilde{\omega}_{z-w}x + \tilde{\omega}_{z-w} \wedge \tilde{\omega}_{z-\mathbf{T}}x,$$

since (3.2.8) is supposed to be exact and therefore is closed. Hence $i^*(\tilde{\omega}_{z-\mathbf{T}}x - \tilde{\omega}_{w-\mathbf{T}}x) = 0$, where i is a function defined by $i(\tau) = (\tau, \tau)$. The next theorem gives the desired result in the case where we have $i^*\tilde{\omega}_{z-\mathbf{T}}x = i^*\tilde{\omega}_{w-\mathbf{T}}x$.

Theorem 3.2.3. (Resolvent identity). *Suppose that* $\tilde{\omega}_{z-\mathbf{T}}x$, $\tilde{\omega}_{w-\mathbf{T}}x$ *and* $\tilde{\omega}_{z-\mathbf{T}} \wedge \tilde{\omega}_{w-\mathbf{T}}x$ *aare representatives for* $\omega_{z-\mathbf{T}}x$, $\omega_{w-\mathbf{T}}x$ *and* $\omega_{z-\mathbf{T}} \wedge \omega_{z-\mathbf{T}}x$, *respectively. Let* $\tilde{\omega}_{z-\mathbf{T}} = m_n$, *where* m_n *is defined in* (3.2.7). *Then the current*

$$\tilde{\omega}_{z-\mathbf{T}} \wedge \tilde{\omega}_{w-\mathbf{T}}x + \tilde{\omega}_{w-\mathbf{T}} \wedge \tilde{\omega}_{z-w}x + \tilde{\omega}_{w-z} \wedge \tilde{\omega}_{z-\mathbf{T}}x,$$

defined on $(\mathbb{C}^n \setminus \sigma_T(\mathbf{T})) \times \mathbb{C}^n \cap \mathbb{C}^n \times (\mathbb{C}^n \setminus \sigma_T(\mathbf{T}))$ *is exact if and only if* $i^*\tilde{\omega}_{z-\mathbf{T}}x = i^*\tilde{\omega}_{w-\mathbf{T}}x$, *where* $i^n_{\mathbb{C}} \to \mathbb{C}^{2n}$ *is defined by* $i(\tau) = (\tau, \tau)$.

Proof. The necessity of having $i^*(\tilde{\omega}_{z-\mathbf{T}}x - \tilde{\omega}_{w-\mathbf{T}}x) = 0$ was already discussed above. The proof then uses Lemma 3.2.2. Details are in [31]. □

Now we give the definition of $f(\mathbf{T})$. If f is a holomorphic function in a neighborhood of $\sigma_T(\mathbf{T})$ then we define $f(\mathbf{T})$ by the formula

$$f(\mathbf{T})x = -\int f\bar{\partial}\phi \wedge \omega_{z-\mathbf{T}}x \quad \text{for all } x \in X, \tag{3.2.9}$$

where $\phi \in C_c^\infty$ is equal to 1 is a neighborhood of $\sigma_T(\mathbf{T})$. This definition is independent of the choice of ϕ. To see this, check [31].

Lemma 3.2.4. *If* $f(\mathbf{T})$ *is defined by the formula* (3.2.9), *then* $f(\mathbf{T}) \in (\mathbf{T})''$.

Thus one can now prove Taylor's theorem.

Theorem 3.2.5. (Taylor). *The mapping*

$$f \mapsto f(\mathbf{T}) : \mathcal{O}(\sigma_T(\mathbf{T})) \to (\mathbf{T})'' \tag{3.2.10}$$

is a continuous algebra homomorphism such that $1(\mathbf{T}) = I$ *and* $z_k(\mathbf{T}) = T_k$.

In the proof of this theorem Sandberg in [31] uses the fact that $f(\mathbf{T})$ is bounded, together with Lemma 3.2.4 and the proof of Proposition 3.2.1.

Taylor also proved the spectral mapping theorem; if $f \in \mathcal{O}(\sigma(T))$ then $f(\sigma(T)) = \sigma(f(T))$. Suppose that $\mathbf{T}$ is a tuple of commuting of operators and that D is an open set such that $\sigma_T(\mathbf{T}) \subset D$. Then there exists a $\delta > 0$ such that $\sigma_T(\mathbf{R}) \subset D$ if $\|\mathbf{T} - \mathbf{R}\| < \delta$. This follows from Lemma 3.1.5.

Newburgh in *The variation of spectra*, Duke Math. J. (1951), proved that the spectrum of one operator is continuous under commutative perturbations; the next proposition says that the same is true for the Taylor spectrum. For its proof see [31].

Proposition 3.2.6. *If* $\mathbf{T}$ *and* $\mathbf{R}$ *are tuples of operators such that* $\mathbf{T}, \mathbf{R}$ *is commuting, then*

$$\sup_{z\in\sigma_T(\mathbf{T})} \inf_{w\in\sigma_T(\mathbf{R})} |z-w| + \sup_{w\in\sigma_T(\mathbf{R})} \inf_{z\in\sigma_T(\mathbf{T})} |z-w| \leq 2 \sup_{z\in\sigma_T(\mathbf{T}-\mathbf{R})} |z| \leq 2\|\mathbf{T}-\mathbf{R}\|.$$

The next theorem says what happens when one has a norm convergent sequence in $\mathcal{L}(X)^n$. Notice that if $\sigma_T(\mathbf{T}) = \sigma_S(\mathbf{T})$ then the conclusion would be that $f(T_k) \to f(\mathbf{T}_0)$ is operator norm. The complete proof is in [31].

Theorem 3.2.7. *Suppose that $\mathbf{T}_k \in \mathcal{L}(X)^n$ are commuting tuples (not necessarily commuting with each other) for $k \geq 0$ and that $\|\mathbf{T}_k - \mathbf{T}_0\| \to 0$ as $k \to \infty$. If f is holomorphic in a neighborhood of $\cup_{k\geq 0}\sigma_T(\mathbf{T}_k)$, then $f(\mathbf{T}_k)x \to f(\mathbf{T}_0)x$ for every $x \in X$.*

3.3 Non-holomorphic functional calculus

In this section we follow Sandberg [31] who extended the holomorphic calculus from the previous section to functions such that $|\bar{\partial} f(z)|$ tends to zero when z approaches the spectrum. If f is a C^1−function with compact support, he defines whenever possible

$$f(\mathbf{T}) = -\int \bar{\partial} f \wedge u_n^x,$$

where u_n^x is a form that represents $\omega_{z-\mathbf{T}}\mathrm{x}$.

The results from [31] are presented below. Suppose $E \supset \sigma_T(\mathbf{T})$ is a compact set such that there exists a sequence u_i on $\mathbb{C}^n \setminus E$ satisfying (3.2.5). Then one has that u_n is operator valued and represents $\omega_{z-\mathbf{T}}$ in $\mathbb{C}^n \setminus E$. The definition of $f(\mathbf{T})$ in this case is

$$f(\mathbf{T}) = -\int \bar{\partial} f \wedge u_n.$$

Define a sequence c_l by

$$c_l = 0, \quad \delta_{z-\mathbf{T},w-\mathbf{T}} c_{l+1} = \bar{c}_l + u_l^2 - u_l^1, \tag{3.3.1}$$

with $u_l^1 = \pi_1^* u_l$ and $u_l^2 = \pi_2^* u_l$ where $\pi_1(z,w) = z$ and $\pi_2(z,w) = w$ are the projections. Thus c_{2n} represents $\omega_{z-\mathbf{T}} \wedge \omega_{w-\mathbf{T}}$. Now one has all the tools for proving the multiplicative property. Complete proof of the following proposition can be checked in [31].

Proposition 3.3.1. *Let u_i be a sequence defined on $\mathbb{C}^n \setminus E$, where $E \supset \sigma_T(\mathbf{T})$ is a compact set, as in* (3.2.5)*, and suppose c_l, $n \leq l \leq 2n$, are forms that satisfy the conditions*

$$i^* c_n = 0, \quad \delta_{z-\mathbf{T},w-\mathbf{T}} c_{l+1} = \bar{c}_l + u_l^2 - u_l^1, \quad c_{2n} = u_n^1 \wedge u_n^2, \tag{3.3.2}$$

where $i(\tau) = (\tau, \tau)$. *Moreover suppose that* $f, g \in C^2_c$ *such that*

$$\int \|\bar{\partial} f \wedge u_n\| < \infty, \quad \int \|\bar{\partial} g \wedge u_n\| < \infty$$

and

$$\int_z \int_w \frac{\|\bar{\partial} f(z) \wedge \bar{\partial} g(w) \wedge c_l\|}{d(z,E)d(w,E)|z-w|^{2(2m-l)-1}} < \infty, \tag{3.3.3}$$

for all l *such that* $n \leq l < 2n$. *Then* $f(\mathbf{T})g(\mathbf{T}) = fg(\mathbf{T})$.

To separate the condition (3.3.3) Sandberg assumes that u_i commute with $\mathbf{T}$. Then one can choose c_i in the following way.

Proposition 3.3.2. *Suppose that* u_i *is a sequence as in* (3.2.5) *and that* $\mathbf{T}u_i = u_i\mathbf{T}$. *Then*

$$c_i = \sum_{k+l=i} u^1_k \wedge u^2_l$$

satisfies (3.3.2).

Proof. One has that $c_l = 0$, and since $\mathbf{T}$ and u_i commute,

$$\delta c_{i+1} - \bar{\partial} c_i = \sum_{k+l=i+1} (\delta u^1_k \wedge u^2_l \quad - \quad u^1_k \wedge \delta u^2_l) \\ - \sum_{k+l=i} (\delta u^1_{k+1} \wedge u^2_l - u^1_k \wedge \delta u^2_{l+1}) = u^2_i - u^1_i,$$

where $\delta = \delta_{z-\mathbf{T}, w-\mathbf{T}}$. Thus c_i satisfies (3.3.2). □

Unfortunately, the sequence c_i in Proposition 3.3.2 does not necessarily satisfy $i^*c_n = 0$. However, by the proof of Theorem 3.2.3 one infers that i^*c_n is exact. There is an explicit choice of sequence that satisfies (3.2.5). Suppose that s satisfies the equalities $\delta_{z-\mathbf{T}}s = I$ and $\mathbf{T}s = s\mathbf{T}$. Then

$$\delta_{z-\mathbf{T}}s = I, \quad \delta_{z*\mathbf{T}}(s \wedge (\bar{\partial}s)^i) = (\bar{\partial}s)^i = \bar{\partial}(s \wedge (\bar{\partial}s)^{i-1})$$

and hence $u_i = s \wedge (\bar{\partial}s)^{i-1}$ satisfies (3.2.5). The sequence c_i of Proposition 3.3.2 is then

$$c_i = \sum_{k+l} s^1 \wedge (\bar{\partial}s^1)^{k-1} \wedge s^2 \wedge (\bar{\partial}s^2)^{l-1}, \tag{3.3.4}$$

where $s^1 = \pi^*_1 s$ and $s^2 = \pi^*_2 s$. Note that if $s \wedge s = 0$ then $s \wedge (\bar{\partial}s) = (\bar{\partial}s) \wedge s$ and hence $i^*c_n = 0$.

Let $E \supset \sigma_T(\mathbf{T})$ be a compact set and let s be a given form such that s is defined on $\mathbb{C}^n \setminus E$, $\delta_{z-\mathbf{T}}s = I$ and $\mathbf{T}s = s\mathbf{T}$. Define the class $S_{\mathbf{T}}$ by

$$S_{\mathbf{T}} = \{f \in C_c^2(\mathbb{C}^n) : \|f\|_{\mathbf{T}} < \infty\}, \tag{3.3.5}$$

where

$$\|f\|_{\mathbf{T}} = \sum_{i=1}^{n} \|\frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{i-1}}{d(z,E)}\|_\infty + \sum_{k+l=n} \|\frac{\bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{k-1} \wedge s \wedge (\bar{\partial} s)^{l-1}}{d(z,E)}\|_\infty.$$

Note that the second sum vanishes if $s \wedge s = 0$. This is always the case if $n = 2$ since then $\delta_{z-\mathbf{T}}(s \wedge s) = s - s = 0$ and $\delta_{z-\mathbf{T}}$ injective. If $n = 1$ then $S_{\mathbf{T}}$ defined by (3.3.5) is a slightly smaller class than S_T defined at the beginning of this chapter. If $f \in S_T$ then $f(T)$ is defined by

$$f(T) = -\int \bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{n-1}.$$

Of course, one has that $f(T) \in \mathcal{L}(X)$ if $f \in S_T$. Note that $S_{\mathbf{T}}$ is an algebra. Next lemma uses Proposition 3.3.1 to prove that $f(\mathbf{T})g(\mathbf{T}) = fg(\mathbf{T})$, if $f, g \in S_{\mathbf{T}}$.

Lemma 3.3.3. *If $f, g \in S_{\mathbf{T}}$ then $f(\mathbf{T})g(\mathbf{T}) = fg(\mathbf{T})$.*

In order to prove that $f(T) \in (T)''$ Sandberg constructs the resolvent $\omega_{z-T,w-R}$ and uses the multiplicative property of the functional calculus of the tuple (T, R), where $R \in \mathcal{L}(X)$ commutes with T.

Lemma 3.3.4. *If $f \in S_{\mathbf{T}}$ then $f(\mathbf{T}) \in (\mathbf{T})''$.*

The complete proofs of these two lemmas can be seen in [31] and with them, a generalization of the holomorphic functional calculus can be proved.

Theorem 3.3.5. (Non-holomorphic functional calculus). *Suppose that $\mathbf{T}$ is an $n-$tuple of commuting operators and that $E \supset \sigma_T(\mathbf{T})$ is compact such that it exists a smooth form s defined on $\mathbb{C}^n \setminus E$ with $\delta_{z-\mathbf{T}}s = I$ and $\mathbf{T}s = s\mathbf{T}$. Let $S_{\mathbf{T}}$ be the class defined by (3.3.5) and let $f(\mathbf{T})$, $f \in S_{\mathbf{T}}$, be the operator defined by*

$$f(\mathbf{T}) = -\int \bar{\partial} f \wedge s \wedge (\bar{\partial} s)^{n-1}.$$

Then the map $f \mapsto f(\mathbf{T}) : S_{\mathbf{T}} \to (\mathbf{T})''$ is a continuous algebra homomorphism that continuously extends the map $f \mapsto f(\mathbf{T}) : \mathcal{O}(E) \to (\mathbf{T})''$.

