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A GAME PERSPECTIVE TO COMPLEX ADAPTIVE SYSTEMS

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Abstract

Understanding the behaviour of a system through the properties of the elements of the system is a central problem in several fields of contemporary research. Appealing approaches for gaining such understanding have been proposed in complex systems studies. One particular approach is based on the scheme of agent-based modelling, in which the elements of the system are described by a set of precise rules which are implemented by computer programs. This dissertation is focused on topics related to two types of agent-based models: minority games and spatial two player games.

The first part of the thesis deals with minority games that have been extensively studied in the physics literature during the past eight years. A minority game describes a society of adaptive individuals with bounded rationality competing for scarce resources. Questions arising from such a model are associated with the efficiency of the system and the success of its individuals in utilizing the scarce resources. Previous studies have indicated that in case the individuals are allowed to evolve, they tend to evolve such that the efficiency of the system improves. However, the actual level of efficiency substantially depends on the type of evolution present in the system. We have applied genetic algorithms to make the system evolving. Our results indicate that natural selection and genetic algorithms can lead the system perform optimally and increase the success of individuals remarkably.

The second part of the thesis describes aspects of games that model strategic interaction situations between individuals. Especially, the focus of this part of the thesis is on models that aim at explaining the emergence and persistence of cooperative behaviour in an animal or human society. Previous studies have indicated that spatial structure of the society largely contributes to the maintenance of cooperation in these models. However, much of the research has been carried out by relying on evolutionary dynamics of the society associated with changes occurring in long times. We have explored a spatial game by allowing the individuals in the system be adaptive and act on short times, and our results show that the characteristic behaviour of the system is different from that observed in studies using evolutionary dynamics.

Preface

This thesis is a result of research work performed in the Complex Systems group at Laboratory of Computational Engineering (LCE), Helsinki University of Technology during the years 2002-2005. The work was initiated during my apprenticeship at LCE as a second-year student in the summer of 2000. My postgraduate studies were funded by the Graduate School on Computational Methods of Information Technology (ComMIT) and the Academy of Finland under the Finnish Centre of Excellence Programme.

There are several persons whom I would like to thank. First, I want to thank my supervisor Acad. Prof. Kimmo Kaski for giving general advice on scientific work, collaboration and providing good facilities for this study. Second, I want to thank Dr. Anirban Chakraborti and Dr. Jari Saramäki for collaboration and help. Discussions with Prof. János Kertész, Dr. Juuso Töyli and Dr. Lászlo Kullman have also been useful and stimulating. Moreover, I want to thank Dr. Péter Szelestey for practical advice and colleague Jukka-Pekka Onnela for general discussions.

Finally, I want to thank my family and friends for attesting with their presence that science and work is not the whole story of life.

Marko Sysi-Aho

List of Publications

I. M. Sysi-Aho, A. Chakraborti, K. Kaski, *Intelligent minority game with genetic crossover strategies*, The European Physical Journal B **34** (2003) 373-377.

II. M. Sysi-Aho, A. Chakraborti, K. Kaski, *Adaptation using hybridized genetic crossover strategies*, Physica A **322** (2002) 701-709.

III. M. Sysi-Aho, A. Chakraborti, K. Kaski, *Biology helps you to win a game*, Physica Scripta **T106** (2003) 32-35.

IV. M. Sysi-Aho, A. Chakraborti, K. Kaski, *Searching for good strategies in adaptive minority games*, Physical Review E **69** (2004) 036125.

V. M. Sysi-Aho, J. Saramäki, K. Kaski, *Invisible Hand Effect in an Evolutionary Minority Game Model*, Physica A **347** (2005) 639-652.

VI. M. Sysi-Aho, J. Saramäki, J. Kertész, K. Kaski, *Spatial snowdrift game with myopic agents*, The European Physical Journal B **44** (2005) 129-135.

Other related publication not included in the thesis: J. Töyli, M. Sysi-Aho, K. Kaski, *Models of asset returns: changes of pattern from high to low event frequency*, Quantitative Finance, volume 4, issue 3, (2004) 373-382.

Throughout the overview these publications are referred through the corresponding Roman numerals.

Authors contribution

The author Marko Sysi-Aho has been the main contributor to Pulications I-VI. He has initiated the research lines leading to these publications, as well as produced and analyzed all the models, data and results related to the publications. The author has also prepared the manuscripts of Publications I-VI, which then have been improved by the coauthors.

Contents

Chapter 1

Introduction

The study of *complex systems* (CS) has recently become an important area of research in natural and social sciences [1, 2]. In general, complex systems are composed of many interacting elements. Often, the elements are non-identical, and their interactions stochastic. Although the elements may follow simple rules, the properties emerging at the higher, collective level of the system itself, the so-called *emergent properties*, can show complex behaviour. Moreover, such a system often evolves towards a state that exhibits organized patterns without any form of centralized control. This phenomenon is known as *self-organization* [3]. If the elements in a complex system are adaptive, such that they can change their reactions as a response to environmental changes, the system is called a *complex adaptive system* (CAS) [4, 5].

There are numerous natural systems that can be considered as CASs, for example the human immune system [6, 7, 8], and several ecological [9], and economical [2, 10, 11, 12] systems. In the human immune system, the interacting elements are specialized cells (e. g. the B-cells). When a pathogen disturbs the homeostasis (see p. 7–12 of Ref. [13]) of our body, individual cells strive to eliminate the pathogen in order to recover the homeostasis. This procedure takes place without centralized control, i. e. , the system of cells self-organizes to defeat the pathogen. As an emergent property, one can observe that the numbers of different types of cells in the body evolve in time, and finally settle down to stable levels [6]. Economical and ecological systems are examples of large-scale systems which are composed of small sub-units. Depending on the level of coarse-graining at which one views these systems, the systems can be divided into units. For example, on a very coarse-grained level, a local ecosystem of a specific continent can represent one unit, and the units interact when some animal species, like butterflies and birds, move across continents due to seasonal changes. On a lower level of coarse-graining, species of an animal can be considered as a unit, and interactions between the units arise when the species populate a specific territory. In the simplest case, one can consider two species, which can be rivals, symbiotic partners, a host and a parasite, or a predator and a prey [14, 15]. For instance, baleen whales and Antarctic krill live in a predator–pray relationship in the Southern Ocean (see p. 400 in Ref. [14]). As an emergent property arising from interactions between these two species, one can see oscillations in their population densities. The co-existence of both species is an instance of self-organization. Also, the free market economy can be viewed as a CAS, in which the elementary units are the people who purchase goods [11, 12]. Self-organization appears such that the numbers of goods on sale are limited but at the same time there is no considerable lack of them in general, that is, the supply and demand of goods are on the average balanced, without centralized control. As an emergent property, one can observe complex patterns in the prices of goods that are traded.

Other examples of natural CASs are ant colonies which efficiently utilize their food resources and optimally organize their traffic in crowded conditions [16, 17]; the Magicicadas insects that synchronize their life-cycles to appear in prime numbers, possibly in order to hinder predators from predicting their appearance [18]; various complex networks, such as the neural network of the worm *C. elegans*, or the collaboration between film actors [19]; cooperative swarming in the bacterium *Myxococcus xanthus* [20]; or competitive interactions among viruses [21].

At present, there is no a formal definition for a CAS, but it is likely that the definition would include features we have discussed above. These features are rather ubiquitous, and thus the number of systems which can be interpreted as CAS around us is astronomical. However, the number of theoretical models that can appropriately describe these CASs is more moderate. A typical problem in modelling a real-world CAS is how to capture the interaction patterns between the elementary units of the system. It may also be problematic to include a multitude of elementary units into the model, so that the modelled system would describe the real CAS in a proper scale. Several classical approaches for modelling CASs lead to equilibrium considerations of the system under study [5]. A famous example is the Lotka-Volterra model [14] for predator–prey interactions. This model, as its more sophisticated modifications [14, 15], examine the existence of equilibrium states or limit cycles [15] in the numbers of predator and prey. In economy, a famous historical example of an equilibrium-oriented consideration is Cournot's model of duopoly [22]. Cournot's model depicts how much of a commodity two firms should produce in order for both firms to gain maximal profits, assuming that the functional connection between price of the commodity and its demand are known. Analysis of the Cournot's model reveals that there is an amount that satisfies both participants. Cournot's model is an archetypical example in conventional economics, about which Arthur states (Ref. [23] p. 108): “Conventional economics thus studies consistent patterns: patterns in behavioural equilibrium that would induce no further reaction.”

Equilibrium-oriented models are convenient for theoretical studies because they are often analytically tractable, but they have been criticized for at least two reasons [5, 9, 23]. First, assumptions about the actors, like humans in economy, tend to be unrealistic in equilibrium-oriented models. For instance, in the classical economic theory a typical assumption concerns the rationality of a human being, which is assumed to be perfect, logical and deductive [10]. Psychologists, on the other hand, have a remarkably different opinion on the human rationality, based on several experiments that have indicated that humans are only moderately good at deductive logic and in addition susceptible to emotions.[1] Second, even if a natural system would converge to equilibrium, the question of how such an equilibrium arises may remain unanswered within equilibrium-oriented models. For instance, concerning human ‘players’ in strategic decision making situations, Camerer (p. 265 in Ref. [24]) states: ”Equilibrium concepts implicitly assume that players either figure out what equilibrium to play by reasoning, follow the recommendation of a fictional outside arbiter (if that recommendation is self-enforcing), or learn or evolve toward the equilibrium”. This statement highlights the fact that equilibrium-oriented considerations often disregard the process of learning [24, 25], which is likely to be of great importance in CAS, at least if these CASs include human or animal actors. Another mechanism that is likely to play a role in CAS is evolution [26]. Like adaptation and learning, evolution may also appear to be difficult to include into models that aim to be analytically feasible. In short, there seems to be a need for modelling CAS with methods that can overcome difficulties related to several conventional, equilibrium-oriented models in various disciplines of science.

One possibility for describing CASs theoretically is to utilize agent-based models, which can be implemented by computer programs. The basic idea in these models is, quoting Srbljinović and Škunca of Croatian Ministry of Defence’s Institute (p. 2 in Ref. [27]): ”to specify the rules of behaviour of individual entities, as well as the rules of their interaction, to simulate a multitude of the individual entities using a computer model, and to explore the consequences of the specified individual-level rules on the level of population as a whole, using results of the simulation runs”. The basic structure of these agent-based models strongly resembles the structure of actual CASs. Furthermore, computerized simulations allow to consider heterogeneous agents and interactions. Thus, some unrealistic assumptions that make equilibrium-oriented models analytically tractable can be relaxed

[1]For experiments about human behaviour in several strategic situations see Ref. [24].

in computerized agent-based models. As an example, Arthur reasoned that in economic theory, the assumption of perfectly rational human agents can be replaced with a more realistic assumption of *boundedly rational* human agents [10, 11, 23].