Proof. The following proof is given by Sandberg in [31].

By Lemma 3.3.4 the map $f \mapsto f(\mathbf{T}) : S_{\mathbf{T}} \to (\mathbf{T})''$ is well defined. The map is continuous and linear. Lemma 3.3.3 shows that the map is multiplicative, and thus the map is an algebra homomorphism. To see that it continuously extends the map $f \mapsto f(\mathbf{T}) : \mathcal{O}(E) \to (\mathbf{T})''$, suppose that we have a sequence $f_n \in \mathcal{O}(U)$, where U is an open neighborhood of E, and that $f_n \to 0$ uniformly on compacts. Then

$$\|f_n \phi\|_{\mathbf{T}} \to 0,$$

where $\phi \in C_c^\infty(U)$ is a function equal to 1 in a neighborhood of E. □

Sandberg proved also the spectral mapping theorem for this functional calculus.

Theorem 3.3.6. (Spectral mapping theorem). *If f is tuple of functions in $S_{\mathbf{T}}$, where $S_{\mathbf{T}}$ is defined by* (3.3.5)*, then $\sigma_T(f(\mathbf{T})) = f(\sigma_T(\mathbf{T}))$.*

The proof uses Theorem 3.1.6 and the next lemma which we will not prove, but see [31] for details.

Lemma 3.3.7. *Suppose that there is an operator valued form s outside E such that $\delta_{z-\mathbf{T}} s = I$ and $\mathbf{T}s = s\mathbf{T}$. Furthermore, suppose that $c \in ((\mathbf{T})'')^m$, $w \in \sigma_T(\mathbf{T})$ and $k \in \Lambda_n^p[w - \mathbf{T}, c, X]$ (with respect to a basis $\mathrm{d}w_1, \ldots, \mathrm{d}w_n, e_{n+1}, \ldots, e_{n+m}$ of $\mathbb{C}^{n+m}$) such that $\delta_{w-\mathbf{T},c} k = 0$. If $f \in S_{\mathbf{T}}$, then*

$$(f(\mathbf{T}) - f(w))k = \delta_{w-\mathbf{T},c} \int_z \bar{\partial} f(z) \wedge \sum_{l=1}^{n} m''_{n+1-l} \wedge s \wedge (\bar{\partial} s)^{l-1} \wedge k,$$

where m''_i is the component of m_i (3.2.7) *with one $\mathrm{d}w$ and no $\mathrm{d}\bar{w}$.*

Further, denote by $\mathrm{co}(E)$ the convex hull of the set E. Then the last result due to Sandberg in [31] is the following.

Theorem 3.3.8. *Let h be a positive decreasing function on $[0, \infty)$. If there is a differential form u^x on $\mathbb{C}^n \setminus \mathrm{co}(\sigma_T(\mathbf{T}))$ such that $\|u^x(z)\| \leq \|x\| h(d(z, E))$ then we define the class $S_h(\mathbf{T})$ by*

$$S_h(\mathbf{T}) = \{f \in C_c^1(\mathbb{C}^n) : \| \, |\bar{\partial} f(z)| h(d(z, \mathrm{co}(\sigma_T(\mathbf{T})))) \, \|_{L^\infty} < \infty\}.$$

Let the norm of functions in $S_h(\mathbf{T})$ be given by

$$\| \, f \, \|_{S_h(\mathbf{T})} = \| \, |\bar{\partial} f(z)| h(d(z, \mathrm{co}(\sigma_T(\mathbf{T})))) \, \|_{L^1} \, .$$

Then the map

$$f \mapsto f(\mathbf{T}) : S_h(\mathbf{T}) \to (\mathbf{T}), \quad \textit{where } f(\mathbf{T})x = -\int \bar{\partial} f \wedge u^x,$$

is a continuous algebra homomorphism. If $f \in S_h(\mathbf{T})$ *then* $\sigma_T(f(\mathbf{T})) = f(\sigma_T(\mathbf{T}))$ *and* $f(\mathbf{T}) = 0$ *if* $f = 0$ *on* $\mathrm{co}(\sigma_T(\mathbf{T}))$*. Furthermore, if* $f \in S_h(\mathbf{T})$, $g \in S_{h_1}(f(\mathbf{T}))$ *(or,* $g \in \mathcal{O}(\sigma_T(f(\mathbf{T})))$*), where* h_1 *is a decreasing function such that* $h(y/\sup|\mathrm{d}f|) \geq Ch_1(y), y \in [0, \infty)$*, and* $g(0) = 0$ *then* $g(f(\mathbf{T})) = g \circ f(\mathbf{T})$.

Chapter 4

Multicentric calculus for holomorphic functions

In this chapter are first presented general results for multicentric calculus for one variable operators and the Riesz spectral projection following O. Nevanlinna's papers on the topic [24], [23] and [25]. Then further estimates are given for a pair of commuting operators and by the end we try to extend the calculus to an $n-$tuple of commuting operators. For this some extra conditions are introduced so that the von Neumann inequality holds true when using a constant.

4.1 Multicentric calculus for single operators

In [24], O. Nevanlinna shows how multicentric representation of functions gives a simple way to generalize the von Neumann result, i.e., the unit disc is a spectral set for contractions in Hilbert spaces. The discussion in his paper regards multicentric representation of holomorphic functions and in short it goes as follows.

If A denotes a bounded operator in a Hilbert space H, denote by $V_p(A)$ the set

$$V_p(A) = \{z \in \mathbb{C} : |p(z)| \leq \|p(A)\|\} \tag{4.1.1}$$

where p is a monic polynomial with distinct roots. It is shown in [24] that these sets are $K-$spectral, whenever the lemniscate does not pass through any critical point of p. Since any compact set can be the spectrum of an operator, it is important that lemniscates have good approximation properties. Furthermore, O. Nevanlinna has provided an algorithm (see [23]) which, for any given A, provides a sequence of monic polynomials p with distinct roots such that the sets $V_p(A)$ squeeze around the polynomially convex hull of the spectrum of A.

To be able to estimate a holomorphic function φ effectively at an operator A, O. Nevanlinna uses in [24] the approach introduced in [25], that is, each polynomial

$$p(z) = \prod_{j=1}^{d}(z - \lambda_j)$$

with simple roots λ_j induces a unique multicentric representation of φ,

$$\varphi(z) = \sum_{k=1}^{d} \delta_k(z) f_k(w) \quad \text{with } w = p(z), \tag{4.1.2}$$

where δ_k denote the polynomials of degree $d-1$ taking the value 1 at λ_k and vanishes at the other roots. In [25] is discussed the practical computation of the Taylor series of f_k. In fact, the coefficients may be computed in a recursion way if the derivatives of the original function φ are available at the local centers λ_k.

O. Nevanlinna in [24] states that the representation (4.1.2) allows an obvious avenue for analysis, estimation and computation in complicated sets. One just treats the functions f_k in discs $|w| \leq \rho$ and combines the estimates for φ in the sets satisfying $|p(z)| \leq \rho$.

In [24] the author demonstrates this approach by generalizing a well-known result of von Neumann on contractions in Hilbert spaces. In order to do this he needs an estimate of the following form

$$\sup_{|w|\leq\rho} |f_k(w)| \leq C(\rho) \sup_{|p(z)|\leq\rho} |\varphi(z)|. \tag{4.1.3}$$

This would then imply that the sets $V_p(A)$ are $K-$spectral sets with some K. Now, let γ_ρ denote the lemniscate

$$\gamma_\rho = \{z \in \mathbb{C} : |p(z)| = \rho\}.$$

For small ρ the lemniscate consists of d separate circular curves, for large ρ it reduces to just one circular curve. In general the lemniscate is smooth except if it contains a critical point, where the derivative of p vanishes. Thus there are at most $d-1$ such exceptional values ρ. Let $s(\rho)$ denote the distance from γ_ρ to the set of critical points.

The key result in [24] is the following theorem, which will just be stated here. For the proof one can check [24].

Theorem 4.1.1. *If p is a monic polynomial of degree d with distinct roots, then there exists a constant C such that if φ is holomorphic for $|p(z)| \leq \rho$, then the functions f_k in (4.1.2) are holomorphic for $|w| \leq \rho$ and if γ_ρ does not contain any critical points of p the estimate (4.1.3) holds with some $C(\rho)$ satisfying*

$$C(\rho) \leq 1 + \frac{C}{s(\rho)^{d-1}}. \tag{4.1.4}$$

Remark 4.1.2. If $C(\rho)$ denotes the smallest constant such that (4.1.3) holds for all φ then $C(\rho) \to 1$ as $\rho \to 0$ or $\rho \to \infty$. Generically the critical points are simple and then the constant is proportional to $1/s(\rho)$.

Some application to the spectral set theory are then given in [24], but first recall some definitions and notations. Denote by $\mathcal{B}(H)$ the space of bounded linear operators in a Hilbert space H.

Definition 4.1.3. *A closed set $\Sigma \subset \mathbb{C}$ is a spectral set for $A \in \mathcal{B}(H)$, if for all rational functions r with poles off Σ there holds*

$$\|r(A)\| \leq \sup_{z\in\Sigma} |r(z)|. \tag{4.1.5}$$

If the equation holds in the form

$$\|r(A)\| \leq K \sup_{z\in\Sigma} |r(z)|,$$

with fixed K, then Σ is called a $K-$spectral set.

Below we state a fundamental result by von Neumann for contractions in Hilbert spaces.

Theorem 4.1.4. *If $A \in \mathcal{B}(H)$, and $\|A\| \leq 1$, then the closed unit disc is a spectral set for A.*

This can be reformulated as follows:

$$\|\varphi(A)\| \leq \sup_{|z|\leq\|A\|} |\varphi(z)| \tag{4.1.6}$$

provided φ is holomorphic in $|z| \leq \|A\|$.

In [24] the results are formulated for holomorphic functions rather than for polynomials or rational functions as the author considers sets which may consist of several simply connected components. In particular, the results apply as such for Riesz spectral projections. The main result of the paper [24] is the following:

Theorem 4.1.5. *Suppose we are given a monic polynomial $p \in \mathbb{P}_d$ with distinct roots and a bounded operator $A \in \mathcal{B}(H)$ in a Hilbert space H. Let $\rho \geq 0$ satisfying $\|p(A)\| \leq \rho$ and be such that the lemniscate γ_ρ contains no critical points of p. Then for all φ which are holomorphic for $|p(z)| \leq \rho$ there holds*

$$\|\varphi(A)\| \leq K \sup_{|p(z)| \leq \rho} |\varphi(z)|, \tag{4.1.7}$$

where the constant K satisfies

$$K \leq C(\rho) \sum_{k=1}^{d} \|\delta_k(A)\|, \tag{4.1.8}$$

with $C(\rho)$ as in Theorem 4.1.1.

A simple but useful application of this theorem is obtained as follows. Suppose γ_ρ consists of several components and is free from critical points. Then one can define φ to be identically 1 in some open neighborhood of some of the components and to vanish in a neighborhood of all the others. If $A \in \mathcal{B}(H)$ is such that $\|p(A)\| \leq \rho$, then the resulting operator is simply the Riesz spectral projection to the invariant subspace with respect to the part of the spectrum where φ equals 1.

Another application is related to the power boundedness. One can apply Theorem 4.1.5 with $\rho = 0$ but then A has to be an algebraic operator and all eigenvalues are nondefective. Requiring $p(A) = 0$ for p with simple zeroes says exactly that. Now let $\mathbb{D}$ denote the open unit disc. Suppose that $V_p(A) \subset \overline{\mathbb{D}}$.

Corollary 4.1.6. *Let p be monic with simple zeros and suppose $\rho \geq 0$ is such that γ_ρ contains no critical points and $\gamma_\rho \subset \bar{\mathbb{D}}$. If $A \in \mathcal{B}(H)$ is such that $\|p(A)\| \leq \rho$, then A is power bounded and with the constant $C(\rho)$ provided by Theorem 4.1.1 we have for all $n \geq 1$,*

$$\|A^n\| \leq C(\rho) \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{4.1.9}$$

Remark 4.1.7. If A is algebraic then one can take p to be the minimal polynomial. Recall that a polynomial is called minimal if it is monic, $p(A) = 0$ and the polynomial is of smallest possible degree. Then Corollary 4.1.6 and Remark 4.1.2 yield

$$\|A^n\| \leq \sum_{k=1}^{d} \|\delta_k(A)\|. \tag{4.1.10}$$

Remark 4.1.8. In [3] one can find a follow up on the papers of O. Nevanlinna [24], [23] and [25]. There we discuss about multicentric holomorphic calculus in which one represents the function φ using a new variable $w = p(z)$ in such a way that when it is evaluated at the operator A, then $p(A)$ is small in norm. As before, p is assumed to have distinct roots.

The separation of a compact set, such as the spectrum, into different components by a polynomial lemniscate is presented with few examples, as well as applications of the Calculus to the computation and estimation of the Riesz spectral projection. It may then be desirable the use of $p(z)^n$ as a new variable and a small dimensional problem is then discussed in [3] to see how the size of coupling can affect on the need of taking a high power of $p(A)$.

4.2 Multicentric calculus for a pair of commuting operators

We start this section by discussing the case when $n = 2$, that is, when we have a pair of commuting operators and then we state some results for the general $n-$tuples.

Let $\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2$ and let $\mathbf{p} = (p_1(z_1), p_2(z_2))$, with p_1, p_2 polynomials of degrees d_1, d_2, respectively, with distinct roots $\lambda_{1,1}, \ldots, \lambda_{d_1,1}$ and $\lambda_{1,2}, \ldots, \lambda_{d_2,2}$, respectively. Let $\mathbf{T} = (T_1, T_2)$ be a pair of commuting operators on a Hilbert space H. Denote by $V_{\mathbf{p}}(\mathbf{T})$ the set

$$V_{\mathbf{p}}(\mathbf{T}) = \{(z_1, z_2) \in \mathbb{C}^2 : |p_j(z_j)| \leq \|p_j(T_j)\| \text{ for } j = 1, 2\}.$$

and note that by the spectral mapping theorem we have the following.

Proposition 4.2.1. *With the above notations we have*

$$\sigma_T(\mathbf{T}) \subset V_{\mathbf{p}}(\mathbf{T}).$$

Proof. Indeed, for $\mathbf{p}(z_1, z_2) = (p_1(z_1), p_2(z_2)) \quad ((z_1, z_2) \in \mathbb{C}^2)$ one applies the spectral mapping theorem twice

$$\sigma_T(\mathbf{p}(\mathbf{T})) = \mathbf{p}(\sigma_T(\mathbf{T})) \Leftrightarrow$$
$$\sigma_T(p_1(T_1), p_2(T_2)) = (p_1(\sigma(T_1)), p_2(\sigma(T_2))) = (\sigma(p_1(T_1)), \sigma(p_2(T_2)))$$

since the Taylor spectrum for single operators equals the usual spectrum, i.e., $\sigma_T(T) = \sigma(T)$. □

Following the same line of proof, one can state a more general result.

Proposition 4.2.2. *Let* $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$ *and let* $\mathbf{p} = (p_1(z_1), \dots, p_n(z_n))$, *with* $p_1, \dots, p_n$ *polynomials of degrees* $d_1, \dots, d_n$, *respectively, with distinct roots* $\lambda_{1,1}, \dots, \lambda_{d_1,1}, \dots, \lambda_{1,n}, \dots, \lambda_{d_n,n}$, *respectively. Let* $\mathbf{T} = (T_1, \dots, T_n)$ *be an* $n-$*tuple of commuting operators acting on a Hilbert space* H. *Denote by* $V_{\mathbf{p}}(\mathbf{T})$ *the set*

$$V_{\mathbf{p}}(\mathbf{T}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |p_j(z_j)| \leq \|p_j(T_j)\| \textit{ for } j = 1, \dots, n\}.$$

Then $\sigma_T(\mathbf{T}) \subset V_{\mathbf{p}}(\mathbf{T})$.

When applying the multicentric calculus described above we treat our two variables separately. First we fix one variable, say we fix z_2, then we fix the first one, z_1, and we apply the calculus as follows.

In multicentric calulus the polynomial is taken as a new variable $w = p(z)$. In this case the polynomials p_j are taken as new variables $w_1 = p_1(z_1)$ and $w_2 = p_2(z_2)$ and functions $\varphi(z_1, z_2)$ are represented with the help of a vector-valued function f, mapping $(w_1, w_2) \mapsto f(w_1, w_2) \in \mathbb{C}^{d_1+d_2}$. Recall that f is assumed holomorphic in a neighbourhood of $V_p(A)$, for the single operator case. For the pair $\mathbf{T}$, we assume $f(w_1, w_2)$ holomorphic in both variables in a neighbourhood of $V_{\mathbf{p}}(\mathbf{T})$.

Note that, when applying φ to the operator $\mathbf{T}$, then φ this will be well defined as a polynomial with commuting T_j's and things hold when going to limit for analytic functions.

Now, denote by $\delta_{k,1} \in \mathbb{P}_{d_1-1}, \delta_{j,2} \in \mathbb{P}_{d_2-1}$ the Lagrange interpolation basis polynomials at $\lambda_{j,1}$ and $\lambda_{k,2}$, respectively, that is

$$\delta_{k,1}(z_1) = \frac{1}{p_1'(\lambda_{k,1})} \prod_{j \neq k} (z_1 - \lambda_{j,1})$$
$$\delta_{j,2}(z_2) = \frac{1}{p_2'(\lambda_{j,2})} \prod_{k \neq j} (z_2 - \lambda_{k,2}).$$

Then the multicentric representation of φ takes the form, when fixing z_2,

$$\varphi(z_1, z_2) = \sum_{j=1}^{d_1} \delta_{j,1}(z_1) f_{j,1}(p_1(z_1), z_2) \tag{4.2.1}$$

which becomes the following when fixing z_1

$$\varphi(z_1, z_2) = \sum_{k=1}^{d_2} \sum_{j=1}^{d_1} \delta_{j,1}(z_1)\delta_{k,2}(z_2) f_{j,k}(p_1(z_1), p_2(z_2)). \tag{4.2.2}$$

Since we treat the two variables separately, let

$$\begin{aligned} V_{p_1}(T_1) &= \{z_1 \in \mathbb{C} : |p_1(z_1)| \le \|p_1(T_1)\|\} \text{ and} \\ V_{p_2}(T_2) &= \{z_2 \in \mathbb{C} : |p_2(z_2)| \le \|p_2(T_2)\|\}, \end{aligned}$$

and let $|p_1(z_1)| = \rho_1$, $|p_2(z_2)| = \rho_2$, be lemniscates which are then mapped onto discs $|w_j| \le \rho_j$, for $j = 1, 2$. Note that ρ_1 and ρ_2 are so that the lemniscate does not pass through any critical points of p_1 or p_2, respectively.

Inserting T_1, T_2 instead of z_1, z_2 in the equation (4.2.2), gives a simple way of defining $\varphi(T_1, T_2)$. This way $\varphi(T_1, T_2)$ is well defined since T_1 and T_2 commute, thus we set

$$\varphi(T_1, T_2) = \sum_{k=1}^{d_2} \sum_{j=1}^{d_1} \delta_{j,1}(T_1)\delta_{k,2}(T_2) f_{j,k}(p_1(T_1), p_2(T_2)). \tag{4.2.3}$$

In this formula $\delta_{j,1}(T_1)$ and $\delta_{k,2}(T_2)$ are polynomials and the key is to estimate $f_{j,k}(p_1(T_1), p_2(T_2))$.

Since the von Neumann's inequality holds for a pair of commuting operators on a Hilbert space H, see Corollary 2.2.12, one can apply it to estimate $f_{j,k}(p_1(T_1), p_2(T_2))$ provided that $p_1(T_1)$ and $p_2(T_2)$ commute. Thus we have for all j, k

$$\|f_{j,k}(p_1(T_1), p_2(T_2))\| \le \sup_{|w_1|,|w_2|\le 1} |f_{j,k}(w_1, w_2)|. \tag{4.2.4}$$

Therefore we get the following

$$\begin{aligned} \|\varphi(T_1, T_2)\| &\le \sum_{k=1}^{d_2} \sum_{j=1}^{d_1} \|\delta_{j,1}(T_1)\| \|\delta_{k,2}(T_2)\| \|f_{j,k}(p_1(T_1), p_2(T_2))\| \\ &\le \sum_{k=1}^{d_2} \sum_{j=1}^{d_1} \|\delta_{j,1}(T_1)\| \|\delta_{k,2}(T_2)\| \sup_{|w_1|,|w_2|\le 1} |f_{j,k}(w_1, w_2)|. \end{aligned} \tag{4.2.5}$$

Further we want to estimate $\sup_{|w_1|,|w_2|\le 1} |f_{j,k}(w_1, w_2)|$ from above by $\sup_{\substack{|p_1(z_1)| \le \rho_1 \\ |p_2(z_2)| \le \rho_2}} |\varphi(z_1, z_2)|$. For this we have the following lemma.

Lemma 4.2.3. *Let $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a polynomial in n variables, with p_j monic polynomials of degree d_j, with distinct roots, for every $j = 1, \ldots, n$. Let ρ_j be the levels of p_j, for every $j = 1, \ldots, n$, that is $|p_j| \leq \rho_j$. Denote $p_j(z_j) = w_j$, $\forall j = 1, \ldots, n$. In the multicentric representation*

$$\varphi(\mathbf{z}) = \sum_{\nu} \delta_\nu(\mathbf{z}) f_\nu(w_1, \ldots, w_n),$$

where $\sum_\nu \delta_\nu(\mathbf{z}) = \sum_{\nu_1=1}^{d_1} \cdots \sum_{\nu_n=1}^{d_n} \delta_{\nu_1}(z_1) \ldots \delta_{\nu_n}(z_n)$, one has the following estimate for $j = 1, \ldots, n$

$$\sup_{\substack{|w_1| \leq 1 \\ \cdots \\ |w_n| \leq 1}} |f_\nu(w_1, \ldots, w_n)| \leq C(\rho_1) \ldots C(\rho_n) \sup_{\substack{|p_1(z_1)| \leq \rho_1 \\ \cdots \\ |p_n(z_n)| \leq \rho_n}} |\varphi(z_1, \ldots, z_n)|. \tag{4.2.6}$$

Moreover, we have for every $j = 1, \ldots, n$

$$C(\rho_j) \leq 1 + \frac{C_j}{s(\rho_j)^{d_j - 1}}, \tag{4.2.7}$$

where C_j is a constant and $s(\rho_j)$ is the distance from the lemniscate of p_j to the set of critical points of p_j.

Proof. We treat the variables separately and we apply the results from the previous section. First we fix all the variables except z_1 and we use the estimate (4.1.3) to get

$$\sup_{|w_1| \leq 1} |f_{\nu_1}(w_1, z_2, \ldots, z_n)| \leq C(\rho_1) \sup_{|p_1(z_1)| \leq \rho_1} |\varphi(z_1, \ldots, z_n)|.$$

For this last estimate we fix all variables except for z_2, then we apply (4.1.3) and we get

$$\sup_{\substack{|w_1| \leq 1 \\ |w_2| \leq 1}} |f_{\nu_1, \nu_2}(w_1, w_2, z_3 \ldots, z_n)| \leq C(\rho_1) C(\rho_2) \sup_{\substack{|p_1(z_1)| \leq \rho_1 \\ |p_2(z_2)| \leq \rho_2}} |\varphi(z_1, \ldots, z_n)|.$$

We continue in the same way until we fix all variables except for z_n. Using then once more the relation (4.1.3) we get the desired result

$$\sup_{\substack{|w_1| \leq 1 \\ \cdots \\ |w_n| \leq 1}} |f_\nu(w_1, \ldots, w_n)| \leq C(\rho_1) \ldots C(\rho_n) \sup_{\substack{|p_1(z_1)| \leq \rho_1 \\ \cdots \\ |p_n(z_n)| \leq \rho_n}} |\varphi(z_1, \ldots, z_n)|.$$

The estimate (4.2.7) follows immediately from Theorem 4.1.1.

□

Applying Lemma 4.2.3 with $n = 2$ to the inequality (4.2.5) one completes the proof of the following theorem.

Theorem 4.2.4. *With the above notations one has*

$$\|\varphi(T_1, T_2)\| \leq \sum_{k=1}^{d_2}\sum_{j=1}^{d_1} \|\delta_{j,1}(T_1)\|\|\delta_{k,2}(T_2)\|C(\rho_1)C(\rho_2) \sup_{\substack{|p_1(z_1)| \leq \rho_1 \\ |p_2(z_2)| \leq \rho_2}} |\varphi(z_1, z_2)|, \tag{4.2.8}$$

where ρ_1 and ρ_2 are the levels of p_1 and p_2, respectively, such that they do not pass through any critical points.

4.3 Multicentric calculus for $n-$tuples of commuting operators

In this section we try to extend the multicentric calculus for an $n-$tuple of commuting contractions. The following is a different approach since we will have a different setup. Some extra conditions must be satisfied and they are presented below.

Let $\mathbf{T} = (T_1, \dots, T_n)$ be an $n-$tuple of commuting contractions on a Hilbert space H. It is known that the von Neumann's inequality fails for $n \geq 3$, but it is still a question whether it holds for a constant C_n, that depends on n.

Whenever the von Neumann's inequality for $n-$tuples of contractions is proven true, we can extend the multicentric calculus discussed in the previous section and in [3], for the general case of $n-$tuples of commuting operators. Now we review some known results on the multidimensional von Neumann inequality.

Definition 4.3.1. *We say that the multidimensional von Neumann inequality holds for a fixed $n-$tuple of commuting operators $\mathbf{T} = (T_1, \dots, T_n)$ if*

$$\|P(T_1, \dots, T_n)\| \leq \|P\|_{\overline{\mathbb{D}}^n}$$

where $\overline{\mathbb{D}}$ denotes the closure of the unit disc, for any polynomial P in n (commutative) variable.

There are some known results when the above holds, but we will list just a few. For more results see [5]. The multidimensional von Neumann inequality holds in the following situations:

i) for a pair of commutative Hilbert space contractions ($n = 2$);

ii) for a commutative family of isometries;

iii) for a family of double commuting (i.e., $T_iT_j = T_jT_i$ for all i and j and $T_i^*T_j = T_iT_j^*$ whenever $i \neq j$) contractions;

iv) for a commutative family $\mathbf{T}$ such that $\sum_{i=1}^n \|T_i\|^2 \leq 1$;

It is an open problem in operator theory to determine whether or not there exists a constant C_n that adjust von Neumann's inequality, more precisely, it is unknown if for every n there exists a constant C_n such that

$$\|P(T_1, \dots, T_n)\| \leq C_n \sup\{|P(z_1, \dots, z_n)| : |z_i| \leq 1\}, \tag{4.3.1}$$

for every polynomial P in n variables and every $n-$tuple $(T_1, \dots, T_n)$ of commuting contractions in $\mathcal{B}(H)$ (the set of all bounded linear operators acting on a Hilbert space H). According to [15], Dixon in [11] gave lower estimates for the optimal C_n and showed that, if such constant verifying (4.3.1) exists, then it must grow faster than any power of n. He did this by considering the problem in the smaller class of $k-$homogeneous polynomials.

Definition 4.3.2. *A $k-$homogeneous polynomial in n variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form*

$$P(z_1, \dots, z_n) = \sum_{\alpha \in \Lambda(k,n)} a_\alpha z^\alpha,$$

where $\Lambda(k, n) := \{\alpha \in \mathbb{N}_0^n : |\alpha| := \alpha_1 + \cdots + \alpha_n = k\}$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $a_\alpha \in \mathbb{C}$.

In [11](Theorem 1.2), the author showed that for fixed k and n growing

$$n^{\frac{1}{2}\left[\frac{k-1}{2}\right]} \ll C_{k,\infty}(n) \ll n^{\frac{k-2}{2}}, \tag{4.3.2}$$

where $[z]$ denotes the integer part of z.

Dixon studied the asymptotic behaviour (as n tends to infinity) of the smallest constant $C_{k,\infty}(n)$ such that

$$\|P(T_1, \dots, T_n)\|_{\mathcal{B}(H)} \leq C_{k,\infty}(n) \sup\{|P(z_1, \dots, z_n)| : |z_i| \leq 1\}, \tag{4.3.3}$$

for every $k-$homogeneous polynomial p in n variables and every $n-$tuple of commuting contractions $\mathbf{T}$.