The strengths and weaknesses of agent-based models are both related to the level of realism one wishes to attain with these models. On one hand, one can try to construct models that reproduce realistic phenomena as accurately as possible, without regard to the complexity of these models. As a drawback of such approach, it may turn out that it is difficult to understand precisely which aspects of the model are responsible for the results. On the other hand, one can study in detail the simplest possible models that capture at least some of the most basic underlying dynamics that may appear in the system under study. Such simplified models, while not necessarily accurately modelling the ‘ingredients’ of any specific system, have the virtue of being controllable and potentially understandable. Still, these simple models can describe the system under study more appropriately than conventional equilibrium-oriented models. Thus, these models can help us address the most basic issues of CASs and give us great insight not only into what types of emergent behaviour we can expect to result from various fundamental underlying dynamics, but also what the appropriate questions are that we can ask of such systems.

In this thesis we shall consider two types of agent-based models: minority game models and spatial two-player game models. These models are simple enough to be analyzed with analytic methods to some extent, but still they can be said to capture factors that are believed to influence behaviour of CASs composed of as complicated creatures as human or animal actors.

In chapter 2 we shall review minority games (MG), in which the interest is focused on self-organization in a population of agents with limited capabilities when they compete for scarce resources. Such competitive situations may arise, for instance, when predators choose turfs for hunting preys, or when routers in the Internet decide how to transfer data packets from one place to another. The minority game is a simple agent-based model reminiscent of these systems [28]. The game consists of N agents who decide between two alternatives, A or B. Those who belong to the minority, win. The agents have access to a global history, in other words to a historical record of the past M winning sides, and they are endowed with S strategies that assign a choice for each possible history. Regardless of the individual agents’ self-interested pursuit, the population of agents shows coordinated behaviour [29]. Previous studies indicate that when the agents are allowed to evolve, they tend to evolve such that the population as a whole performs optimally [30, 31]. In order to allow evolution, we have applied genetic algorithms to the MG. Our results show that natural selection and genetic algorithms are efficient methods for boosting the performance of the population as well as that of the individual agents in this toy world (Publications I–V).

In chapter 3 we shall describe spatial two-player games, and focus in particular on the so-called snowdrift game [32]. Two-player games are well-studied in the field of traditional game theory [33] that deals with rational strategic behaviour, mainly related to human behaviour or the behaviour of human societies. Later on, two-player games have also been applied to various problems in biology, due to the maturation of evolutionary aspects of game theory [15, 34]. For example, competitive interactions among viruses [21], evolution of ATP-producing pathways [35], or cooperative swarming in the bacterium *Myxococcus xanthus* [20] have been modelled with two-player games. One of the central problems in biology and socioeconomics is to understand the emergence and persistence of cooperative behaviour between unrelated individuals. Standard metaphors for investigating this problem are two particular two-player games: the so-called prisoner's dilemma game and its variant, the so-called snowdrift game [15, 22, 34]. In relation to these games, several studies have proven that spatially structured populations are in an important role in explaining the emergence and persistence of cooperative behaviour [36, 37]. In these studies, the viewpoint has largely been that of biological evolution, which is typically associated with time scales that are longer than the lifetime of an individual. In order to study whether conclusions of cooperative behaviour in spatially structured two-player games remain the same also in the case where the individuals act adaptively for short time scales, we studied a spatially structured snowdrift game in (Publication VI). Our results show that there are differences in the results obtained by considering dynamics at different time scales.

Chapter 2

Minority Games

The minority game (MG) was introduced for modelling emergent properties in a system of individuals with bounded rationality and inductive thinking. The original work of Challet and Zhang [28] was inspired by Arthur's El Farol bar problem (BP) [10, 23].

In his 1994 paper, "Inductive reasoning and bounded rationality", Arthur gave reasons for why conventional assumptions about perfectly rational and deductively thinking agents, widely adapted in economics, decision making and game theory [33], may not be appropriate for describing real human behaviour. Based on psychological experiments carried out by others, Arthur argued ([10],p. 406) that "Modern psychology tells us that as humans we are only moderately good at deductive logic, and we make only moderate use of it. ... We carry out localized deductions based on our current hypotheses and act on them. As feedback from the environment comes in, we may strengthen or weaken our beliefs ..." In order to model this type of behaviour, Arthur proposed the BP, inspired by the El Farol bar in Santa Fe, which offers Irish music on Thursday nights. In the BP, N people independently decide each week whether to go to the bar in which space is limited. The evening is enjoyable only if fewer than 60 percent of the possible N people are present, but in advance there is no sure way to tell the numbers coming to the bar, and therefore a person goes if he/she expects fewer than 60 percent to show up, or stays at home if he/she expects more than 60 percent to attend. The choices are affected only by the available information of the numbers of visitors who came to the bar in the past weeks. Of interest in this context is the dynamics of the number of people attending each week. Arthur constructed a computational model for this problem in which agents are given a certain number of predictors, in the form of functions that map the attendance figures of the few past weeks to the attendance figure of the next week. Then, each agent keeps track of the accuracy of his/her predictors and uses the one that is, by some criterion, the most accurate at the present moment. Arthur found that in this simulated BP, the mean attendance always converges to 60 percent. Furthermore, the predictors were found to *self-organize* dynamically such that 40 percent of them anticipate that more than 60 percent of the agents will attend the bar next week, and 60 percent of the predictors predict the opposite [10].

Arthur's BP model has at least two notable features. The first one is that the agents in the BP can adaptively react to changes in their environment by switching from one strategy to another. The second one is that there is no universally good predictor that could always predict the attendance better than other predictors in the game in all possible circumstances. This is due to the fact that the attendance depends fully on the predictions of the other predictors. Unfortunately, the BP model is considerably 'large' problem to be analyzed thoroughly. One reason for this is that the class of predictors is not restricted in any way, and another reason is that the information provided to the BP agents, the number of past weeks attendees, is quite fine-grained.

Later Challet and Zhang simplified Arthur's BP model such that the resulting model, the minority game, was more appropriate for detailed analysis but still captured the important features of the original BP model [28]. Since their 1997 paper, "Emergence of cooperation and organization in an evolutionary game", the MG has been a subject of intensive study, and the original model has been modified in various ways. The MG as such is an interesting model with the basic idea being associated with competition for scarce resources. The game has some potential applications in biology, technology, economics and social sciences. For instance, predators occupying turfs for hunting prey, data packets sent to a mobile network, taxi drivers choosing routes between two places, or students who aim at educating themselves into fields with high demand of labour but shortage of workers all benefit from being in the minority. The rules of the game are simple, yet the system of agents produces interesting emergent properties such as coordination and self-organization.

Several MG papers have been published during the past eight years. The aim of this chapter is to give an overview of the various studies on MGs. Our presentation will not be detailed, and the reader is encouraged to check the original references. For an extensive review of minority game research, the reader may wish to consult the MG web page [38], which contains an excellent collection of links to various papers, theses and other reviews with short descriptions of their contents. The topics and papers to be discussed in this chapter have been selected from the main lines of MG research, but the selection is necessarily somewhat subjective and some papers containing original and important ideas may have been left out. To give a continuous view of the subject, the author's and collaborators' own publications are presented in the text shortly, without highlighting them later on in a separate chapter. For a more detailed view on these publications and the author's contribution, the original publications are included at the end of this thesis.

The outline of this chapter is as follows. In section 2.1 the MG is described in its original form. In section 2.2 typical features of interest in the MG are discussed and concepts that are used to quantify these features presented. In section 2.3 the Nash Equilibria in minority games are shortly discussed and section 2.4 provides a short review of the results of the original MG, with emphasis on the quantities described in section 2.2. In section 2.5 the reasons why the results of section 2.4 emerge are pondered and in section 2.6 some works that approach the MG analytically are discussed. Sections 2.7-2.9 will focus on modifications and applications of the MG such that section 2.7 introduces some financial MG models, section 2.8 reviews evolutionary extensions to the MG, and section 2.9 discusses other modifications of the game, including multichoice MG models and games where the information available to the agents is different from the information in the basic MG. In section 2.10 a look at recent experiments that have been conducted with human players is taken. Finally, section 2.11 concludes the chapter with a critical view on the applicability of MGs to describe real-world social phenomena.

2.1 Description of the basic minority game

In the MG, N players, henceforth called agents, compete with each other and act based on induction and adaptation. At each time step of the game, each agent joins one of two groups labeled -1 (say group A) or 1 (group B). Each agent that is in the minority group at that time step is awarded one point, while each agent belonging to the majority group loses one point. At a given step an agent chooses the group to join based on the prediction of a strategy, which is defined to be the next action (to be in -1 or 1) given a specific binary signal, u_M, of length M comprising the minority groups of the previous M time steps. The parameter M denotes the length of the the memory, and a total of $P = 2^M$ bit strings, the so-called histories, can be constructed using these M bits. At the beginning of the game each agent i is randomly assigned S of the possible 2^P strategies, which shall be denoted by lower case s. The response of the sth strategy of an agent i to a history u_M is $a_{i,s}^{u_M} \in \{+1, -1\}$. Agents keep track of the performance of their strategies such that at time t, the accumulated score of the sth strategy of agent i is

$$U_{i,s}(t) = U_{i,s}(t-1) - a_{i,s}^{u_M(t)} \chi\left(A(t)\right) \tag{2.1}$$

points, where $A(t)$ is the difference between number of agents who chose 1 and -1, and $\chi(x)$ is an odd, increasing function of x. Typically, $\chi(x) = \text{sgn}(x)$ or

$\chi(x) = x$. The agents are inductive, such that at each moment of the game an agent uses the strategy which has been the most rewarding, i. e. , the strategy with the highest score. Ties among the strategies of an agent are decided by a coin toss. The strategy $s_i(t)$ that agent i actually plays at time step t is called the active strategy of that agent. Due to the fact that the agents have more than one strategy, the game is adaptive in that the agents can choose to play different strategies at different moments of the game in response to changes in their environment. The course of the game is illustrated in Fig. 2.1.

2.2 Typical observables and features of concern

The behaviour of the MG can be viewed both at the system-level and at the level of an individual agent. A basic observable quantity at the system-level is the *attendance*, the difference between the numbers of agents who choose side 1 and those who choose side -1 at time t, that is,

$$A(t) = \sum_{i=1}^{N} a_i(t), \tag{2.2}$$

where $a_i(t) = \pm 1$ indicates the actual choice of an agent i, i. e. , the prediction of his/her active strategy at time t. A simulated example of $A(t)$ in the basic MG is shown in Fig. 2.2. Alternatively, one can measure the number of agents who attend one of the two sides, say side 1, at time step t, $A_1(t)$. Note that this can sometimes lead to confusion, as both definitions have been used in MG literature. For instance, the most frequently studied characteristic in MG research, σ^2/N of Eq. (2.4), takes different values depending on the definition used (cf. , e. g. , Fig. 2.3 and Fig. 4 in [29]). One can also measure the size of the minority group at time step t and compare it to the maximal size of the minority group (see, e. g. , Eq. (1) in P4).

The most widely studied system-level property of minority games is coordination among the agents. One way to quantify the amount of coordination is to measure fluctuations of the attendance, Eq. (2.2),

$$\sigma^2 = \left\langle A^2 \right\rangle, \tag{2.3}$$

where the average is taken over time. The smaller σ^2 is, the larger the typical minority group is, and thus the choices of the agents are better coordinated. In literature, it is common to report the values of the *per capita fluctuations*

$$\sigma^2/N = \frac{\left\langle A^2 \right\rangle}{N}, \tag{2.4}$$

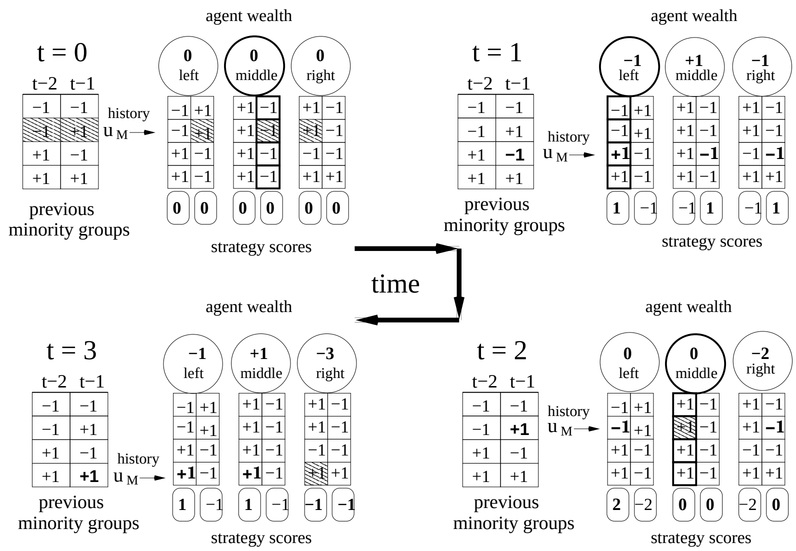

Figure 2.1: Schematic diagram to illustrate the MG. The game consists of $N = 3$ agents: left, middle and right. Each agent has two strategies that indicate an action, +1 or -1, for each possible history u_M comprising the minority groups of the previous $M = 2$ time steps. The number of such histories is $P = 2^M = 4$, and accordingly the length of the strategies of the agents is $P = 4$. At the outset of the game, the strategies of each agent are drawn at random such that each entry in a strategy is -1 or 1 with probability $1/2$. Initially, the scores of the strategies—declared in the 'legs' of the agents, the rounded boxes below the strategies of the agents— are set to zero. In subsequent time steps, the agents always use the strategy with the highest score, and in case of a tie, they flip a coin. In order to start the game, one needs to draw the first history at random. Random choices are pointed out by shading in the figure. For instance, at $t = 0$ the history and the choices of the agents between their two strategies are all drawn at random. After each agent has selected his/her strategy, the minority group can be identified, and thereby the new values for the strategy scores as well as the new history can be determined. The 'head' of the winning agent at time t, and his/her active strategy are circled with heavy lines in the figure. One point is added to (subtracted from) the scores of those strategies that had predicted the minority group correctly (incorrectly). The previous minority group in each history, the score of active strategies, and the actions determined by the active strategies are marked with bold typeface in the figure. The wealth of an agent is declared in the 'head' of the agent. The wealth of each agent is increased (decreased) by one unit, if the agent succeeds in choosing the minority (majority) group.

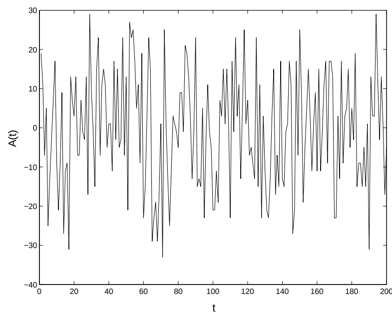

Figure 2.2: A simulated attendance time series, $A(t)$, from the basic MG with $N = 101$, $M = 5$, and $S = 3$. The attendance values fluctuate around zero.

which shows specific dependence on the control parameter $z = P/N$, a property that will become clear below. In order to state whether a system of agents shows coordinated behaviour or not, one needs a reference system for comparison. A natural choice for the reference system is the random choice game (RCG), where each agent chooses his/her action, -1 or 1, independently, randomly and with equal probability [29, 39]. The variance of $A = \sum_{i=1}^{N} a_i$, where $P(a_i = 1) = P(a_i = -1) = 1/2$, is $\sigma^2 = N$. Thus, $\sigma^2/N = 1$ is the level of per capita fluctuations in the RCG. Then if $\sigma^2/N < 1$ in a game, the game is more coordinated than the RCG, whereas in the opposite case, $\sigma^2/N > 1$, the game is less coordinated than the RCG.

Another important question at the system-level concerns the predictability of the next minority choice, that is, whether an agent can extract such information from the record of minority groups that helps him predict the next winning side. During the course of the game, it could happen that after a particular history u_M which, of course, can occur several times, the winning side would be 1 more often than -1. Thus, an agent who used a strategy with response 1 to u_M, would win in the long run.[1] In order to find out whether such predictable patterns exist, one can study statistical properties of the time series of the minority groups, G. Even though this time series might contain clear patterns, an agent cannot necessarily utilize this information if they appear in periods k that are longer than the

[1]Strictly speaking this is true only if the agent who observed such patterns would not be involved in the game, but who could profit from playing the game. An action taken by such an external agent would not contribute to the attendance $A(t)$. This assumption is crucial, because if $|A(t)| = 1$, it is enough for one agent to switch his/her action to the opposite to change the minority group sign.

length of the memory of the agent, M. Savit *et al.* [29] and Manuca *et al.* [39] presented a straightforward method to study the existence of such patterns. They suggested that one can estimate the conditional probabilities $P(1|u_k)$ of having a minority group 1 immediately following each of the binary strings u_k of length k in G. If the agents efficiently used the information accessible to them, they should be able to eliminate predictable patterns for $k \leq M$. If there are no predictable patterns, the next minority choice can not be guessed, and the conditional probabilities $P(1|u_k)$ should be about $1/2$ for each u_k. If, instead, the conditional probabilities are rugged, i. e. , they are not equal for each u_k, some histories are more likely followed by a particular winning (losing) side than others. Since the conditional probabilities are estimated for each u_k, there are 2^k numbers $P(1|u_k)$ to be estimated. For moderate values of k, this is easily done by inspecting a graph where the conditional probabilities are plotted against their corresponding history string labels, (labeled, for instance, by decimal presentation of the binary strings, see, e. g. , Fig. 3 in [29]). However, if k is large, such graphs may be inconvenient to use. Then one may wish to compress the information presented by the conditional probabilities into one number. One possibility is to use the *predictability* introduced by Challet and Marsili [40] [2]:

$$\theta = E\left[\langle A \rangle^2\right] = \sum_{u_M} P(u_M) \langle A|u_M \rangle^2 , \tag{2.5}$$

where $E[\cdot]$ denotes the expectation over the history strings u_M which occur with probability $P(u_M)$, and the sum runs over all strings. Values $\theta > 0$ indicate that an agent may succeed in predicting the next minority group, and win in the long run by continually playing $-\langle \mathrm{sgn} A|u_M \rangle$ in response to the history u_M.

At the level of individual agents, the focus of MG studies often lies on the wealth of the agents. A straightforward way to define the *agent wealth* is to follow the original idea by Challet and Zhang [28], such that the wealth of an agent at time t is defined to be the number of times the agent has won minus the number of times the agent has lost. In some financial extensions of the MG the agent wealth may be defined differently, e. g. , by allocating the initial capital of an agent between risky and riskless assets with prices evolving during the game. There are numerous possibilities for defining the capital evolution process, some of which will be discussed in section 2.7, but no further account will be presented in this thesis. The reader may wish to consult Refs. [44, 45, 46, 47, 48, 49] and references therein for more details.

It may also be of interest how similarly the agents in the game act. Since the strategies determine the actions of each agent, the question basically concerns

[2] For other choices that convey the same idea as Eq. (2.5) see, e. g. , Refs. [39, 41], or for more complicated measures Refs. [42, 43].

the similarity between the strategies of the agents. In order to quantify similarity between strategies which are represented as binary strings, it is customary to use the Hamming distance (HD). The normalized HD between a pair of strategies (s_i, s_j) of the same length P is the proportion of bits that are different at the corresponding positions in the two strategies:

$$HD(s_i, s_j) = \frac{1}{2P} \sum_{k=1}^{P} |s_i(k) - s_j(k)|. \tag{2.6}$$

The larger the HD is, the further the two strategies are from each other. The strategy $\bar{s}$ whose every bit is different from the bits of a given strategy s is said to be anticorrelated to s, $HD(\bar{s}, s) = 1$, and if $HD(s_i, s_j) = 1/2$, the strategies s_i and s_j are said to be uncorrelated with each other. In practice, the HD is often used to measure differences between the strategies of a single agent, in which case we will call it the intra-agent HD, or between the active strategies of two different agents at a given time step, in which case we will call it the inter-agent HD.

Sometimes it is useful to select strategies for the agents from a reduced set of strategies (RSS) [41]. A RSS consists of uncorrelated and anticorrelated strategies, that is, for each strategy pair (s_i, s_j) in a particular RSS the HD is $1/2$ or 1. Such strategies are kept markedly different from each other. The use of a RSS makes it possible to analyze some characteristics of the MG with ease. For instance, if the strategies of agents are drawn from a RSS it is easy to understand why fluctuations of Eq. (2.3) are of the certain order for different values of the control parameter z, as will be discussed in section 2.5. Furthermore, the number of strategies in a RSS is only $2P$, as opposed to the 2^P strategies in the full strategy space.

There are also other quantities that are useful for characterizing the behaviour of an MG [40, 50], but in this thesis we will mainly confine ourselves to the quantities described above. For instance, Challet and Marsili [40] have studied the autocorrelation of the minority group choices conditional to the histories. This measure gives valuable information about the persistence of the actions of agents, indicating whether the agents tend to repeat their last actions [3]. In addition, they have studied a quantity ϕ, which tells how big fraction of the strategies of an agent population is not used.