Further, Montero and Tonge in [20], for $1 \leq q < \infty$, they consider $C_{k,q}(n)$, the smallest constant such that

$$\|P(T_1, \dots, T_n)\|_{\mathcal{B}(H)} \leq C_{k,q}(n) \sup\{|P(z_1, \dots, z_n)| : \sum_{i=1}^n |z_i|^q \leq 1\}, \tag{4.3.4}$$

for every $k-$homogeneous polynomial p in n variables and every $n-$tuple of commuting contractions $\mathbf{T}$ with $\sum_{i=1}^{n}\|T_i\|^q \le 1$. Their upper and lower estimates for the growth of $C_{k,q}(n)$ are the following

$$n^{\frac{k-1}{q'}-\frac{1}{2}\left[\frac{k}{2}\right]} \ll C_{k,q}(n) \ll n^{\frac{k-2}{q'}} \text{ for } 1 \le q \le 2, \tag{4.3.5}$$

$$n^{\frac{k}{2}-\frac{1}{2}\left(\left[\frac{k}{2}\right]+1\right)} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}} \text{ for } 2 \le q < \infty, \tag{4.3.6}$$

where q' denotes the conjugate of q, i.e., $\frac{1}{q}+\frac{1}{q'}=1$. In all the estimates k is considered to be fixed.

Note that the upper bounds here hold for every $n-$tuple $\mathbf{T}$ satisfying $\sum_{i=1}^{n}\|T_i\|^q \le 1$, not necessarily commuting. If one does not ask for the contractions to commute, this bound is shown to be optimal in [20](Proposition 15).

Based on the combinatorial methods from [11], the authors in [15] change the construction of the Hilbert space and the operators given there to find the exact asymptotic growth of $C_{k,\infty}(n)$, answering a question posed by Dixon. Their main result in [15] is the following theorem.

Theorem 4.3.3. *For $k \ge 3$, and $1 \le q < \infty$, let $C_{k,q}(n)$ be the smallest constant such that*

$$\|Q(T_1,\dots,T_n)\|_{\mathcal{B}(H)} \le C_{k,q}(n)\sup\{|Q(z_1,\dots,z_n)| : \|(z_i)_i\|_q \le 1\},$$

for every $k-$homogeneous polynomial Q in n variables and every $n-$tuple of commuting contractions $\mathbf{T}$ with $\sum_{i=1}^{n}\|T_i\|^q \le 1$. Then

(i) $C_{k,\infty} \sim n^{\frac{k-2}{2}}$

(ii) for $2 \le q < \infty$ we have

$$\log^{-3/q}(n)n^{\frac{k-2}{2}} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}}.$$

In particular, $n^{\frac{k-2}{2}-\varepsilon} \ll C_{k,q}(n) \ll n^{\frac{k-2}{2}}$ for every $\varepsilon > 0$.

One can extend now the multicentric calculus for an $n-$tuple of communting contractions $\mathbf{T}=(T_1,\dots,T_n)$. We will do so by considering Dixon's estimate (4.3.3) and later we will comment on Theorem 4.3.3.

Recall the following, $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$, $p(\mathbf{z})=(p_1(z_1),\dots,p_n(z_n))$ with p_j polynomial of degree d_j with distinct roots, for every $j=1,\dots n$ and denote $w=p(\mathbf{z})$ with $w=(w_1,\dots,w_n)$.

Consider the decomposition

$$\varphi(\mathbf{z}) = \sum_{\nu}\delta_\nu(\mathbf{z})g_\nu(p(\mathbf{z})) \tag{4.3.7}$$

where $g_\nu(\mathbf{z}) = \sum_{k=0}^{m} \sum_{|\alpha|=k} a_\alpha p(\mathbf{z})^\alpha = \sum_{k=0}^{m} Q_{k,\nu}(p(\mathbf{z}))$, that is, g_ν is a sum of m $k-$homogeneous polynomials in n variables, denoted by $Q_{k,\nu}$ and by $\sum_\nu \delta_\nu(\mathbf{z})$ we mean $\sum_{\nu_1=1}^{d_1} \cdots \sum_{\nu_n=1}^{d_n} \delta_{\nu_1}(z_1) \cdots \delta_{\nu_n}(z_n)$.

Then we can define

$$\varphi_k(\mathbf{z}) = \sum_\nu \delta_\nu(\mathbf{z}) \sum_{|\alpha|=k} a_\alpha p(\mathbf{z})^\alpha = \sum_\nu \delta_\nu(\mathbf{z}) Q_{k,\nu}(w). \tag{4.3.8}$$

With the above notation it is obvious that (4.3.7) becomes

$$\varphi(\mathbf{z}) = \sum_{k=0}^{m} \varphi_k(\mathbf{z}). \tag{4.3.9}$$

The goal is to estimate $\|\varphi(\mathbf{T})\|$. Inserting $\mathbf{T}$ and then applying the norm to (4.3.9) we see that

$$\|\varphi(\mathbf{T})\| \leq \sum_{k=0}^{m} \|\varphi_k(\mathbf{T})\|, \tag{4.3.10}$$

and using (4.3.8) we get

$$\|\varphi(\mathbf{T})\| \leq \sum_{k=0}^{m} \sum_{\nu_1=1}^{d_1} \cdots \sum_{\nu_n=1}^{d_n} \|\delta_{\nu_1}(T_1)\| \cdots \|\delta_{\nu_n}(T_n)\| \|Q_{k,\nu}(p(\mathbf{T}))\|. \tag{4.3.11}$$

Since $Q_{k,\nu}(p(\mathbf{T}))$ are $k-$homogeneous polynomials, from (4.3.3) it follows that

$$\|Q_{k,\nu}(p(\mathbf{T}))\| \leq C_{k,\infty}(n) \sup_{\|p_j(z_j)\|_q \leq 1} |Q_{k,\nu}(p(\mathbf{z}))|. \tag{4.3.12}$$

Note that the constant $C_{k,\infty}(n)$ does not depend on the degree of the polynomial $Q_{k,\nu}$, thus, if for simplicity we denote $\sum_{\nu_1=1}^{d_1} \cdots \sum_{\nu_n=1}^{d_n} \|\delta_{\nu_1}(T_1)\| \cdots \|\delta_{\nu_n}(T_n)\|$ by A_k, we have

$$\|\varphi(\mathbf{T})\| \leq \sum_{k=0}^{m} A_k C_{k,\infty}(n) \sup_{\|p_j(z_j)\|_q \leq 1} |Q_{k,\nu}(p(\mathbf{z}))|. \tag{4.3.13}$$

Now we can use Lemma 4.2.3 to each decompositions $\varphi_k(\mathbf{z})$ to get

$$\sup_{\|p_j(z_j)\|_q \leq 1} |Q_{k,\nu}(p(\mathbf{z}))| \leq C(\rho_1) \ldots C(\rho_s) \sup_{|p_j(z_j)| \leq \rho_j} |\varphi_k(\mathbf{z})|, \tag{4.3.14}$$

where $s \leq n$ and $\rho_j \leq 1$ is the level of p_j, such that it does not pass through any critical points.

Again, for simplicity we denote $C(\rho_1)\dots C(\rho_s)$ by B_k, which is a constant depending on the levels ρ_j and is finite whenever the lemniscate $|p_j(z_j)| \leq \rho_j$ does not pass through any critical points. Hence we have

$$\|\varphi(\mathbf{T})\| \leq \sum_{k=0}^{m} A_k C_{k,\infty}(n) B_k \sup_{|p_j(z_j)|\leq 1} |\varphi_k(\mathbf{z})|. \tag{4.3.15}$$

Further we need to be able to estimate $\sup_{|p_j(z_j)|\leq 1} |\varphi_k(\mathbf{z})|$ from above by $\sup_{\mathbf{z}} |\varphi(\mathbf{z})|$, for all $z_j \in V_{p_j}(T_j) = \{z \in \mathbb{C} : |p_j(z)| \leq \|p_j(T_j)\|\}$.

For this we define the norm $|||\varphi||| = \max_k \sup_{\mathbf{z}} |\varphi_k(\mathbf{z})|$. This is indeed a norm since $|||\varphi||| \geq 0$ and if $|||\varphi||| = 0$ then $\varphi = 0$, hence for any $a \in \mathbb{C}^n$ we have $|||a\varphi||| = |a||||\varphi|||$. It is also easy to see that the following holds

$$\begin{aligned} |||\varphi + \psi||| &= \max_k \sup_{\mathbf{z}} |\varphi_k(z) + \psi_k(z)| \\ &\leq \max_k \sup_{\mathbf{z}} |\varphi_k(z)| + \max_k \sup_{\mathbf{z}} |\psi_k(z)| \\ &= |||\varphi||| + |||\psi|||. \end{aligned}$$

We denote the dimension of the space of polynomials where φ lies by (d, m) where d is the vector containing the degrees of the polynomials p_j. Using the fact that all spaces are norm equivalent, then the space where φ lies is finite dimensional and thus there exists a constant C such that

$$\frac{1}{C}|||\varphi||| \leq \sup_{\mathbf{z}} |\varphi(\mathbf{z})| \leq C|||\varphi|||. \tag{4.3.16}$$

Since $\sup_{\mathbf{z}} |\varphi_k(\mathbf{z})| \leq \max_k \sup_{\mathbf{z}} |\varphi_k(\mathbf{z})| = |||\varphi|||$ for every k, then using the above estimate we get

$$\sup_{\mathbf{z}} |\varphi_k(\mathbf{z})| \leq C \sup_{\mathbf{z}} |\varphi(\mathbf{z})|. \tag{4.3.17}$$

Therefore from (4.3.17) and (4.3.15) we obtain an estimate of $\|\varphi(\mathbf{T})\|$ from above by $\sup_{\mathbf{z}} |\varphi(\mathbf{z})|$, that is,

$$\|\varphi(\mathbf{T})\| \leq K \sup_{\mathbf{z}} |\varphi(\mathbf{z})|, \tag{4.3.18}$$

with K being a constant that depends on the levels ρ_j's of the polynomials p_j's, on δ_ν at the operator $\mathbf{T}$ and on the degree of φ but not on φ itself.

Therefore we have proven the following theorem.

Theorem 4.3.4. *Let $\mathbf{z} \in \mathbb{C}^n$, $p(\mathbf{z}) = (p_1, \ldots, p_n)$ be a polynomial in n variable such that for $j = 1, \ldots, n$, p_j is a monic polynomial of degree d_j with distinct roots and let $\rho_j \geq 0$ satisfying $|p_j(z_j)| \leq \rho_j$ be such that the lemniscate contains no critical points of p_j. Let $k \geq 3$ and let $\mathbf{T} = (T_1, \ldots, T_n)$ be an $n-$tuple of commuting contractions.*

Then for all φ which are holomorphic for $|p_j(z_j)| \leq \rho_j$, for $j = 1, \ldots, n$, there holds

$$\|\varphi(\mathbf{T})\| \leq K \sup_{\mathbf{z}} |\varphi(\mathbf{z})|, \tag{4.3.19}$$

with K being a constant that depends on the levels ρ_j's of the polynomials p_j's, on δ_ν at the operator $\mathbf{T}$ and on the degree of φ but not on φ itself.

Remark 4.3.5. Note that the above estimates work for any q such that $1 \leq q < \infty$ due to Theorem 4.3.3.

Chapter 5

Multicentric calculus for non-holomorphic functions

This chapter presents the latest results due to O. Nevanlinna [26] regarding the multicentric calculus without assuming the functions to be analytic. The calculus is worked out for one variable, where the operator is a matrix, and we present it in details (although proofs are skipped) so that later one can extend it for more than one variable. When working with $n-$variables the operator is then considered as an $n-$tuple of commuting matrices. Thus basic notions on commuting matrices are covered.

If one wants to determine which matrices commute with a given set of $n \times n$ matrices two tools appear to be useful, that is, a standard form for a given matrix (Jordan canonical form) and restrictions on the form of the commuting matrices. The latter is a canonical form called the H-form introduced by K.C. O'Meara and C. Vinsonhaler in [28]. We will shortly discuss both this tools.

An attempt on extending the calculus for n-tuple of commuting operators is worked out by the end of this chapter.

5.1 Multicentric calculus for a matrix

The discussion in this section follows [26]. Given any square matrix or a bounded operator A in a Hilbert space such that $p(A)$ is normal (or similar to normal), the author of [26] constructs a Banach algebra, depending on the polynomial p, for which a simple functional calculus holds. The algebra deals with $\mathbb{C}^d-$valued functions, when the polynomial p is of degree d. The functions are defined on the spectrum $\sigma(A)$. The calculus provides a natural approach to deal with nontrivial Jordan blocks and one does not need differentiability at such eigenvalues.

As an application there are considered situations in which $p(A)$ is diagonalizable or similar to normal. The aim thus is to remove the Jordan blocks by moving from T to $p(A)$. If $D = \mathrm{diag}\{\alpha_j\}$ is a diagonal matrix and φ is a continuous function, the any reasonable functional calculus satisfies $\varphi(D) = \mathrm{diag}\{\varphi(\alpha_j)\}$, while if A is diagonalizable so that with a similarity T one has $A = TDT^{-1}$, then one sets

$$\varphi(A) = T\varphi(D)T^{-1}. \tag{5.1.1}$$

When A has an eigenvalue with nontrivial Jordan block, then the customary approach is to assume that φ is smooth enough at the eigenvalues so that the off-diagonal elements can be represented by derivatives of φ.

In [26] it is shown that there is a simple way to parametrize continuous functions which slow down at those places where extra smoothness is needed. This allows a functional calculus which agrees with the holomorphic functional calculus if applied to holomorphic functions, but is defined for functions that do not need to be differentiable at any point.

As in previous chapter, here also the starting point for the calculus is taking $w = p(z)$ as a new variable and replacing the scalar function $\varphi : z \mapsto \varphi(z) \in \mathbb{C}$ with a vector valued function $f : w \mapsto f(w) \in \mathbb{C}^d$, where d is the degree of the polynomial p. The multicentric representation of φ is then of the form

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(p(z)), \tag{5.1.2}$$

where δ_j's are the Lagrange interpolation polynomials such that $\delta_j(\lambda_j) = 1$ while $\delta_j(\lambda_k) = 0$ for $k \neq j$, see [25].