2.3 Nash equilibria

Generally, in strategic games like the MG, the agents have a number of strategies from which they can choose one at any time step. If an agent always uses a uniquely

[3] $c(t, t+\tau) = E[\langle \mathrm{sgn} A(t) \mathrm{sgn} A(t+\tau) | u_M \rangle]$, where $E[\cdot]$ denotes the expectation over the histories u_M and $\langle \cdot | u_M \rangle$ denotes the expectation over time conditional to $u_M(t) = u_M(t+\tau) = k$.

determinded strategy in response to a particular state of his/her environment, the agent is said to play pure strategies. If, instead, for each state of the game, an agent may probabilistically choose among several strategies, the agent is said to play mixed strategies. In the context of games, a pertinent concept is that of the Nash equilibrium (NE). The Nash equilibria can be said to be those states which are stable under the payoff incentives, given the choices available to the agents (for the formal definition see, e. g. , Ref. [22], p. 8, or Ref. [51], p. 481). This means that the predicted action of an agent must be the best response of the agent to the predicted actions of the other agents. If such a state is obtained, no agent will have an incentive to deviate from that state.

A good description of NE in minority games is given by Marsili *et al.* in Ref. [52], or by Marsili and Challet in Ref. [47]. In order to illustrate NE in an MG, let us restrict our attention to a stage-game[4] and one history, $P = 1$. On one hand, if the agents play mixed strategies, the MG has a unique NE in which each agent chooses his/her action, 1 or -1, with equal probability $1/2$. The agents in the random choice game described in section 2.2 play such mixed strategy NE, and in that case $\sigma^2 = N$. On the other hand, if the agents play pure strategies the MG has multiple NE. These are the states for which $|A| = 1$, and the number of these states is the number of ways $(N-1)/2$ agents out of N can choose one side, and the rest $(N+1)/2$ agents the other side. If the MG is in a pure strategy NE, the fluctuations are minimal, $\sigma^2 = 1$.

2.4 Results from the basic minority game

The most prominent feature of the basic MG is its behavioural dependence on the control parameter $z = P/N$. There are three regimes, low, intermediate and high values of z, where the game clearly shows different behaviour. In this section we describe results from the basic MG, trying to illuminate how the characteristics like fluctuations, predictability, and agent wealth introduced in section 2.2 depend on z.

2.4.1 Fluctuations

Savit *et al.* [29] were the first who pointed out that for a fixed number of strategies per agent S, the per capita fluctuations, σ^2/N, show universal behaviour as a function of the control parameter z. They simulated the MG for a fixed number of strategies per agent[5], S, by varying the values of M and N, and observed that

[4] A stage-game is a game that is played only once, not iteratively.

[5] They reported the case $S = 2$.

σ^2/N as a function of z always followed the same curve. It turned out that for small values of z, the level of coordination among the agents is low, the fluctuations are large, $\sigma^2 \sim N^2$, which is clearly higher than the RCG (random choice game, see section 2.2) limit $\sigma^2 = N$. On the other hand, for large z the fluctuations were found to be of the same order as the fluctuations in the RCG, $\sigma^2 \sim N$. Furthermore, it was noted that transition between these two behaviours of σ^2 occurs in the intermediate-z region wherein the best emergent coordination is achieved, by σ^2/N having the minimum value at $z = z_c$. Values of σ^2/N at around z_c are notably lower than the RCG limit, $\sigma^2/N = 1$, and approach this limit as z grows large. The behaviour of σ^2/N as a function of z is illustrated in Fig. 2.3 for several values of S.

Numerical results show that if the agents in a basic MG use strategies that are in a RSS (reduced set of strategies, see section 2.2), the behaviour of σ^2/N does not qualitatively change as compared to the situation where the agents use strategies from the full strategy space [41, 50]. The qualitative behaviour of σ^2 is also otherwise robust. For instance, the minimum of σ^2/N at some z, which need not to be $z = z_c$ where the minimum occurs in the basic MG for a fixed S, persists under various circumstances: the real history that comprises the previous M winning groups can be replaced with a random signal [53], the agents can use different types of payoff functions $\chi(x)$ of Eq. (2.1) when they update the scores of their strategies [54], the information (history) to the agents need not be the same for all the agents [55, 56], the interactions among the agents may change [57], or the strategies of the agents may evolve over time [30, 31].

2.4.2 Predictability

Regarding the predictability of the next minority group, Savit *et al.* [29] and Manuca *et al.* [39] found that in the low-z region the conditional probabilities $P(1|u_M)$ are about $1/2$, independent of u_M, and fluctuate inversely proportionally to the simulation time.[6] Thus, in the low-z region, the agents can not extract predictive information from the time series of the minority group, G. However, G is not a random sequence with no regularities. There is information in G that can be detected by studying histories of length larger than M, but this information cannot be detected by the agents with memory length M. By contrast, in the high-z region, the probabilities $P(1|u_M)$ appear to be rugged, indicating that some histories are more likely followed by a particular winning (losing) side than others. Because of this asymmetry there is significant information available to the agents, and some agents are more likely to win than others. The transition from even probabilities

[6]It is notable that for the RCG the conditional probabilities $P(1|u_M)$ would be also about $1/2$, but fluctuations would be larger, inversely proportional to the square root of the simulation time.

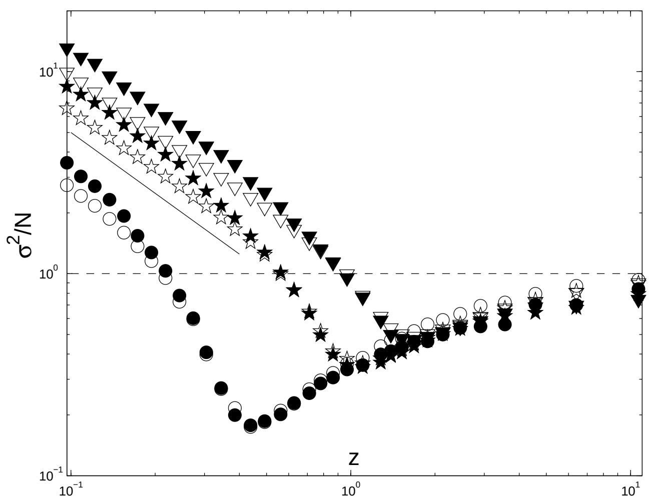

Figure 2.3: Per capita fluctuations σ^2/N versus the control parameter $z = P/N$ ($P = 2^M$) for $S = 2$ (circles), $S = 3$ (stars), and $S = 4$ (triangles) obtained from numerical simulations of the MG. We have used $M = 5$, and let the simulation run for a warm-up period of $t = 200P = 6400$ time steps, after which the σ^2/N value was calculated using the subsequent $t = 300P = 9600$ time steps. The symbols in the figure represent average values of σ^2/N over 200 sample runs. The solid symbols correspond to the basic MG simulation with real histories u_M, i. e. , histories that comprise the previous M minority groups, whereas the open symbols correspond to simulations with random histories, i. e, , histories that are drawn from a uniform distribution of numbers $\{1, \ldots, P\}$. The straight line depicts the rate of decay of σ^2/N in the low-z region, and it is $\propto 1/z$. The dashed line, $\sigma^2/N = 1$, depicts the level of fluctuations in the RCG. In general, the fluctuations are high in the low-z region, smallest in the intermediate-z region, and approach the RCG limit in the high-z region. The behaviour of σ^2/N as a function of z does not seem to be sensitive to replacing real histories with random ones. For each S, there is a $z = z_c$ at which the curve obtains a minimum value. When S is increased, z_c increases, as does the minimum value of σ^2/N. For large S the shape of the minimum around z_c smooths out such that after the initial decay $\propto 1/z$, the curves directly approach the RCG limit (see Fig. (2.6) for $S = 5$ and $S = 21$). Due to the universal behaviour of σ^2/N versus z, the results do not depend on the value of M.

to rugged ones occurs in the neighbourhood of $z \sim z_c$, i. e. ,roughly in the same region where the coordination among the agents is the highest. This point was further clarified by Challet and Marsili in Ref. [40]. They demonstrated that the predictability, θ of Eq. (2.5), starts to deviate from zero after z exceeds a critical value, z_c. This behaviour of θ as a function of the control parameter z is illustrated in Fig. 2.4 for several values of S.

2.4.3 Agent wealth

The evolution of the wealth of agents in the basic MG is quite monotonic. Challet and Zhang [28] observed that the gap between the rich and the poor agents appears to increase linearly with time. Manuca *et al.* [39] analyzed the wealth of agents separately in the low-z region and in the high-z region. They found that in the low-z region the agents tend to be the wealthier the smaller their intra-agent HD (Hamming distance, see section 2.2) is, without regard to the bits of those strategies, or to the bits of the strategies of the other agents. By contrast, in the high-z region the intra-agent HD did not seem to correlate with the agent wealth. Instead, it turned out that the agents whose inter-agent HD is large, that is, their strategies are maximally distant from all other strategies, gain wealth fastest. Thus, in the high-z region the success of the strategy of an agent is highly dependent on the other strategies in the game.

2.4.4 Nash equilibria

Generally, the dynamics of the basic MG does not lead the game into a Nash equilibrium. One could think that in the high-z region the agents might play the mixed strategy NE, in which each agents chooses 1 or -1 with probability $1/2$, since the fluctuations, Eq. (2.3), are close to the RCG limit. However, as Manuca *et al.* argue (Ref. [39], p. 605): "Although the agents do choose between their strategies randomly, they do not choose the minority groups randomly. It would be possible, therefore, for some agents to choose some other deterministic ordering of choices for his strategies and increase his wealth. That this is possible, in principle, follows from the fact that G is not an IID sequence." The basic MG does not either converge to any pure strategy NEa, which would be characterized by minimal fluctuations, $\sigma^2 = 1$. The situation may dramatically change if the basic MG is slightly modified, for instance, by introducing agents that can take into account their impact on the attendance, Eq. (2.2), when they update the scores of their strategies [58]. For the time being we will postpone the discussion of such modifications until section 2.5.4.

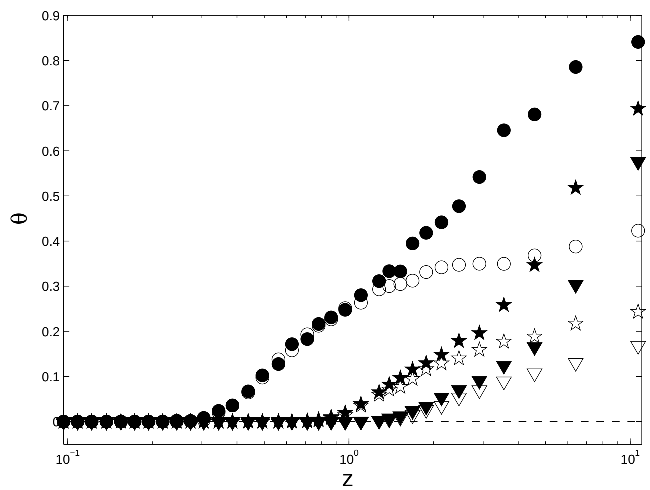

Figure 2.4: The predictability θ versus the control parameter $z = P/N$ ($P = 2^M$) for $S = 2$ (circles), $S = 3$ (stars), and $S = 4$ (triangles) obtained from numerical simulations of the MG. We have used $M = 5$, and let the simulation run for a warm-up period of $t = 200P = 6400$ time steps, after which the value of θ was calculated based on the subsequent $t = 300P = 9600$ time steps. The symbols in the figure represent the average values of θ over 200 sample runs. The solid symbols correspond to simulations with real histories u_M, i. e. , histories that comprise the previous M minority groups, whereas the open symbols correspond to simulations with random histories, i. e. , histories that are drawn from a uniform distribution of numbers $\{1, \ldots, P\}$. The dashed line indicates the zero level. Behaviour of θ as a function of z is quantitatively sensitive to replacement of real histories with random signals. However, in both cases, $\theta = 0$ in the low-z region, indicating that the agents in the game can not predict the next minority group. After a critical value, z_c, which increases with S, θ starts to deviate from zero, and appears to grow monotonically as a function of z. Thus, in the high-z region the outcome of the game is not symmetric, but some histories are more likely followed by a particular winning side than others. The behaviour of θ as a function of z for a fixed S is universal, as it does not depend on the value of M (see [40]).

2.5 Understanding the results

The above results indicate that both the system-level and the individual agent level behaviour depends on the control parameter z. The generic pattern is as follows: in the low-z region a characteristic quantity of the game behaves in one way and in the high-z region the same quantity behaves in another way, the transition between the two behaviours taking place in the intermediate-z region. In this section we will review some explanations that aim at providing understanding to these results.

2.5.1 Fluctuations

Savit *et al.* proposed that the large fluctuations in the low-z region, $\sigma^2 \sim N^2$, are due to the so-called period-2 dynamics [29]. A detailed description of this dynamics is provided by Manuca *et al.* in Ref. [39]. Basically, the period-2 dynamics is related to how the scores of the agents' strategies evolve over time, and thus which actions the agents take at consecutive time steps. In brief, the agents' responses to an odd occurrence of a given history look random and cause, within statistical fluctuations, about half of the agents to choose one side.[7] In contrast, the next even occurrences of the same history is followed by a large number of agents choosing just the opposite side to that which happened to be the winning side at the last odd occurrence of the same history. Accordingly, the fluctuations grow large, $\sigma^2 \sim N^2$. The agents' strategies tend to be similar due to the fact that for low $z = P/N$, a large number of agents N share only a few, 2^P, strategies. Numerical results reported by Challet and Marsili in Ref. [40] indicate that in the low-z region the autocorrelation function of the minority groups conditional to the histories is periodic with period $2P$. Thereby, the agents tend to switch their responses to a particular history with period P, a result that is in harmony with the explanation of the period-2 dynamics.

The crowd-anticrowd theory (CAT) of Johnson *et al.* provides another appealing explanation for the behaviour of σ^2 [59, 60, 61]. The basic idea of the CAT is to investigate how many agents are using the same, or similar, strategies at a time. The subset of agents n_i using a particular strategy, say s_i, will all act in the same way and thus constitute a crowd. However, at the same time there may be a number of agents n_i^* who are using the opposite, or at least very dissimilar, strategies to the subset n_i. This second group constitutes the anticrowd. Then, by utilizing random-walk analysis with step size $N_i = n_i - n_i^*$ one can approximate the order of σ^2 for different z. The actual analysis of CAT utilizes the RSS. This is conve-

[7] Odd (even) occurrence of a strategy refers to the numbers of times the strategy has appeared during the game. For example, the odd (even) occurrence of strategy u_M means that the strategy appears the 1st (2nd), 3rd (4th), 5th (6th), etc. time.

nient for the reason that when the strategies in the RSS are ranked as best, second best, third best, etc. , according to their score, the rth strategy is anticorrelated with the $(2P - r + 1)$th strategy. This makes it easier to find out the probabilities with which the rth strategy and the strategy anticorrelated to it are played in the game, and thereby to estimate how many agents play similarly. In the low-z region where M is small and N is large, for most values of S each agent carries a considerable fraction of all possible strategies. In this region there are practically no anticrowds, and the crowds dominate. Therefore, N_i is large, yielding $\sigma^2 \sim N^2$ [59].

In the high-z region the fluctuations are close to the RCG limit, $\sigma^2 \sim N$. Savit *et al.* [29] and Manuca *et al.* [39] argued that such fluctuations emerge as a consequence of the agents' decreasing capability to coordinate their choices. Difficulties to coordinate are due to the large number of possible histories, P, and the large number of possible strategies in the strategy space, 2^P. In terms of the CAT, when the strategy space is very large the agents will have a low chance of holding the same strategy. Moreover, the probability that an agent's best strategy is anticorrelated to another agent's best strategy is small. Accordingly, the crowds and anticrowds effectively disappear and the agents act independently, yielding $\sigma^2 \sim N$.

In the intermediate-z region σ^2/N is the smallest. The CAT suggests that the crowds and the anticrowds are of the same order in this region, and thus their effects tend to cancel out.

2.5.2 Predictability

The period-2 dynamics of Savit *et al.* also explains why there is no predictable information available to the agents in the low-z region [29, 39]. As mentioned in section 2.4, the conditional probabilities $P(1|u_k)$ lie around $1/2$, and the predictability $\theta = 0$ in this region. These results emerge because the agents tend to participate opposite groups in the even and odd occurrences of a particular history. Thus, the level of participation in 1 and -1 is quite regular, and thereby $P(1|u_k) = 1/2$, with variance that is even smaller than the variance of these probabilities for IID random series of binary numbers.

Challet *et al.* explained the behaviour of θ by describing the MG as a spin-glass model [40, 58]. Their method of inference is sketched in sections 2.5.4 and 2.6 below. In regard to the predictability, an important result is that the dynamics of the basic MG tends to minimize θ, which, however, is *a priori* biased by the term Ω^{u_M} that appears in Eq. (2.11) (see section 2.5.4). In the low-z region there are enough agents to overcome this bias term and force the predictability to zero, whereas in the high-z region there are not enough agents to do this. In the intermediate-z region the compensating effect of the agents on the bias term is of the order of the bias term.

2.5.3 Agent wealth

Manuca *et al.* provided a good explanation for the observed dependence of agent wealth on the intra-agent HD and on the inter-agent HD [39]. In the low-z region there is no predictive information available to the agents, and consequently, no emergent coordination among the agents' choices. Therefore, any attempt to use the apparent information encoded in the histories leads to ill-adaptive herding behaviour. Agents do best when they ignore the information in the histories and do not adapt to the changing environment. This effectively happens when the strategies of an agent are as similar as possible, since in that case, the choices made by the agent practically make no difference. On the other hand, in the high-z region the agents can make use of the available information. In this case, the agents possessing strategies that allow them to behave maximally differently from the other agents will win more frequently. This is the reason of the positive correlation between an agent's wealth and the intra-agent HD.

Recent results reported by Yip *et al.* support the above explanations [62]. They found that the wealth of individual agents, and also the system performance, can be enhanced in the low-z region, if the agents are allowed to participate the game in a random fashion. The reason for this is that randomly participating agents can avoid ill-adaptive herding.

2.5.4 Nash equilibria

In order to understand why the basic MG does not converge to NEa one needs to study the dynamics of the game. In particular, the following questions should be addressed: what is the long term outcome of the game and how does it depend on the agents' ability (or disability) to take into account their own impact on the attendance, Eq. (2.2). Several authors have been working on these questions, for instance, Marsili *et al.* [47, 63], Challet *et al.* [58, 64], and De Martino and Marsili [65]. Their reasoning is build on similarity between the MG model and the spin glass model [66], and their works are based on game-theoretic [47, 63, 64] and statistical mechanics [58, 65] approaches. Here, we will not review these works in detail, but will limit the discussion to three central concepts underlying each of these works: 1) sophisticated agents, 2) the probabilistic strategy selection rule, and 3) continuous time and state variables.

1. *Sophisticated agents*. Agents in the basic MG are naive in that they are not able to take into account the impact of their own action on the attendance

$A(t)$ when they update scores of their strategies. In other words, an agent i does not understand that if he/she had played the action $a_{i,s'}^{u_M}(t)$ of his/her strategy s' instead of the action $a_i(t)$ of his/her active strategy $s_i(t)$ as a response to the history u_M, the attendance $A(t)$ of Eq. (2.2) would have changed in case $a_{i,s'}^{u_M}(t) \neq a_i(t)$. Consequently, the agents underestimate the success of their active strategies as compared to the success of their other strategies. Thus, it would be useful to modify the strategy update rule of Eq. (2.1) such that the inactive stragies of an agent are penalized in order to take into account the fact that the winning group, or the amount of points gained by each agent at any time step, could have changed if the agent had played one of his/her inactive strategies instead of his/her active strategy. For that purpose, one can introduce a parameter $\eta \in \mathbb{R}$, which regulates how strongly all but the active strategy of an agent are penalized, such that the scores of agent i's strategies are updated according to the map

$$U_{i,s}(t+1) = U_{i,s}(t) - \frac{1}{P} a_{i,s}^{u_M(t)} \chi \left(A(t) - \eta(a_{i,s_i(t)}^{u_M(t)} - a_{i,s}^{u_M(t)}) \right) . \quad (2.7)$$

If $\eta = 0$, one recovers the basic MG, and if $\eta = 1$ each agent updates the score of his/her strategy s as if he/she had actually played s *and* as if all the other agents had kept their strategies fixed. If $\eta > 0$, the agents are said to be sophisticated, and due to that the behaviour of the MG changes dramatically. It is also typical, for mathematical convenience, to choose $\chi(x) = x$ instead of $\chi(x) = \text{sgn}(x)$. This is a minor change and it does not alter the qualitative behaviour of the MG, as is proven in several studies (see, e. g. , Refs. [50, 54, 58]).

2. *The probabilistic strategy selection rule*. In order to analyze the long time dynamics of the MG, it is useful to get rid of the discrete characteristics of the game, such as the 'always play the best strategy' rule. Instead, one can introduce a probabilistic strategy selection rule that favours well performing strategies. One possibility is to impose that agent i adopts his/her strategy s with probability

$$P_i(s) = \frac{\exp[\Gamma U_{i,s}]}{\exp[\sum_{s'} \Gamma U_{i,s'}]}, \quad (2.8)$$

where $\Gamma > 0$ is the learning rate ('the inverse temperature') that regulates how strongly the good strategies are favoured over the bad ones. For example, when $\Gamma = 0$ the strategies are played randomly, and when $\Gamma \to \infty$ one recovers the original 'always play the best strategy' rule. Intermediate values of Γ interpolate between these two extremes.

This rule was first time applied to the MG by Cavagna, whose work [67] will be discussed in section 2.6 below. In papers that map the MG to spin glasses, it is common that the treatment is limited to $S = 2$ strategies, and these strategies are marked with signs $S = \pm 1$. Note that this is an indexing convention, and it does not refer to agent's actions which are denoted by $a_{i,s}^{u_M} = a_{i,\pm 1}^{u_M}$.