If $p(A)$ is diagonalizable one can apply the known functional calculus to represent $f_j(p(A))$, but since δ_j's are polynomials, $\delta_j(A)$ is well defined and differentiability of φ is not needed.

Now consider the Banach space of continuous functions $f_M \to \mathbb{C}^d$ and associate with it a product, called in [26], "polyproduct" $\circledcirc$, such that it becomes a Banach algebra, denoted by $C_\Lambda(M)$. Here Λ denotes the set of zeros of the polynomial p. Then the functions φ can be viewed as Gelfand transforms $\hat{f}$ of functions $f \in C_\Lambda(M)$.

Recall the following notations and assumptions. Assume we are given a polynomial $p(z) = (z - \lambda_1)\cdots(z - \lambda_d)$ with distinct zeros $\Lambda = \{\lambda_j\}_{j=1}^d$ mapping the $z-$plane onto $w-$plane: $w = p(z)$. Denote by $\Lambda_1 = \{z : p'(z) = 0\}$ the set of all critical points. Recall that by the Gauss-Lucas theorem Λ_1 is in the convex hull of Λ. The Lagrange interpolation polynomials $\delta_j(z)$ are given by

$$\delta_j(z) = \frac{p(z)}{p'(\lambda_j)(z - \lambda_j)} = \prod_{k \neq j} \frac{z - \lambda_k}{\lambda_j - \lambda_k}.$$

Assume then we are given a function f mapping a compact $M \subset \mathbb{C}$ into $\mathbb{C}^d$, which determines a unique function φ on $K = p^{-1}(M)$ if one sets

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(p(z)) \quad \text{for } z \in K.$$

We say that φ is given on K by a multicentric representation and denote it in short

$$\varphi = \mathcal{L}f.$$

In the reverse direction, suppose it is given a scalar function φ on a set K_0. Then a necessary condition for f to be uniquely determined is that K_0 is balanced with respect to Λ in the sense that $K_0 = p^{-1}p(K_0)$. Assume throughout that $K_0 \subset K = p^{-1}(M)$ is such that $p(K_0) = M$.

Denote the roots of $p(z) - w = 0$ by $z_j = z_j(w)$. Away from the critical values these are analytic and it is assumed a fixed numbering so that $z_j(w) \to \lambda_j$ is $z_1(w) \to \lambda_1$ (when $w \to 0$). Let $\delta_j(\zeta; w)$ denote the interpolation polynomials, with w fixed, which takes the value 1 at $\zeta = z_j(w)$ while vanished at other $z_k(w)'$s:

$$\delta_j(\zeta; w) = \frac{p(z) - w}{p'(z_j(w))(\zeta - z_j(w))}, \tag{5.1.3}$$

so that in particular $\delta_j(\zeta; 0) = \delta_j(\zeta)$.

Proposition 5.1.1. *Suppose K is a balanced compact set with respect to local centers Λ. Assume that φ is given pointwisely in K. Then f is uniquely* *defined for all noncritical values* $w \in M \setminus p(\Lambda_1)$ *by*

$$f_k(w) = \sum_{j=1}^{d} \delta_j(\lambda_k; w)\varphi(z_j(w)). \tag{5.1.4}$$

The functions f_k *inherit the smoothness of* φ, *and additionally, if* $\lambda_c \in \Lambda_1$ *is an interior point of* K *and* φ *is at that point analytic, then the singularities of each* f_k *at the critical value* $p(\lambda_c)$ *are removable.*

Proof. See the discussions in [24] and [25]. □

So, one can use the expression $f = \mathcal{L}^{-1}\varphi$ at least when the components of f are determined by (5.1.4) for noncritical values w provided φ is given in a balanced set. This is natural when φ is analytic in a balanced domain, however, the topic in [26] is in functions which are perhaps given only on discrete sets, such as the set of eigenvalues of a matrix and then some extra care is needed in considering the possible lack of injectivity of $\mathcal{L}$. Therefore it is built a Banach algebra and $\mathcal{L}$ is viewed as performing the Gelfand transformation $\hat{f} = \mathcal{L}f$.

Further we consider continuous functions f mapping M into $\mathbb{C}^d$ and aim to define a Banach algebra structure into $C(M)^d$. We define the "polyproduct" $\circledcirc$ as multiplication in $C(M)^d$ and we want that $\mathcal{L}$ takes the vector functions into scalar functions in such a way that $\mathcal{L}$ becomes an algebra homomorphism

$$\mathcal{L}(f \circledcirc g) = (\mathcal{L}f)(\mathcal{L}g)$$

where the multiplication of scalar functions $\mathcal{L}f$ is pointwise. Next we define the product $\circledcirc$.

Definition 5.1.2. *Let* f *and* g *be pointwise defined functions from* $M \subset \mathbb{C}$ *into* $\mathbb{C}^d$. *Then their "polyproduct"* $f \circledcirc g$ *is a function defined on* M, *taking values in* $\mathbb{C}^d$ *such that*

$$(f \circledcirc g)(w) = (f \circ g)(w) - w(L \circ \Box f(w) \circ \Box g(w))l,$$

where L *is a matrix which has zero diagonal and* $L_{ij} = 1/(\lambda_i - \lambda_j)$ *for* $i \neq j$, l *is a vector in* $\mathbb{C}^d$ *which has components* $l_j = 1/p'(\lambda_j)$, $\circ$ *is the Hadamard (or Schur, elementwise) product and for* $a \in \mathbb{C}^d$ *we set*

$$\Box : a \mapsto \Box a = \begin{pmatrix} 0 & a_1 - a_2 & \dots & a_1 - a_d \\ a_2 - a_1 & 0 & \dots & a_2 - a_d \\ \dots & \dots & \dots & \dots \\ a_d - a_1 & \dots & a_d - a_{d-1} & 0 \end{pmatrix}$$

and call it boxing the vector a.

The following short notation will be use for simplicity:

$$f \circledcirc g = f \circ g - w(L \circ \square f \circ \square g)l.$$

For the powers we write $f^n = f \circledcirc f^{n-1}$ and the inverse as f^{-1} where it exists: $f \circledcirc f^{-1} = \mathbf{1}$, ($\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{C}^d$, is the unit in the algebra).

Next are presented results from [26] without proofs.

Proposition 5.1.3. *The vector space of functions*

$$f : M \subset \mathbb{C} \to \mathbb{C}^d$$

equipped with the product $\circledcirc$ *becomes a complex commutative algebra with* $\mathbf{1}$ *as the unit.*

Theorem 5.1.4. *Let* f *and* g *be defined in* M *and* $K = p^{-1}(M)$. *Then if* φ *and* ψ *are functions defined on* K *by* $\varphi = \mathcal{L}f$ *and* $\psi = \mathcal{L}g$, *then* $\varphi\psi$ *is given by*

$$\varphi\psi = \mathcal{L}(f \circledcirc g).$$

The key in proving the above theorem is a lemma in [26] stating that for $w = p(z)$ one has

$$\delta_i^2(z) = \delta_i(z) - w \sum_{j \neq i} [\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)]$$

while for $i \neq j$ one has

$$\delta_i(z)\delta_j(z) = w[\sigma_{ij}\delta_i(z) + \sigma_{ji}\delta_j(z)]$$

where σ_{ij} is defined to be

$$\sigma_{ij} = \frac{1}{p'(\lambda_j)} \frac{1}{\lambda_i - \lambda_j}.$$

For the algebra it is then defined a norm. The uniform norm $|f|_M = \max_{w \in M} |f(w)|_\infty$ where $|a|_\infty = \max_{1 \leq j \leq d} |a_j|$, is not an algebra norm in general, so one needs to move into the "operator norm".

Definition 5.1.5. *For* $f \in C(M)^d$ *we set*

$$\| f \| = \sup_{|g|_M \leq 1} |f \circledcirc g|_M.$$

This is clearly a norm in $C(M)^d$ and it is in fact equivalent with $| \cdot |_M$.

Proposition 5.1.6. *There is a constant C, depending on M and on Λ such that*

$$\|f \circledcirc g\| \leq \|f\|\|g\| \tag{5.1.5}$$
$$|f|_M \leq \|f\| \leq C|f|_M. \tag{5.1.6}$$

A complete proof of the above proposition can be found in [26].

Since the polyproduct $\circledcirc$ is uniquely determined by Λ, the algebra is denoted in short $C_\Lambda(M)$.

Theorem 5.1.7. *The Banach space $C(M)^d$ equipped with the polyproduct $\circledcirc$, and denoted by $C_\Lambda(M)$, is a commutative unital Banach algebra. The algebra norm $\|\cdot\|$ is equivalent with $|\cdot|_M$ and functions with components given by polynomials $p(w, \overline{w})$ are dense in $C_\Lambda(M)$.*

To apply the Gelfand theory one needs to know all characters in the algebra $C_\Lambda(M)$.

Definition 5.1.8. *A nontrivial linear functional $\chi : C_\Lambda(M) \to \mathbb{C}$ is called a character if it is additionally multiplicative:*

$$\chi(f \circledcirc g) = \chi(f)\chi(g).$$

The set of all characters is the character space, denoted here by $\mathcal{X}$. Thus $\mathcal{X}$ is the set of all characters χ_z given by

$$\chi_z : f \mapsto \sum_{j=1}^{d} \delta_j(z) f_j(p(z)), \quad z \in p^{-1}(M).$$

When φ is non-holomorphic hence with less smoothness it is easy to preserve its role because f, the vector function that parametrize φ, can be taken as any continuous vector function, while the behaviour of φ is in general complicated near critical points.

One approach is to consider $C_\Lambda(M)$ as the defining algebra while the functions φ appear as Gelfand transforms.

Now we recall some basic properties of Gelfand theory. Let $\mathcal{A}$ be a commutative unital Banach algebra with unit e and denote by h a character such that $h(ab) = h(a)h(b)$ for all $a, b \in \mathcal{A}$. Denote by $\Sigma_\mathcal{A}$ the set of characters of $\mathcal{A}$. Then every $h \in \Sigma_\mathcal{A}$ has norm 1 and $\Sigma_\mathcal{A}$ is compact in the Gelfand topology: one gives $\Sigma_\mathcal{A}$ the relative weak*−topology it has as a subset of the dual of $\mathcal{A}$.

The Gelfand transform of $a \in \mathcal{A}$ is

$$\hat{a} : \Sigma_\mathcal{A} \to \mathbb{C} \text{ where } \hat{a}(h) = h(a).$$

The function $\hat{a}$ is then always continuous in the Gelfand topology and this allows one to study the algebra $\mathcal{A}$ by studying continuous functions on $\Sigma_{\mathcal{A}}$. Thus one treats $\hat{a} \in C(\Sigma_{\mathcal{A}})$ as a continuous norm with the sup-norm. Recall that the spectrum $\sigma(a)$ of an element $a \in \mathcal{A}$ consists of all those $\lambda \in \mathbb{C}$ for which $\lambda e - a$ does not have an inverse in $\mathcal{A}$. Denote by $\rho(a)$ the spectral radius of a given by $\rho(a) = \max\{|\lambda| : \lambda \in \sigma(a)\}$.

In the following theorem are presented basic results on the Gelfand theory.

Theorem 5.1.9. (Gelfand representation theorem) *Let $\mathcal{A}$ be a commutative unital Banach algebra. Then for all $a \in \mathcal{A}$*

i) $\sigma(a) = \hat{a}(\Sigma_{\mathcal{A}}) = \{\hat{a}(h) : h \in \Sigma_{\mathcal{A}}\}$;

ii) $\rho(a) = \|\hat{a}\|_\infty = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \|a\|$;

iii) $a \in \mathcal{A}$ *has an inverse if and only if* $\hat{a}(h) \neq 0$ *for all* $h \in \Sigma_{\mathcal{A}}$;

iv) $\mathrm{rad}\mathcal{A} = \{a \in \mathcal{A} : \hat{a}(h) = 0$ *for all* $h \in \Sigma_{\mathcal{A}}\}$.

Now consider $C_\Lambda(M)$ and recall the following notations, $f^n = f \circledcirc f^{n-1}$, f^{-1} is the inverse of f and $\mathcal{X}$ the character space of $C_\Lambda(M)$

$$\mathcal{X} = \{\chi_z : z \in p^{-1}(M)\}$$

where

$$\chi_z(f) = \sum_{j=1}^{d} \delta(z) f_j(p(z)).$$

This allows us to identify χ_z with z and consequently $\mathcal{X}$ with $p^{-1}(M)$, hence one can view the Gelfand transform $\hat{f}$ as a function of $z \in p^{-1}(M)$.

Definition 5.1.10. *Given $f \in C_\Lambda(M)$ we set*

$$\hat{f} : p^{-1}(M) \to \mathbb{C}, \quad \hat{f} : z \mapsto \hat{f}(z) = \sum_{j=1}^{d} \delta(z) f_j(p(z)).$$

Thus one can view the multicentric representation operator $\mathcal{L}$ as performing the Gelfand transformation $\mathcal{L} : f \mapsto \hat{f}$.

Denote this Gelfand transformation by $\mathcal{L}$ to remind that for constant vectors $a \in \mathbb{C}^d$ this is just the Lagrange interpolation polynomial (restricted into $p^{-1}(M)$). Also denote $|\hat{f}|_K = \sup_{z\in K} |\hat{f}(z)|$. Thus the general Gelfand representation theorem for the algebra $C_\Lambda(M)$ [26] is the following.

Theorem 5.1.11. (Multicentric representation as Gelfand transform) *For $f \in C_\Lambda(M)$ the following hold with $K = p^{-1}(M)$:*

i) $\sigma(f) = \{\hat{f}(z) : z \in K\}$;

ii) $\rho(f) = |f|_K = \lim_{n\to\infty} \|f^n\|^{1/n} \leq \|f\|$;

iii) f *has an inverse if and only if* $\hat{f}(z) \neq 0$ *for all* $z \in K$;

iv) $\mathrm{rad}C_\Lambda(M) = \{f \in C_\Lambda(M) : \hat{f}(z) = 0$ *for all* $z \in K\}$.

Recall, that an algebra $\mathcal{A}$ is semi-simple is $\mathrm{rad}\mathcal{A} = \{0\}$.

Theorem 5.1.12. $C_\Lambda(M)$ *is semi-simple if and only if* M *contains no isolated critical values of* p.

From Theorem 5.1.11 conclude that if φ is given by multicentric representation $\varphi = \mathcal{L}f$ where f is continuous and bounded, then $1/\varphi = \mathcal{L}g$ with a bounded and continuous g if and only if $\varphi(z) \neq 0$ for $z \in p^{-1}(M)$. Thus one has the following theorem that we state here without proof. For proof one can check [26].

Theorem 5.1.13. *There exists a constant* C *depending on* M *and* Λ *such that the following holds. If* $f \in C_\Lambda(M)$ *is such that for all* $z \in p^{-1}(M)$

$$|\mathcal{L}f(z)| \geq \eta > 0,$$

then there exists $g \in C_\Lambda(M)$ *such that* $f \circledcirc g = \mathbf{1}$ *and*

$$\|g\| \leq C\frac{\|f\|^{d-1}}{\eta^d}. \tag{5.1.7}$$

In order to describe the functional calculus in [26] we need to recall notions of the quotient algebra $C_\Lambda(M)\backslash\mathcal{I}_{K_0}$. When one applies the functional calculus described later on, the natural requirement for φ is that it is well defined at the spectrum $\sigma(A)$ of the operator A. This means that f representing φ must be well defined on a set which includes $p(\sigma(A))$. However, $p^{-1}(p(\sigma(A)))$ is likely to be properly larger than $\sigma(A)$ which in practice shows up in lack of uniqueness in representing φ.

Let $K_0 \subset \mathbb{C}$ be compact, put $p(K_0) = M$ and denote as before $K = p^{-1}(M)$. Assume here that the inclusion $K_0 \subset K$ is proper. Let $\mathcal{I}_{K_0}$ be the closed ideal in $C_\Lambda(M)$

$$\mathcal{I}_{K_0} = \{f \in C_\Lambda(M) : \hat{f}(z) = 0 \text{ for } z \in K_0\}.$$

Then the set of elements one is dealing with can be identified with the cosets $[f]$:

$$C_\Lambda(M) \setminus \mathcal{I}_{K_0} = \{[f] : [f] = f + \mathcal{I}_{K_0}\}.$$

This is a unital Banach algebra with norm given by

$$\|[f]\| = \inf_{g\in\mathcal{I}_{K_0}} \|f + g\|.$$

Definition 5.1.14. *Given a closed ideal $J \subset \mathcal{A}$ the hull of the ideal is the set of all characters which vanish at every element in the ideal.*

Definition 5.1.15. *Given a closed ideal J in a commutative Banach algebra $\mathcal{A}$, the character space of the quotient algebra $\mathcal{A} \setminus J$ is the hull of J.*

Now one can identify the character space of the quotient algebra $C_\Lambda(M) \setminus \mathcal{I}_{K_0}$.

Corollary 5.1.16. *The quotient algebra $C_\Lambda(M) \setminus \mathcal{I}_{K_0}$ is a Banach algebra with unit and the character space can be identified with K_0, so that the Gelfand transformation becomes $[f] \mapsto \hat{f}_{|K_0}$.*

We continue with presenting the functional calculus as in [26]. There it is discussed the functional calculus related to $C_\Lambda(M)$ for matrices. Denote by $\mathbb{M}_n$ complex $n \times n-$matrices with the norm $\|A\| = \sup_{|x|_2=1} |Ax|_2$ and denote by $\sigma(A) = \{\alpha_k\}$ the eigenvalues of A and by m_A the minimal polynomial of A, that is, the monic polynomial q of smallest degree such that $q(A) = 0$:

$$m_A(z) = \prod_{k=1}^{m} (z - \alpha_k)^{n_k+1}.$$

The usual way to formulate the class of functions φ for which $\varphi(A)$ is well defined, asks the following to be known at every eigenvalue α_k

$$\varphi(\alpha_k), \dots, \varphi^{(n_k)}(\alpha_k).$$

Based on this information one can then construct an Hermite interpolation polynomial p and set $\varphi(A) = p(A)$.

Definition 5.1.17. *Given $A \in \mathbb{M}_n$ we call all monic polynomials p such that $p(A)$ is similar to a diagonal matrix as simplifying polynomials for A.*

If $\mathcal{K}$ denotes those indices k for which $n_k > 0$ in the minimal polynomial, then setting

$$s_A(z) = \int_0^z \prod_{k \in \mathcal{K}} (\zeta - \alpha_k)^{n_k} \mathrm{d}\zeta + c$$

one has a polynomial of minimal degree such that $s_A^{(j)}(\alpha_k) = 0$ for $j = 1, \dots, n_k$ and $k \in \mathcal{K}$. Clearly then $s_A(A)$ is similar to the diagonal matrix $\mathrm{diag}(s_A(\alpha_k))$. Since one can add an arbitrary constant to s_A, one may assume as well that s_A has distinct roots.

Now let p be a simplifying polynomial for A with distinct roots. Assume that φ is given on $\sigma(A)$ as

$$\varphi(z) = \sum_{j=1}^{d} \delta_j(z) f_j(p(z)).$$

If $B = p(A)$ one can define for $f_j \in C(\sigma(B))$ the matrix function $f_j(B)$ either by Lagrange interpolation at $p(\alpha_k)$ or by assuming the similarity transformation to the diagonal form $B = TDT^{-1}$ by given and setting $f_j(B) = Tf_j(D)T^{-1}$, both yielding the same matrix $f_j(B)$ which commute with A. Then the following matrix is well defined

$$\varphi(A) = \sum_{j=1}^{d} \delta_j(A) f_j(B).$$

In follows immediately that if one has two functions $f, g \in C_\Lambda(\sigma(B))$, and denote $\varphi = \mathcal{L}f$, $\psi = \mathcal{L}g$ and $\varphi\psi = \mathcal{L}(f \circledcirc g)$, then this definition yields

$$(\varphi\psi)(A) = \varphi(A)\psi(A).$$

However, O. Nevanlinna in [26] formulates the exact statement using a different notation to underline that knowing the values of φ at the spectrum of A need not determine f uniquely, and hence not $\varphi(A)$, either.

Definition 5.1.18. *Assume p is a simplifying polynomial for $A \in \mathbb{M}_n$ with distinct roots Λ. Then we denote by χ_A the mapping $C_\Lambda(p(\sigma(A))) \to \mathbb{M}_n$ given by*

$$f \mapsto \chi_A(f) = \sum_{j=1}^{d} \delta_j(A) f_j(B). \tag{5.1.8}$$

Theorem 5.1.19. *The mapping χ_A is a continuous homomorphism $C_\Lambda(p(\sigma(A))) \to \mathbb{M}_n$.*

For the proof see [26].

Now it can be concluded that one can formulate a spectral mapping theorem. Let $M = p(\sigma(A))$ and $\sigma(A)$ is a proper subset of $p^{-1}(M)$. Then it follows from Corollary 5.1.16 that the spectrum of $[f]$ in $C_\Lambda(M) \setminus \mathcal{I}_{K_0}$ is $\sigma([f]) = \{\hat{f}(z) : z \in \sigma(A)\}$.

Theorem 5.1.20. (Theorem 3.4 in [26]) *We have for $[f] \in C_\Lambda(p(\sigma(A)))\setminus\mathcal{I}_{K_0}$ and $\chi_A(f) \in \mathbb{M}_n$*

$$\sigma(\chi_A(f)) = \sigma([f]).$$

Now consider bounded operators A in a Hilbert space H. Let $\|A\|$ denote the norm of $A \in \mathcal{B}(H)$.

Definition 5.1.21. *We call $A \in \mathcal{B}(H)$ polynomially normal, if there exists a nonconstant monic polynomial p such that $p(A)$ is normal. The polynomial p is then called a simplifying polynomial for A.*

Theorem 5.1.22. *Let H be separable and $A \in \mathcal{B}(H)$ such that $p(A)$ is normal for some nonconstant polynomial p. Then there exist reducing subspaces $\{H_n\}_{n=0}^{\infty}$ for A, such that $H = \oplus_{n=0}^{\infty} H_n$ and $A_{|H_0}$ is algebraic while $A_{|H_n}$ are for $n \geq 1$ similar to normal.*

Let $N = p(A)$ be normal and assume that p has simple roots. Then the first task is to define $f_j(N)$ in a consistent way. Recall the following results.

Lemma 5.1.23. *Let $M \subset \mathbb{C}$ be compact. Then the closure of polynomials of the form $q(w, \overline{w})$ in the uniform norm over M equals $C(M)$.*

Since N commutes with N^* the operator $q(N, N^*)$ is well defined and the following holds.

Lemma 5.1.24. *If $N \in \mathcal{B}(H)$ is normal, then*

$$\|q(N, N^*)\| = \max_{w \in \sigma(N)} |q(w, \overline{w})|.$$

Given a normal operator N and a continuous function f_j on $\sigma(N)$ one approximates f_j by a sequence $\{q_{j,n}\}$ such that

$$|f_j - q_{j,n}|_\infty = \max_{w \in \sigma(N)} |f_j(w) - q_{j,n}(w, \overline{w})| \to 0$$

and sets

$$f_j(N) = \lim_{n \to \infty} q_{j,n}(N, N^*) \tag{5.1.9}$$

Then $f_j(N) \in \mathcal{B}(H)$ is normal, with $\|f_j(N)\| = |f_j|_\infty \leq \|f\|$.

One can apply Definition 5.1.18 for $A \in \mathcal{B}(H)$ and p a simplifying polynomial for A with distinct roots, so that $N = p(A)$ is normal. Thus one has χ_A as the mapping $C_\Lambda(p(\sigma(A))) \to \mathcal{B}(H)$ given by

$$f \mapsto \chi_A(f) = \sum_{j=1}^{d} \delta_j(A) f_j(N). \tag{5.1.10}$$

Note that $\delta_j(A)$ and $f_j(N)$ commute. In fact, A commutes with $N = p(A)$ and since N commutes with N^*, the operator A commutes with N^* as well, by Fuglede's theorem. Then O. Nevanlinna in [26] combines the construction in the following theorem.

Theorem 5.1.25. *Let $A \in \mathcal{B}(H)$ and a simplifying polynomial p be given as above. Then the mapping χ_A is a continuous homomorphism from $C_\Lambda(p(\sigma(A)))$ to $\mathcal{B}(H)$. In particular,*

$$\chi_A(f \circledcirc g) = \chi_A(f)\chi_A(g)$$

and

$$\|\chi_A(f)\| \leq C\|f\| \text{ with } C = \sum_{j=1}^{d} \|\delta_j(A)\|.$$

We therefore conclude this section, as in [26], by stating the spectral mapping theorem for operators. Its complete proof can be found in [26]. If $A \in \mathcal{B}(H)$ is such that $p(A)$ is similar to normal, then one has $\chi_A(f) = T\chi_V(f)T^{-1}$ and therefore $\chi_A(f)$ and $\chi_V(f)$ have the same spectrum. Hence one may as well assume that A is polynomially normal.

Theorem 5.1.26. *Suppose p has simple zeros and $A \in \mathcal{B}(H)$ is such that $p(A)$ is normal. Then for all $[f] \in C_\Lambda(p(\sigma(A))) \setminus \mathcal{I}_{\sigma(A)}$ we have*

$$\sigma(\chi_A(f)) = \sigma([f]).$$

5.2 Commuting matrices

Before attempting to extend the above calculus we want to get familiar with how to find a matrix that commutes with a given one or a given set of $n \times n$ commuting matrices. For this, as mentioned before we have two different tools, which are the Jordan canonical form and the H-form. We start from the first one and continue with the second one by shortly describing them.

From the equation $AX = XB$, where A and B are two square matrices of different orders one can find the unknown rectangular matrix X and then replacing B by A one finds all the solutions of the equation $AX = XA$, that is, all matrices X that commute with A. The computations and results presented below follow [14] and [17] unless otherwise specified.

It is known that any square matrix $A \in \mathbb{M}_n$ can be expressed in the Jordan canonical form

$$T^{-1}AT = J = \operatorname{diag}(J_1, J_2, \ldots, J_t)$$

with

$$J_k = J_k(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{M}_{m_k}$$

where T is non-singular, $m_1+m_2+\cdots+m_t = n$ and $\lambda_1, \lambda_2, \ldots, \lambda_u$ are distinct eigenvalues of A.

We can write J_k also in the following form

$$J_k = J_K(\lambda_k) = \lambda_k I^{(m_k)} + N^{(m_k)},$$

with $I^{(m_k)}$ the identity matrix of dimension m_k and $N^{(m_k)}$ the m_k dimension square matrix with superdiagonal of 1s and zero otherwise.

Following Chapter VIII in Gantmacher book [14] we will find the solution for the equation $AX = XB$, where A and B are two square matrices of different orders, say n and m, respectively. Then, by replacing B with A, one gets all the matrices that commute with A.

To solve the equation

$$AX = XB \tag{5.2.1}$$

we write A and B in the Jordan canonical form:

$$A = TJ_AT^{-1}, \quad B = SJ_BS^{-1} \tag{5.2.2}$$

where T and S are square non-singular matrices of orders n and m, respectively, and J_A and J_B are the Jordan matrices

$$\begin{aligned} J_A &= \operatorname{diag}(\lambda_1 I^{(t_1)} + N^{(t_1)}, \lambda_2 I^{(t_2)} + N^{(t_2)}, \ldots, \lambda_u I^{(t_u)} + N^{(t_u)}) \\ J_B &= \operatorname{diag}(\mu_1 I^{(t_1)} + N^{(s_1)}, \mu_2 I^{(s_2)} + N^{(s_2)}, \ldots, \mu_v I^{(s_v)} + N^{(s_v)}). \end{aligned}$$

Then equation (5.2.1) becomes $TJ_AT^{-1}X = XSJ_BS^{-1}$, and after multiplying on the left by T^{-1}, on the right by S and denoting

$$X_{AB} = T^{-1}XS \Leftrightarrow X = TX_{AB}S^{-1} \tag{5.2.3}$$

we get

$$J_AX_{AB} = X_{AB}J_B. \tag{5.2.4}$$

Thus we have reduced equation (5.2.1) by the equation (5.2.4), of the same form, in which the given matrices are in the Jordan canonical form, thus they are quasi-diagonal.