3. *Continuous state and time variables*. For mathematical convenience, it is also customary to introduce the continuous time variable $t^* = t/P$, when $P \to \infty$, such that one can consider a simple differential equation instead of the discrete time equation for updating state variables of the MG. At time t the state of the basic MG is determined if one knows each agent's active strategy $s_i(t)$, $i = 1, \dots, N$, and the history $u_M(t)$. Similarly, when one wants to know at what state the game is in the long time stationary limit, one needs to know what strategy each agent tends to play. For that, one can reduce the actually played strategies, $s_i(t) = \pm 1$ for $i = 1, \dots, N$, to their time averages
$$m_i = \langle s_i \rangle, \quad i = 1, \dots, N, \tag{2.9}$$
and consider the dynamics of these continuous 'strategy index' variables.

Moreover, it is convenient to introduce the auxiliary variables [40]

$$\omega_i^{u_M} = \frac{a_+^{u_M} + a_-^{u_M}}{2}, \quad \text{and} \quad \xi_i^{u_M} = \frac{a_+^{u_M} - a_-^{u_M}}{2}. \tag{2.10}$$

With these notations the attendance of Eq. (2.2) reads

$$A(t) = \Omega^{u_M} + \sum_{i=1}^{N} \xi_i^{u_M(t)} s_i(t), \tag{2.11}$$

where $\Omega^{u_M} = \sum_{i=1}^{N} \omega_i^{u_M}$. In the end, it turns out that the stationary states of the MG correspond to the minima of the function

$$\begin{aligned} H_\eta \quad = \quad & \sum_{i \neq j}^{1,N} E[\xi_i^{u_M} \xi_j^{u_M}] m_i m_j + 2 \sum_{i=1}^{N} E[\Omega^{u_M} \xi_i^{u_M}] m_i \\ & + \eta \sum_{i=1}^{N} E[(\xi_i^{u_M})^2](1 - m_i^2) + E[(\Omega^{u_M})^2], \end{aligned} \tag{2.12}$$

where $E[\cdot]$ means expectation over the different histories. For details, the reader may wish to check, e. g. , Refs. [58, 65].

Thus, the basic MG case, $\eta = 0$, corresponds to minimizing $H_0 = E[\langle A \rangle^2]$, which is just the predictability θ of Eq. (2.5). By contrast, if one were to obtain NEa with the MG dynamics, the minimized quantity should be σ^2 of Eq. (2.3), which in turn corresponds to setting $\eta = 1$ and minimizing $H_1 = E[\langle A^2 \rangle]$. In other words, the basic MG does not converge to NEa because the dynamics minimizes the predictability instead of σ^2. However, the latter is minimized if the agents can take into account their own impact on the attendance as shown by the form of H_1.

2.6 Analytic approaches

Analytic treatment of the MG has proceeded progressively. The crowd-anticrowd theory by Johnson *et al.* [68] provided an explanation for behaviour of σ^2 as a function of z. The geometric viewpoint of Challet *et al.* that utilizes the RSS gave also insight into the behaviour of σ^2 [41, 50], and D'hulst and Rodgers [69] studied the success rates of agents and their strategies by applying probabilistic arguments to the inter-agent Hamming distances. Manuca *et al.* [39], in turn, used a mean-field description to depict the game behaviour in the low-z region. Nowadays, the analytic understanding of the basic MG, with slight modifications like the ones mentioned in section 2.5.4, is rather solid. In this section we shall briefly review some important works that have promoted this understanding.

One of the major obstacles to an analytic study of the basic MG is the presence of an explicit time feedback via the memory M. One way to overcome this problem was provided by Cavagna in 1999 [53]. He suggested that instead of using the signal that emerges endogenously from the agents' responses to the last M outcomes of the game, one could use an exogenous signal, drawn from a uniform distribution, that represents a piece of common information to the agents. Based on numerical studies, Cavagna reasoned that (Ref. [53],p. R3785) "in order to obtain all the crucial features of the minority game, the presence of an individual memory of the agents is irrelevant". By memory he meant that the parameter M was irrelevant, and only the dimensionality of the strategy space $P = 2^M$ was meaningful. As shown in Fig. (2.3), at least the behaviour of σ^2/N as a function of z is similar both by using real histories and a random signal. Later, however, it turned out that this statement was specious. In a reply to Cavagna's comment [70] on their earlier paper [29], Savit pointed out [71] that even though the MG showed the same behaviour both by using endogenous and exogenous information, the memory is not irrelevant, since in both cases the agents have memory which is reflected in the relative rankings of their strategies that give rise to the observed behaviour of fluctuations σ^2. Furthermore, using De Bruijn graphs, Challet and Marsili proved that the memory plays a significant role in the minority game [72]. This is especially true if the basic minority game is modified, e. g. , in such a way that the agents can take into account their own impact [52, 64], $\eta > 0$ in Eq. (2.12). Also, the behaviour of θ in Fig. (2.4) seems to depend on whether the information provided for the agents is the real history or a randomly generated sequence of events. Despite of some deficiencies, it is helpful to use a randomly generated sequence of events instead of the real history when studying minority games analytically.

In 1999 Cavagna *et al.* introduced the so-called thermal minority game (TMG) model that is a continuous version of the basic MG [67]. The TMG differs from the basic MG in the following three aspects: i) The real history u_M is replaced with a randomly drawn signal, $\vec{\eta}(t)$, from a continuous uniform distribution. The signal lies in $\mathbb{R}^P$ and its norm is set to be one. ii) The deterministic 'always use the best strategy' rule has been replaced with a probabilistic strategy selection rule. The probabilities are determined according to Eq. (2.8). iii) The agents' P-dimensional binary string strategies are replaced with continuous real-valued strategies $\vec{R}$ of length $\sqrt{P}$ in $\mathbb{R}^P$. An agent's response to a given history is then defined to be the scalar product of the strategy and the information, $b(\vec{R}) = \vec{R} \cdot \vec{\eta}(t)$. The sum of responses, $\widetilde{A} = \sum_{i=1}^{N} b_i(t)$ over the agents i is analogous to the attendance, Eq. (2.2).

The TMG produces qualitatively similar behaviour of σ^2/N versus z as does the basic MG. Interestingly, however, the level of σ^2/N falls below the levels obtained by the basic MG or by the RCG within a certain interval of the parameter Γ, i. e. , the level of coordination is higher. This property was further studied by Jefferies *et al.* [73] and Hart *et al.* [74]. Jefferies *et al.* analyzed how the fluctuations, σ^2, depend on the fraction of players who play the TMG, while the others play the basic MG. On one hand, they found that if the fraction of TMG agents is below a certain threshold value, σ^2 is larger than the RCG limit, $\sigma^2 = N$, regardless of the inverse temperature Γ. On the other hand, if the fraction of TMG agents increases, or if the probability by which the TMG agents play other than their best strategy increases, σ^2 decreases. Hart *et al.* [74] reasoned that the improved coordination is due to the probabilistic strategy selection rule, which results in a cancellation between the actions of a crowd, with agents acting collectively and making the same decision, and those of an anticrowd, with agents acting collectively by making the opposite decision to that of the crowd. So, the uncertainty in strategy selection partially smooths out the harmful crowding effect (see section 2.5). Moreover, Garrahan *et al.* [75] have provided a rather theoretical study of continuous time dynamics of the TMG. Their description is based on the formalism introduced by Challet and Marsili in Ref. [40] and later adopted in many works, see Refs. [47, 52, 58, 64, 65].

Another analytic approach to solve the MG was presented by Challet *et al.* in Ref. [58] where they showed that the stationary state of the MG is described by the ground state of a spin model which can be solved analytically. They analyzed the MG by means of the concepts described in section 2.5.4, and they utilized the following three assumptions: i) a random sequence of events is drawn from a uniform distribution instead of using the real histories u_M, ii) the agents adopt strategies according to the probabilities of Eq. (2.8), and iii) the time is rescaled to $t^* = t/P$, so that when P is large, the model is continuous in time. For analyzing the long time dynamics of the game, they also used the continuous variables of Eq. (2.9) that tell what strategies the agents play on average in the stationary state. One of their important results, which is the basis of their analytic solution, was to show that the dynamics of the MG minimizes the function H_η of Eq. (2.12). Their actual analytic solution deals with solving the minima of H_η by using replica method [66]. For more details, the reader may wish to consult Refs. [47, 52, 58, 64, 65, 66]. In regard to the success of replica method, the results of Challet *et al.* indicate that the minima of H_η obtained with the replica method are in good agreement with the minima obtained by numerical minimization of H_η. This work is central to the MG research, providing answers on how the agents tend to play in the long run, why the basic MG does not converge to NEa, and why the subsequent actions of the agents are negatively correlated in the low-z region, or positively correlated in the high-z region, and only short-time correlated in the intermediate-z region. Their work reveals that in the basic MG with $\eta = 0$ the agents minimize predictability of Eq. (2.5), and for $\eta > 0$ the agents start to growing order minimize the fluctuations σ^2.

Other analytic solutions to the MG include those of the Heimel and Coolen [76] and Jefferies *et al.* [77]. Heimel and Coolen applied generating functional analysis in their work and they assumed that the histories are randomly generated. Their results consist of calculating the location of the phase transition point z_c, and solving the game for $z > z_c$ exactly for large N. Furthermore, they demonstrate that for $z < z_c$ the stationary state of the system is not unique, and that depending on the initial scores of the strategies the fluctuations σ^2 can either grow large or disappear totally as $z \to 0$. Jefferies *et al.* focused on the microscopic dynamical properties, as opposed to global statistics, of the game. They studied, for instance, the dynamics of the strategy scores and the dynamics of the real histories. They showed that the MG can be viewed as a stochastically disturbed deterministic system, and that this deterministic system can be described concisely by coupled mapping equations.

Currently known analytic methods to solve the MG may fail if rules of the game are modified. As an example, recently Challet *et al.* considered the basic MG with a simple change: scores of the strategies have a finite memory, as opposed to the infinite memory that accumulates points from the beginning of the game in the basic MG model [78]. Introducing the finite memory for the strategies is reasonable since in reality individuals tend to forget, or at least the memory of happenings may fade slowly, as time goes on. Moreover, recent information about previous minority groups might tell more about the current course of the game to the MG agents than information on minority groups far in the past. If this is the case, it is reasonable to put more weight on the contribution of recent information when deciding on the next predicted minority group. Such weighting can be implemented, e. g. , by multiplying the strategy scores of an agent up to the previous time step $t-1$, the $U(t-1)$ in Eq. (2.1), by a 'discount' factor $(1-\lambda/P)$, and admitting points to his/her strategies in usual fashion. Interestingly, this simple modification to the MG rules has induced complications to the analytic treatment of the game insomuch that Challet *et al.* concluded that ([78],p. 149) "all the analytical tools used so far to study minority games fail when $\lambda > 0$". Thus, there is still need for general and robust analytic methods to solve the MG.

2.7 Minority game and financial markets

In economics, the price of a good is typically explained by its demand and supply [12]. Usually, increase (decrease) in demand or decrease (increase) in supply leads to higher (lower) prices. Thus, in simplistic terms, one is better off buying (selling) when most of the others want to sell (buy), so it pays off to be in the minority. This feature is readily encoded in the MG, if the choices of the agents are interpreted such that +1 (-1) means selling (buying) one unit of a commodity whose price is related to the attendance, Eq. (2.2), which depicts the difference in the selling and buying. This is the analogy that metaphorically sets minority games into financial or economic context. Typically, financial MG models include one or all of the following features: i) a price forming mechanism, ii) possibility for agents not to play the game in rounds that appear unfavourable for them[8], and iii) several types of agents with different characteristics are involved in the game. The aim of a financial MG model is to provide answers to questions like what the relationship between different types of agents in the market is, or does the price time series produced with the model resemble real-world financial price time series. One can speculate with answers to the former question, because even though one could draw conclusions about the relationship between different types of agents in the model, there may not be adequate information available from the real-world to validate one's conclusions. However, one can easily investigate the second question, because financial price time series, or their transformations, the return time series[9], are easily available. Moreover, it is known that real-world

[8] A MG with this property is called a grand canonical MG.

[9] Typically, the return-series r_t is derived from the price-series p_t with one of the following ways: 1) $r_t = p_t - p_{t-1}$, 2) $r_t = (p_t - p_{t-1})/p_{t-1}$, or 3) $r_t = \log(p_t) - \log(p_{t-1})$.

financial time series show widely observed empirical regularities called stylized facts [79]. The most prominent of these regularities are 'fat-tailed' return distributions, long-ranged volatility autocorrelations, and clusters in volatility and trading volume. It is desirable for a financial MG model, and of course for any other financial model, that the price time series generated by the model would include as many of these empirical properties as possible.

2.7.1 Symbiosis between producers and speculators

Slanina and Zhang were the first who applied an MG for modelling an economic system [44]. Their model depicts capital flow and price fluctuations in a society of N agents, divided into N_p producers and N_s speculators. By the rules of their game, the capital flows to the producers and from them to the speculators. At each time step, some amount of stock is traded. The stock price, $x(t)$, is the output signal of the market. Each player is characterized by two dynamical variables, the amount of stock $S_i(t)$ and the amount of money $B_i(t)$, where index i denotes the player. So, the total capital owned at time t by the ith player is $W_i(t) = B_i(t) + x(t)S_i(t)$. The producers follow a fixed strategy of buying and selling, irrespective of the current or the past price. Each producer has his/her own randomly chosen period τ_i and time scale T_i during which he/she invests. At the outset of the game, the investment of producer i, $a_i(t)$, is drawn from a uniform distribution in the interval $[-1, 1]$ with the restriction $\sum_{t=0}^{\tau_i - 1} a_i(t) = 0$, that is, the investment is balanced over i's own investment period τ_i. The producers participate in the game only if they have positive capital. In that case, they attempt to buy a certain number of stocks. The actual number they are willing to buy depends on $a_i(t)$, τ_i, T_i, $x(t)$, and $\langle W \rangle$, i. e. , the average wealth of the players. As opposed to the producers, the speculators are able to analyze the past price time series. They observe the M previous values of the price and simplify this information into two signs, one denoting rise in price and the other denoting fall in price between two consecutive time steps, such that the information to the speculators is analogous to the history in the basic MG. The strategies of the speculators define a buy or a sell decision for each possible history. If the scores of the strategies of a speculator are low, the speculator can abstain from playing. Otherwise, the speculator is willing to buy or sell a number of stocks depending on his/her strategy. The total numbers of stocks that the speculators and the producers are willing to sell and buy define the supply and demand, which, in turn, induce a new stock price for the next time step. The new price depends on the imbalance between supply and demand, as well as on the actual capacity of the agents to trade. The emerging price time series is the basic observable quantity in Slanina's and Zhang's model, and its analysis enables one to conclude something about the relationship between the two agent groups. It turns out that the producers and speculators live in symbiosis. The producers make it possible for the speculators to gain profits, whereas the speculators even out fluctuations in the price, thus yielding stable gains for both the producers and the speculators. Slanina's and Zhang's model is somewhat tangential to 'conventional' MG models and their work has gained little attention compared to some other financial MG models, like the one presented by Challet *et al.* in 2000 [58].

Like Slanina and Zhang [44], also Challet *et al.* [58] divided the agents into two main groups, the producers and the speculators whose characteristics strongly resemble those introduced by Slanina and Zhang. In Challet's *et al.* 's version the producers are MG agents who possess one strategy only, and they feed information into the system. The speculators, on the other hand, are regular MG agents that possess several strategies, and they try to make use of the information that the producers inject into the system. Challet *et al.* did not concentrate on any price time series but they draw conclusions about the evolution of the wealth of the agents and the relationship between the two agent groups. The strategies of the agents in their model are similar to the strategies in the basic MG, and agent i's gain at time step t is defined to be $g_i(t) = -a_{i,s_i(t)}^{u_M(t)} A(t)$ with the notations of section 2.5.4. Again, it turns out that the producers and the speculators live in symbiosis. Benefits to each group depend on the parameters. For example, when the number of producers is large compared to the number of speculators, the bias term Ω^{u_M} of Eq. (2.11) is large, and consequently the predictability θ of Eq. (2.5) tends to be non-zero. Note that only speculators have various strategies, and thus only they can overcome the bias effect of Ω^{u_M}. If $\theta > 0$, the speculators can make use of the information encoded in the histories (see section 2.5) and they tend to win, whereas producers who are endowed with only one strategy can not make use of this information and they tend to lose. Challet *et al.* also investigated what happens if an agent can freely choose whether to attend the game or not [58]. They allowed each speculator to abstain from playing if the scores of the strategies of the agent were less than or equal to zero. As a result, the game is in the asymmetric phase, where the predictability $\theta > 0$, but almost at the transition point, such that the average losses of the producers are extremely small. They found that when the number of producers is increased, the a priori asymmetry of the outcome increases (due to Ω^{u_M} in Eq. (2.11)), and due to that more and more agents actually play the game. If the game includes also noise traders that toss coin, the per capita fluctuations of Eq. (2.4) grow and the payoffs of the agents are reduced by their presence. However, deep in the symmetric phase (where $\theta = 0$), noise traders reduce the fluctuations. Moreover, Challet *et al.* studied the effect of privileged agents and insider-trading on the outcome of the game. The priviledged agents are endowed either with more strategies $S' > S$ or with a larger memory parameter $M' > M$ than the other agents. The insider-traders, in turn, know in advance the choice which some group of agents will make. The gain of an agent with $S' > S$ was found to be $\propto \sqrt{\ln S'}$, in other words very slowly. Large memories $M' > M$ turned out to be disadvantageous in the asymmetric region ($\theta > 0$), rather indifferent in the transition region, and advantageous in the symmetric region ($\theta = 0$). An agent with insider information was always found to perform better than other agents, except at a critical point characterized by short-term time correlations.

2.7.2 Majority-minority games

The previously discussed models assume two types of agents with apparently different interests and functions in the market. However, if one considers speculative markets, such as stock markets, it appears more convincing that the agents in the market all share the common interest of making maximal profits by trading stocks. For instance, investors may want to buy a stock at low price and later sell it at high price. In such markets, the heterogeneity of agents may not appear explicitly as different functions of the agents in the market but it may be hidden in the agents' different beliefs about the behaviour of price evolution of the stock. Majority-minority games are aimed for modelling speculative markets with two groups of agents, the trend followers and the contrarians, which differ in their beliefs about the price but share the common interest of making profits by trading. The trend followers play the majority game, because they believe that when most agents buy a stock, its price will increase in the future. Beliefs of this kind can be justified, for example, by looking at the continual rise of stock-market indices in late 1990s and early 2000. On the other hand, the contrarians behave as minority game players who believe that the stock has a 'fundamental' price which reflects its real value. If there are more agents who want to buy the stock than there are agents who want to sell, the contrarians believe that the price of the stock will temporarily rise beyond the fundamental price, after which it will inevitably come down. The fall in stock-price indices after early 2000 is a reminiscent feature justifying beliefs of this kind.

In Ref. [46] Jefferies *et al.* introduced contrarian and trend follower agents in a financial MG model, but their discussion was not very detailed. However, about the same time, Marsili published a paper with the sole focus being on majority-minority game models [63]. Marsili considered a situation where a certain number of agents is set to play the majority game and the rest of the agents are set to play the minority game. On one hand, fixing the numbers of each agent type in the game allows controlled study of the model. On the other hand, real financial agents may be able to temporarily change their beliefs if they detect changes in the market that favour one type of belief over the other. Such additional degree of freedom was introduced into the majority-minority game by Andersen and Sornette in [49]. Their work, which also includes a sensibly defined price forming mechanism, is a serious attempt to make the majority-minority game based toy model behave like a real financial market. Their numerical simulations indicate that, within certain assumptions, the characteristics of price time series emerging from their model are quite similar to typical real-world financial price time series with several stylized facts.

Other studies of the majority-minority games are provided by De Martino *et al.* in Refs. [80, 81], and Kozlowski and Marsili have studied the pure majority game in [82].

2.7.3 MG based market models and stylized facts

Jefferies *et al.* were among the first who studied whether an MG based market model is an adequate tool for modelling some empirically observed financial phenomena [46]. In their model each agent is assigned $S_i(0)$ risky assets and $B_i(0)$ riskless assets at the outset of the game. Agents in the game are heterogeneous in terms of wealth, investment size and investment strategy. Each trade made by an agent is the exchange of one quantum of a riskless asset for one quantum of a risky asset, irrespective of the wealth of the agent or price of the asset. Moreover, the agents can abstain from playing the game if they have no trust in their possibilities to make profits. Agent i's confidence is measured by his/her historical success, r_i, and the variability of his/her success, $\mathrm{std}(r_i)$, such that the agent plays only if the score of his/her best strategy is larger than $r_{\mathrm{min},i} = \max[0, -(r_i - \lambda \mathrm{std}(r_i)]$, where λ is a coefficient of risk-aversion. When a trade is made, it is made at the market price $p(t)$, which is determined by a market-making mechanism. The market-making mechanism must be separately defined for the model. In the real-world markets, the so-called market-maker is responsible for setting a reasonable price according to the demand-supply imbalance [83]. In Ref. [46] Jefferies *et al.* used two approaches for modelling the effect of the market-maker on the price $p(t)$. In the first one it is assumed that the supply and demand are in equilibrium at each time step, and in the other one this unrealistic assumption is relaxed. The numbers of stocks that an agent trades are proportional to the agent's wealth and his/her confidence in his/her strategies. If an agent loses all his/her assets, he/she can no longer trade. This represents the bankruptcy of that agent. The investment strategies of agents can fall into two broad classes, value and trend. At each time step, a value investor aims at making profits from buying low and selling high. A trend investor, in turn, considers the movement of the stock and aims at making profits from buying an upward moving asset and selling a downward mover. A population of only value investors have a minority game character, whereas a population of trend investors create a majority game of self-fulfilling prophecies (see section 2.7.2). In general, the population of traders is a combination of these types. Using a model with these ingredients, Jefferies *et al.* [46] were able to produce return time series reminiscent of real financial return time series with fat-tailed distributions, clustered volatility and high-volume autocorrelation. They also showed that this type of agent model can be used to predict price movements of the asset, predictions being better if an ensemble of individual agents' predictions is combined together in a proper way instead of using only individual agent's predictions. Moreover, they applied their model to measure and control risk in a portfolio management setting.