We can then partition X_{AB} into blocks corresponding to the quasi-diagonal matrices J_A and J_B, that is, $X_{AB} = \{X_{\alpha\beta}\}_{\alpha\beta}$, with $\alpha = 1, 2, \ldots, u$ and $\beta = 1, 2, \ldots, v$ where $X_{\alpha\beta}$ is a rectangular matrix of dimension $t_\alpha \times s_\beta$. Therefore we see that equation (5.2.4) breaks up into uv matrix equations

$$(\lambda_\alpha I^{(t_\alpha)} + N^{(t_\alpha)})X_{\alpha\beta} = X_{\alpha\beta}(\mu_\beta I^{(s_\beta)} + N^{(s_\beta)})$$

for $\alpha = 1, 2, \ldots, u,$ and $\beta = 1, 2, \ldots, v$, so

$$(\mu_\beta - \lambda\alpha)X_{\alpha\beta} = N^{(t_\alpha)}X_{\alpha\beta} - X_{\alpha\beta}N^{(s_\beta)}. \tag{5.2.5}$$

Two cases appear in solving equation (5.2.5), either $\lambda_\alpha \neq \mu_\beta$ or $\lambda_\alpha = \mu_\beta$. If $\lambda_\alpha \neq \mu_\beta$ then $X_{\alpha\beta} = 0$ (see [14]). If $\lambda_\alpha = \mu_\beta$ then $X_{\alpha\beta}$ is a rectangular upper triangular matrix. Further, it is determined in [14] the structure of $X_{\alpha\beta} \in \mathbb{M}_{t_\alpha \times s_\beta}$ when $\lambda_\alpha = \mu_\beta$ and that is the following

- if $t_\alpha < s_\beta$ then
$$X_{\alpha\beta} = \begin{pmatrix} 0 & T_{t_\alpha} \end{pmatrix}, \tag{5.2.6}$$
- if $t_\alpha > s_\beta$ then
$$X_{\alpha\beta} = \begin{pmatrix} T_{s_\beta} \\ 0 \end{pmatrix}, \tag{5.2.7}$$

  where $T_{t_\alpha} \in \mathbb{M}_{t_\alpha}$ and $T_{s_\beta} \in \mathbb{M}_{s_\beta}$ are upper triangular Toeplitz matrices (i.e. the diagonal and all superdiagonals are constants while all the other elements are zero, for example
$$T_{t_\alpha} = \begin{bmatrix} c_1 & c_2 & \cdot & \cdot & \cdot & c_{t_\alpha} \\ 0 & c_1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & c_2 \\ 0 & \cdot & \cdot & \cdot & 0 & c_1 \end{bmatrix}$$
  with $c_1, c_2, \ldots, c_{t_u}$ arbitrary parameters).

Then if we set $T = S$, $J_A = J_B$ and denoting $X_{AB} = X_A$ in (5.2.4) we get all solutions for $AX = XA$, i.e., all matrices that commute with A, in the above form:
$$X = TX_AT^{-1}, \tag{5.2.8}$$
where $X_A = \{X_{\alpha\beta}\}_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, u)$ and $X_{\alpha\beta}$ is either the null matrix or a regular upper triangular matrix given by (5.2.6) or (5.2.7), depending on whether $\lambda_\alpha \neq \lambda_\beta$ or $\lambda_\alpha = \lambda_\beta$.

Definition 5.2.1. *A set of matrices $A_1, A_2, \ldots, A_n$ is said to commute if they commute pairwise.*

A property of commuting matrices is that they preserve each other's eigenspaces, so they map the same invariant subspaces. If both matrices are diagonalizable, then they can be simultaneously diagonalized. Moreover, if one of the matrices has the property that its minimal polynomial coincide with its characteristic polynomial, i.e. the characteristic polynomial has only simple roots, then the other matrix can be written as a polynomial in the first matrix.

Two Hermitian matrices (or self-adjoint matrices) commute if their eigenspaces coincide. In particular, two Hermitian matrices without multiple eigenvalues commute if they share the same set of eigenvectors.

Example 5.2.2. (of commuting matrices) The unit matrix commute with all matrices. Diagonal matrices commute. Jordan blocks commute with upper triangular matrices that have the same values along the diagonal and superdiagonals. If the product of two symmetric matrices is symmetric, then they must commute.

Now, given a set of $n \times n$ matrices one can find matrices that commute with the one in the set by using the H-form. This form was introduced by K.C. O'Meara and C. Vinsonhaler in [28] as a tool for preserving the upper triangularity property in commuting matrices since the Jordan canonical form fails to accommodate this. Upper triangular matrices are easier to work with in determining commuting properties. It is well known that a set of commuting matrices can be simultaneously triangularized (see, for example, R. A. Horn and C. R. Johnson *Matrix Analysis*, 1985). Hence the H-form for an $n \times n$ matrix over an algebraically closed field will let one assume that all commuting matrices are also upper triangular.

Following the description given in [28], a basic H-matrix is a blocked-matrix generalization of a basic Jordan matrix, with associated eigenvalue λ, where one replaces the eigenvalues by scalar matrices and the 1's by full column rank matrices in reduced raw echelon form. The H-form does not allow multiple basic H-matrices for the same eigenvalue λ, as the Jordan form does.

Definition 5.2.3. (Definition 4.1. in [28]) *A basic H-matrix with eigenvalue λ is an $n \times n$ matrix A of the following form: There is a partition $n_1 + n_2 + \cdots + n_r = n$ of n with $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$ such that when A is viewed as a blocked matrix with diagonal blocks of size $n_1, n_2, \ldots, n_r$, the diagonal blocks are the $n_i \times n_i$ scalar matrices λI and the first super-diagonal blocks are full rank $n_i \times n_{i+1}$ matrices in reduced echelon form (i.e. an identity matrix followed by zero rows). All other blocks of A are zero. In this case, we say that A has an H-block structure $(n_1, n_2, \ldots, n_r)$.*

Too see how H-matrices look like we will present a few simple examples following [28]. The matrices

$$\begin{bmatrix} \lambda & 0 & 1 & 0 & & & \\ & \lambda & 0 & 1 & & & \\ & & \lambda & 0 & 1 & & \\ & & & \lambda & 0 & & \\ & & & & \lambda & 1 & \\ & & & & & \lambda & 1 \\ & & & & & & \lambda \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \lambda & 0 & 0 & 1 & 0 & 0 \\ & \lambda & 0 & 0 & 1 & 0 \\ & & \lambda & 0 & 0 & 1 \\ & & & \lambda & 0 & 0 \\ & & & & \lambda & 0 \\ & & & & & \lambda \end{bmatrix}$$

are basic H-matrices with block structure $(2,2,1,1,1)$ and $(3,3)$ respectively.

Next we check the commutativity. For this let A_1 and A_2 be H-matrices given by

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} \lambda_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix},$$

then with a simple calculation we get

$$A_1A_2 = \begin{bmatrix} \lambda_1\lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_1\lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_1\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_1\lambda_2 \end{bmatrix} = A_2A_1$$

Hence they commute.

Definition 5.2.4. *Let A be a square matrix over an algebraically closed field F, and let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be distinct eigenvalues of A. We say that A is in H-form if A is a direct sum of basic H-matrices, one for each eigenvalue. In other words, A has the form*

$$\begin{bmatrix} H_1 & & & & \\ & H_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & H_r \end{bmatrix},$$

where H_i is a basic H-matrix with eigenvalue λ_i for $i = 1, 2, \dots, r$.

These forms has nice properties that we state below without any proofs, but for details one can check [28].

Theorem 5.2.5. (Theorem 4.7. in [28]) *Let $A_1, A_2, \dots, A_k$ be commuting $n \times n$ matrices over an algebraically closed field F. Then there is a similarity transformation which puts A_1 in H-form and simultaneously puts $A_2, \dots, A_k$ in upper triangular form.*

The following proposition establishes the uniqueness of the H-form, by making use of the nullities of the powers of $A - \lambda I$.

Proposition 5.2.6. *If A is a basic H-matrix with eigenvalue λ and block structure $(n_1, \dots, n_r)$, then*

$$\begin{aligned} r &= \textit{nilpotent index of } A - \lambda I, \\ n_1 &= \textit{nullity } (A - \lambda I), \\ n_i &= \textit{nullity } (A - \lambda I)^i - \textit{nullity } (A - \lambda I)^{i-1} \textit{ for } i = 2, \dots, r. \end{aligned}$$

Consequently, each square matrix is similar to a unique matrix in H-form (ignoring permutations of basic blocks).

Recall, that the nullity of a matrix A is the number of free variables in the equation $Ax = 0$, in other words, is the dimension of the kernel of A.

Below we present the connection between the H-form and the Jordan form as stated in [28].

Proposition 5.2.7. *The H and Jordan structures of any nilpotent $n \times n$ matrix A (more generally, a matrix with a single eigenvalue) are conjugate ("dual" or "transpose") partitions of n. Moreover, the H-form and Jordan form of any square matrix are conjugate under a permutation transformation.*

Proposition 5.2.8. *Let J be a Jordan matrix with just two basic $\lambda-$blocks and Jordan structure (r, s) with $r \geq s$ (i.e. the block sizes are $r \times r$ and $s \times s$). Let A be the (basic) H-form of J. Then*

(1) If $r = s$, A has H-block structure $(n_1, \dots, n_r)$ with $n_1 = n_2 = \cdots = n_r = 2$. Thus, as a partitioned matrix,

$$A - \lambda I = \begin{bmatrix} 0 & I_t \\ 0 & 0 \end{bmatrix},$$

where $t = 2s - 2$ is even. Here I_t denotes the $t \times t$ identity matrix, and 0 denotes a variable size zero matrix.

(2) If $r > s$, A has H-block structure $(n_1, \ldots, n_r)$ with $n_1 = n_2 = \cdots = n_s = 2$, and $n_i = 1$ for $i = s+1, \ldots, r$. Thus, as a partitioned matrix,

$$A - \lambda I = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_u \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $t = 2s - 1$ is odd and $u = r - s - 1$.

(3) Combined, the two statements say A has the form in (2) with

$$t = 2s - 1 - \delta_{rs} \text{ and } u = r - s - 1 + \delta_{rs}.$$

Example 5.2.9. The $(5, 3)$ Jordan Matrix J has the H-form A, where

$$J = \begin{bmatrix} \lambda & 1 & & & & & & \\ & \lambda & 1 & & & & & \\ & & \lambda & 1 & & & & \\ & & & \lambda & 1 & & & \\ & & & & \lambda & 0 & & \\ & & & & & \lambda & 1 & \\ & & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{bmatrix} \text{ and } A = \begin{bmatrix} \lambda & 0 & 1 & & & & & \\ & \lambda & 0 & 1 & & & & \\ & & \lambda & 0 & 1 & & & \\ & & & \lambda & 0 & 1 & & \\ & & & & \lambda & 0 & 1 & \\ & & & & & \lambda & 0 & 0 \\ & & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{bmatrix}.$$

Following Proposition 5.2.8, here we have $t = 2s - 1 = 2 \times 3 - 1 = 5$ and $u = r - s - 1 = 5 - 3 - 1 = 1$.

5.3 Multicentric calculus for $n-$tuples of commuting matrices

To extend the calculus presented at the beginning of this chapter we start by looking at the case when $n = 2$, that is, we consider two variables $(z_1, z_2) \in \mathbb{C}^2$ and using results from Chapter 4 on multicentric calculus for $n-$tuples of commuting operators we work out a formula similar to the one in Theorem 5.1.4 for one variable case. In a similar fashion, a formula for $n = 3$ is provided but not one for a general n since for this the calculations are messy and the formula seems to be too lengthy.

Let $(w_1, w_2) = (p_1(z_1), p_2(z_2))$ with p_1, p_2 monic polynomials of degree d_1, d_2, respectively. Denote the roots of p_1 by λ_j and the roots of p_2 by

μ_k, for $j = 1, 2, \ldots, d_1$ and $k = 1, 2, \ldots, d_2$. Let $\delta_{j,1}(z_1)$ and $\delta_{k,2}(z_2)$ be the corresponding Lagrange interpolation polynomials,

$$\delta_{j,1}(z_1) = \frac{w_1}{p_1'(\lambda_j)(z_1 - \lambda_j)} \text{ and } \delta_{k,2}(z_2) = \frac{w_2}{p_2'(\mu_k)(z_2 - \mu_k)}.$$

From the multicentric representations of two functions, $\varphi(z_1, z_2)$ and $\psi(z_1, z_2)$, one works out the product $(\varphi\psi)(z_1, z_2)$ as follows.

Let

$$\varphi(z_1, z_2) = \sum_{k=1}^{d_2} \sum_{j=1}^{d_1} \delta_{k,2}(z_2)\delta_{j,1}(z_1) f_{j,k}(w_1, w_2)$$

and

$$\psi(z_1, z_2) = \sum_{m=1}^{d_2} \sum_{l=1}^{d_1} \delta_{m,2}(z_2)\delta_{l,1}(z_1) g_{l,m}(w_1, w_2).$$

Then, in short,

$$\begin{aligned} \varphi\psi &= \sum_{j,k,l,m} \delta_{k,2}\delta_{j,1} f_{j,k} \delta_{m,2}\delta_{l,1} g_{l,m} \\ &= \sum_{j,l,k,m} \delta_{j,1}\delta_{l,1}\delta_{k,2}\delta_{m,2} f_{j,k} g_{l,m}, \end{aligned} \tag{5.3.1}$$

for $j, l = 1, 2, \ldots, d_1$ and $k, m = 1, 2, \ldots, d_2$.

From the one variable case we have

$$\sum_{k,m=1}^{d_2} \delta_{k,2}\delta_{m,2} f_{j,k} g_{l,m} = \sum_{k=1}^{d_2} \delta_{k,2} \left[f_{j,k} g_{l,k} - w_2 \sum_{m \neq k} \sigma_{k,m}^{(2)} (f_{j,k} - f_{j,m})(g_{l,k} - g_{l,m}) \right], \tag{5.3.2}$$

where

$$\sigma_{k,m}^{(2)} = \frac{1}{p_2'(\mu_m)} \frac{1}{\mu_k - \mu_m}.$$

We insert (5.3.2) in equation (5.3.1) and we get

$$\begin{aligned} \varphi\psi &= \sum_{j,l=1}^{d_1} \delta_{j,1}\delta_{l,1} \sum_{k=1}^{d_2} \delta_{k,2} \left[f_{j,k} g_{l,k} - w_2 \sum_{m \neq k} \sigma_{k,m}^{(2)} (f_{j,k} - f_{j,m})(g_{l,k} - g_{l,m}) \right] \\ &= \sum_{j,l,k} \delta_{j,1}\delta_{l,1}\delta_{k,2} \left[f_{j,k} g_{l,k} - w_2 \sum_{m \neq k} \sigma_{k,m}^{(2)} (f_{j,k} - f_{j,m})(g_{l,k} - g_{l,m}) \right] \end{aligned}$$

$$= \sum_{k=1}^{d_2} \delta_{k,2} \sum_{j,l=1}^{d_1} \delta_{j,1}\delta_{l,1} f_{j,k} g_{l,k} - \sum_{k=1}^{d_2} \delta_{k,2} w_2 \sum_{m\neq k} \sigma_{k,m}^{(2)} \sum_{j,l=1}^{d_1} \delta_{j,1}\delta_{l,1}(f_{j,k}-f_{j,m})(g_{l,k}-g_{l,m}). \tag{5.3.3}$$

For simplicity we will work the two sums in (5.3.3) separately. Denote them by S_1 and S_2.

For S_1 we use

$$\sum_{j,l=1}^{d_1} \delta_{j,1}\delta_{l,1} f_{j,k} g_{l,k} = \sum_{j=1}^{d_1} \delta_{j,1} \left[f_{j,k} g_{j,k} - w_1 \sum_{l\neq j} \sigma_{j,l}^{(1)} (f_{j,k}-f_{l,k})(g_{j,k}-g_{l,k}) \right],$$

where

$$\sigma_{j,l}^{(1)} = \frac{1}{p_1'(\lambda_l)} \frac{1}{\lambda_j - \lambda_l},$$

thus

$$S_1 = \sum_{j,k} \delta_{k,2}\delta_{j,1} \left[f_{j,k} g_{j,k} - w_1 \sum_{l\neq j} \sigma_{j,l}^{(1)} (f_{j,k}-f_{l,k})(g_{j,k}-g_{l,k}) \right]. \tag{5.3.4}$$

For S_2 we see that

$$\sum_{j,l=1}^{d_1} \delta_{j,1}\delta_{l,1}(f_{j,k}-f_{j,m})(g_{l,k}-g_{l,m}) = \sum_{j=1}^{d_1} \delta_{j,1} \Big[(f_{j,k}-f_{j,m})(g_{j,k}-g_{j,m}) - \\ -w_1 \sum_{l\neq j} \sigma_{j,l}^{(1)} \left((f_{j,k}-f_{j,m})-(f_{l,k}-f_{l,m})\right)\left((g_{j,k}-g_{j,m})-(g_{l,k}-g_{l,m})\right) \Big],$$

so we get

$$S_2 = \sum_{j,k} \delta_{k,2}\delta_{j,1} w_2 \sum_{m\neq k} \sigma_{k,m}^{(2)} \Big[(f_{j,k}-f_{j,m})(g_{j,k}-g_{j,m}) - w_1 \sum_{l\neq j} \sigma_{j,l}^{(1)} \left((f_{j,k}-f_{j,m})-(f_{l,k}-f_{l,m})\right)\left((g_{j,k}-g_{j,m})-(g_{l,k}-g_{l,m})\right) \Big]. \tag{5.3.5}$$

After inserting (5.3.4) and (5.3.5) in equation (5.3.3) we get

$$\begin{aligned}\varphi\psi &= \sum_{j,k}\delta_{k,2}\delta_{j,1}\Bigg[f_{j,k}g_{j,k} - w_1\sum_{l\neq j}\sigma^{(1)}_{j,l}(f_{j,k}-f_{l,k})(g_{j,k}-g_{l,k})\\ &- w_2\sum_{m\neq k}\sigma^{(2)}_{k,m}(f_{j,k}-f_{j,m})(g_{j,k}-g_{j,m})\\ &+ w_1w_2\sum_{m\neq k}\sum_{l\neq j}\sigma^{(2)}_{k,m}\sigma^{(1)}_{j,l}\left((f_{j,k}-f_{j,m})-(f_{l,k}-f_{l,m})\right)\\ &\qquad\left((g_{j,k}-g_{j,m})-(g_{l,k}-g_{l,m})\right)\Bigg]. \end{aligned} \tag{5.3.6}$$

Hence $\varphi\psi = \mathcal{L}_1\mathcal{L}_2(f\circledcirc g)$ where $f=(f_{j,k})_{jk}\in\mathbb{M}_{d_1\times d_2}$ and $g=(g_{l,m})_{lm}\in\mathbb{M}_{d_1\times d_2}$.

In a similar way one can work out the product $\varphi\psi$ where φ and ψ have three variables, that is, one gets the following

$$\begin{aligned}\varphi\psi &= \sum_{i,k,m}\delta_{m3}\delta_{k2}\delta_{i1}\Bigg[f_{ikm}g_{ikm} - w_1\sum_{j\neq i}\sigma^{(1)}_{ij}(f_{ikm}-f_{jkm})(g_{ikm}-g_{jkm})\\ &-w_2\sum_{l\neq k}\sigma^{(2)}_{kl}(f_{ikm}-f_{ilm})(g_{ikm}-g_{ilm})\\ &-w_3\sum_{n\neq m}\sigma^{(3)}_{mn}(f_{ikm}-f_{ikn})(g_{ikm}-g_{ikn})\\ &+w_1w_2\sum_{l\neq k}\sum_{j\neq i}\sigma^{(2)}_{kl}\sigma^{(1)}_{ij}\left((f_{ikm}-f_{ilm})-(f_{jkm}-f_{jlm})\right)\\ &\qquad\left((g_{ikm}-g_{ilm})-(g_{jkm}-g_{jlm})\right)\\ &+w_1w_3\sum_{n\neq m}\sum_{j\neq i}\sigma^{(3)}_{mn}\sigma^{(1)}_{ij}\left((f_{ikm}-f_{ikn})-(f_{jkm}-f_{jkn})\right)\\ &\qquad\left((g_{ikm}-g_{ikn})-(g_{jkm}-g_{jkn})\right)\\ &+w_2w_3\sum_{n\neq m}\sum_{l\neq k}\sigma^{(3)}_{mn}\sigma^{(2)}_{kl}\left((f_{ikm}-f_{ikn})-(f_{ilm}-f_{iln})\right)\\ &\qquad\left((g_{ikm}-g_{ikn})-(g_{ilm}-g_{iln})\right)\\ &-w_1w_2w_3\sum_{n\neq m}\sum_{l\neq k}\sigma^{(3)}_{mn}\sigma^{(2)}_{kl}\sigma^{(1)}_{ij}\\ &\qquad\left(((f_{ikm}-f_{ikn})-(f_{ilm}-f_{iln}))-((f_{jkm}-f_{jkn})-(f_{jlm}-f_{jln}))\right)\\ &\qquad\left(((g_{ikm}-g_{ikn})-(g_{ilm}-g_{iln}))-((g_{jkm}-g_{jkn})-(g_{jlm}-g_{jln}))\right)\Bigg]. \end{aligned} \tag{5.3.7}$$

Although the formulas are long and messy one can easily write a more general formula for n variables. Extra care is needed in this case when handling the indices.

We continue with an example to see how formula (5.2.6) looks like.

Example 5.3.1. Let $p_1(z_1)$ be a monic polynomial of degree 2 with distinct roots λ_1 and λ_2, and $p_2(z_2)$ be a monic polynomial of degree 3 with distinct roots μ_1, μ_2 and μ_3. Denote $w_1 = p_1(z_1)$ and $w_2 = p_2(z_2)$. Then $(f \circledcirc g)(w_1, w_2)$ is a 2×3 matrix with the following elements

$$\begin{aligned}
(f \circledcirc g)_{11} &= f_{1,1}g_{1,1} - w_1\sigma^{(1)}_{1,2}(f_{1,1} - f_{2,1})(g_{1,1} - g_{2,1}) \\
&- w_2[\sigma^{(2)}_{1,2}(f_{1,1} - f_{1,2})(g_{1,1} - g_{1,2}) + \sigma^{(2)}_{1,3}(f_{1,1} - f_{1,3})(g_{1,1} - g_{1,3})] \\
&+ w_1w_2[\sigma^{(2)}_{1,2}\sigma^{(1)}_{1,2}((f_{1,1} - f_{1,2}) - (f_{2,1} - f_{2,2}))((g_{1,1} - g_{1,2}) - (g_{2,1} - g_{2,2})) \\
&\quad +\sigma^{(2)}_{1,3}\sigma^{(1)}_{1,2}((f_{1,1} - f_{1,3}) - (f_{2,1} - f_{2,3}))((g_{1,1} - g_{1,3}) - (g_{2,1} - g_{2,3}))]
\end{aligned}$$

$$\begin{aligned}
(f \circledcirc g)_{12} &= f_{1,2}g_{1,2} - w_1\sigma^{(1)}_{1,2}(f_{1,2} - f_{2,2})(g_{1,2} - g_{2,2}) \\
&- w_2[\sigma^{(2)}_{2,1}(f_{1,2} - f_{1,1})(g_{1,2} - g_{1,1}) + \sigma^{(2)}_{2,3}(f_{1,2} - f_{1,3})(g_{1,2} - g_{1,3})] \\
&+ w_1w_2[\sigma^{(2)}_{2,1}\sigma^{(1)}_{1,2}((f_{1,2} - f_{1,1}) - (f_{2,2} - f_{2,1}))((g_{1,2} - g_{1,1}) - (g_{2,2} - g_{2,1})) \\
&\quad +\sigma^{(2)}_{2,3}\sigma^{(1)}_{1,2}((f_{1,2} - f_{1,3}) - (f_{2,2} - f_{2,3}))((g_{1,2} - g_{1,3}) - (g_{2,2} - g_{2,3}))]
\end{aligned}$$

$$\begin{aligned}
(f \circledcirc g)_{13} &= f_{1,3}g_{1,3} - w_1\sigma^{(1)}_{1,2}(f_{1,3} - f_{2,3})(g_{1,3} - g_{2,3}) \\
&- w_2[\sigma^{(2)}_{3,1}(f_{1,3} - f_{1,1})(g_{1,3} - g_{1,1}) + \sigma^{(2)}_{3,2}(f_{1,3} - f_{1,2})(g_{1,3} - g_{1,2})] \\
&+ w_1w_2[\sigma^{(2)}_{3,1}\sigma^{(1)}_{1,2}((f_{1,3} - f_{1,1}) - (f_{2,3} - f_{2,1}))((g_{1,3} - g_{1,1}) - (g_{2,3} - g_{2,1})) \\
&\quad +\sigma^{(2)}_{3,2}\sigma^{(1)}_{1,2}((f_{1,3} - f_{1,2}) - (f_{2,3} - f_{2,2}))((g_{1,3} - g_{1,2}) - (g_{2,3} - g_{2,2}))]
\end{aligned}$$

$$\begin{aligned}
(f \circledcirc g)_{21} &= f_{2,1}g_{2,1} - w_1\sigma^{(1)}_{2,1}(f_{2,1} - f_{1,1})(g_{2,1} - g_{1,1}) \\
&- w_2[\sigma^{(2)}_{1,2}(f_{2,1} - f_{2,2})(g_{2,1} - g_{2,2}) + \sigma^{(2)}_{1,3}(f_{2,1} - f_{2,3})(g_{2,1} - g_{2,3})] \\
&+ w_1w_2[\sigma^{(2)}_{1,2}\sigma^{(1)}_{2,1}((f_{2,1} - f_{2,2}) - (f_{1,1} - f_{1,2}))((g_{2,1} - g_{2,2}) - (g_{1,1} - g_{1,2})) \\
&\quad +\sigma^{(2)}_{1,3}\sigma^{(1)}_{2,1}((f_{2,1} - f_{2,3}) - (f_{1,1} - f_{1,3}))((g_{2,1} - g_{2,3}) - (g_{1,1} - g_{1,3}))]
\end{aligned}$$

$$\begin{aligned}
(f \circledcirc g)_{22} &= f_{2,2}g_{2,2} - w_1\sigma^{(1)}_{2,1}(f_{2,2} - f_{1,2})(g_{2,2} - g_{1,2}) \\
&- w_2[\sigma^{(2)}_{2,1}(f_{2,2} - f_{2,1})(g_{2,2} - g_{2,1}) + \sigma^{(2)}_{2,3}(f_{2,2} - f_{2,3})(g_{2,2} - g_{2,3})] \\
&+ w_1w_2[\sigma^{(2)}_{2,1}\sigma^{(1)}_{2,1}((f_{2,2} - f_{2,1}) - (f_{1,2} - f_{1,1}))((g_{2,2} - g_{2,1}) - (g_{1,2} - g_{1,1})) \\
&\quad +\sigma^{(2)}_{2,3}\sigma^{(1)}_{2,1}((f_{2,2} - f_{2,3}) - (f_{1,2} - f_{1,3}))((g_{2,2} - g_{2,3}) - (g_{1,2} - g_{1,3}))]
\end{aligned}$$

$$\begin{aligned}
(f \circledcirc g)_{23} &= f_{2,3}g_{2,3} - w_1\sigma_{2,1}^{(1)}(f_{2,3}-f_{1,3})(g_{2,3}-g_{1,3}) \\
&- w_2[\sigma_{3,1}^{(2)}(f_{2,3}-f_{2,1})(g_{2,3}-g_{2,1}) + \sigma_{3,2}^{(2)}(f_{2,3}-f_{2,2})(g_{2,3}-g_{2,2})] \\
&+ w_1w_2[\sigma_{3,1}^{(2)}\sigma_{2,1}^{(1)}((f_{2,3}-f_{2,1})-(f_{1,3}-f_{1,1}))((g_{2,3}-g_{2,1})-(g_{1,3}-g_{1,1})) \\
&\quad +\sigma_{3,2}^{(2)}\sigma_{2,1}^{(1)}((f_{2,3}-f_{2,2})-(f_{1,3}-f_{1,2}))((g_{2,3}-g_{2,2})-(g_{1,3}-g_{1,2}))].
\end{aligned}$$

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