Challet *et al.* have also constructed a financial MG model with the aim of producing stylized facts of financial price time series [84, 85, 86, 87]. Their model is simple and minimalistic, yet it is able to produce interesting results. Similarly to their earlier work in Ref. [58], Challet *et al.* assume that two groups of agents, the producers (N_p) and the speculators (N_s), with different interests exist. Some versions of their model also include noise traders that choose their actions randomly, like agents in the RCG. However, we will not discuss this additional feature here. The producers have one strategy only and they play the MG every time step, whereas the speculators have several strategies and they play the game only if they have a strategy which gives an average gain larger than some 'risk free' interest rate ϵ. Thus, each speculator has also an inactive strategy and at each time step of the game only part of the speculators are active and actually play the game. Otherwise, the rules of their financial MG model are very similar to the rules of the basic MG. The price time series is defined in a particularly simple fashion,

$$\log p(t+1) = \log p(t) + A(t)/\lambda, \tag{2.13}$$

where λ is a tunable parameter that describes the market liquidity. It turns out that this model, as the basic MG, shows a phase transition, and the behaviour of $p(t)$ depends on in what phase the game is simulated. The phases are determined by the numbers of speculators and producers as well as the number of histories P. We will not discuss the phase structure here, but a good description of it can be found in Ref. [86]. A particularly interesting region is the the symmetric phase, where the predictability, Eq. (2.5), is zero. It turns out that within this model, four prominent characteristics that are present in real financial price time series emerge in the symmetric phase in which $\theta = 0$. The first one is that the price time series, Eq. (2.13), intermittently shows crashes [79]. In reality, it is troublesome to extract the factors that give rise to crashes but in the model the frequency of crashes has been found to increase with N_s and decrease with ϵ. The second interesting feature is the clustering of $p(t)$ and the clustering of trading activity, measured by the number of active speculators at any one time step. The third interesting result is that the autocorrelation function of the return time series decays slowly, and the fourth result is that the probability distribution function of the return time series is fat tailed. There is some evidence that real financial return time series could follow a power-law distribution [88] but it is speculative what factors affect the exponent. In the model by Challet *et al.* the exponent depends on N_s, N_p and ϵ. It is also interesting that in the symmetric phase of the game, many speculators refrain from playing and the level of active speculators N_s^a is just barely sufficient to exploit the information injected into the market by producers, thus making the market efficient in the long run. By contrast, locally in time, the market may not be efficient. This behaviour reflects efficient usage of information, a subject that has been under intensive study in financial literature [89, 90]. However, all these features crucially depend on the fact that agents neglect their market impact. If the agents can account even approximatively for their market impact, the results change dramatically: the phase transition disappears, and the dynamics converges to a state where each speculator plays one strategy at all times or does not play at all. Also volatility clustering and fat-tailed distribution of returns disappear if the agents can take into account their market impact.

The two previously discussed models are meant to suggest explanations for several stylized facts with one model. Giardina *et al.* , for their part, tried to explain one particular stylized fact, the long ranged volatility correlations, with several models [45]. They assumed that the volatility clustering and the volume clustering emerge as a consequence of a generic pattern where financial market agents compare performance of different strategies on a signal that looks random. A common strategy for each agent at all times is the inactive 'do nothing' strategy. Relying on such an assumption they consider a simple case where each agent has two strategies, one of them being the inactive strategy, and the difference between the score of these two strategies will behave, as a function of time, like a random walk. The survival time of any of these strategies will be given by the return time of a random walk to zero. Since these return times follow a power-law distribution, the non-trivial volume autocorrelation will emerge. Volatility and the volume, in turn, are strongly correlated in financial markets [91], and thus the explanation should apply to volatility clustering as well. Giardina *et al.* tested their hypothesis with an MG that included an inactive strategy. They found that the volume activity of the agents is in accordance with their generic explanation. Furthermore, they noted that a similar mechanism might also be present in other models where agents can switch between different strategies or classes of strategies. For example, in Ref. [49] the agents may switch from contrarian behaviour to trend follower behaviour, and in Ref. [84] the speculator agents can be inactive or active.

Later on Giardina *et al.* introduced an elaborate artificial market model that was based on the MG but included several additional parameters depicting various real-world financial market phenomena [48, 92]. Although the number of parameters in their model is large, only two parameters play a crucial role; one determines how sensitive the market price of their model is to the agents' actions, and the other determines whether an agent behaves as a contrarian or as a trend follower. Depending basically on the values of these two parameters the market shows three different behaviours: oscillatory, intermittent and stable. One of their major results is a rough phase diagram that shows the ranges of these two parameters that correspond to each behaviour. Their model produces particularly interesting results in the intermittent parameter region. Emerging properties in this region include, for instance, volatility clustering, bubbles, crashes and fat-tailed return distributions. The emergence of volatility clustering in their model is due to the generic mechanism discussed above [45]. Overall, their model is a serious attempt to depict the behaviour of real financial markets, and Refs. [48, 92] include several good references for developing artificial market models.

Common to MG based market models for producing stylized facts seems to be that they are at least as complicated as the basic MG. However, Liu *et al.* [93] gave recently evidence that also the basic MG can produce real-market like behaviour with several stylized facts in the low-z region, if the time horizon during which the score of each strategy is updated is limited.

2.7.4 Predicting time series

The MG has also been applied to time series prediction. Jefferies *et al.* introduced a method to predict the sign of the next price change in a time series [46]. Individual agents are provided a historical record of the previous signs of the price changes and they predict the next sign according to their best strategy like the basic MG agents predict the next minority group. Then, the actual prediction of the sign of price change is based on a composition of the individual agents' predictions. Johnson *et al.* tested this prediction method in a controlled setting [94]. They asked whether one can successfully predict the sign of price change from a price time series that is produced by another MG. The answer turned out to be positive. Moreover, Lamper *et al.* showed that the level of predictability increases prior to a large change in price [95], and thus large changes arise as a predictable consequence of information encoded in the global state of the system.

2.8 Evolutionary extensions

All the MG models discussed so far are adaptive in the sense that agents can change their actions by selecting strategies in response to changes in their environment.

However, the models do not allow for evolution of the strategies, and an interesting question concerns what would change in the behaviour of MGs if evolutionary mechanisms were applied to the agents' strategies. This question is not a new one. Challet and Zhang have explored the effect of applying natural selection and evolution to the strategies of the MG agents on the game behaviour at a very early stage of the MG research [28, 41]. However, they did not study the problem very deeply. In this section, we review MGs with evolutionary attributes. The models can be roughly divided into two groups: one in which the agents choose either their strategies or their actions stochastically, and one in which the agents choose them deterministically.

2.8.1 Models with probabilistic features

The first evolutionary MG (EMG) model was presented by Johnson *et al.* in 1999 [68]. Their model is different from the basic MG model in several aspects. Firstly, each agent in the EMG has access to a common register that contains the outcomes from the most recent occurrences of all 2^M possible history bit strings of length M, that is, the register contains information about the last winning side corresponding to every possible history. In case $M = 2$, the register could be (111-1), for example. Reading from left to right, these numbers tell what were the last winning sides following the possible histories (-1-1), (1-1), (-1,1) and (11) correspondingly. Secondly, the strategy of an agent is a single number, p, which is the probability to play the previous winning side corresponding to a particular history, stored in the common register. With probability $1 - p$ the agent plays the opposite action. Each time an agent chooses the righ minority (majority) group, he/she gains (loses) one point. If the agent's score falls below a value $d < 0$, then his/her strategy is modified; i. e. , a new value for p is randomly drawn from a uniform distribution so that $p \in [p_0 - R, p_0 + R]$, where p_0 is the current value of p. The boundaries $p = 0$ and $p = 1$ are reflective. Numerical simulations and analytic inference that Johnson *et al.* carried out with this model reveal that the agents tend to self-segregate into opposing groups characterized by extreme behaviour. About half of the agents end up having strategy values near $p = 0$ and about half of the agents end up having values near to $p = 1$. Cautious agents, those for which $p \approx 1/2$, perform poorly and tend to become rare.

The evolutionary MG model by Johnson *et al.* [68] led to a number of other studies. Ceva [96] pointed out that the results of Johnson *et al.* [68] depend on the type of update rules used. In another paper, Johnson *et al.* [97] studied the same evolutionary mechanism in more general setting, where the fraction of agents who can win per round is not fixed to 50 percent as in the MG, but is an adjustable parameter. Burgos and Ceva [98] simplified the model such that it does not make use of memories at all, and found that this simplified model produces the same results as the original model of Johnson *et al.* [68]. Lo *et al.* [99], in turn, endowed each agent in the evolutionary MG with one strategy that is similar to the strategies in the basic MG, and analytic consideration of the evolutionary MG was carried out by Lo *et al.* in Ref. [100].

In 2002, Hod and Nakar [101] presented an appealing idea that they applied to the evolutionary MG of Johnson *et al.* by proposing that the reward of winning should not necessarily be equal to the fine of losing in the game. In the extreme situation, the fine may be larger than the reward. For instance, a successful hunt of a predator could maintain the predator for a few days, but an unsuccess in hunting could lead to the death of the predator. Hod and Nakar introduced the reward to fine ratio parameter, R, such that $R = 1$ corresponds to the original MG model, whereas for $R < 1$ the reward is smaller than the fine, and for $R > 1$ the opposite holds. With this modification they showed that the self-segregation is a special feature that occurs only if the reward to fine ratio is sufficiently high. For $R < 1$, cautious strategies $p \approx 1/2$ tend to become most popular, in other words, under hard conditions individuals tend to behave similarly and do the same as the majority. In addition, they found that the distribution of the p-values of the agents display temporal oscillations around $p = 1/2$ such that the smaller the value of the reward to fine ratio, the farther the system is from a steady state distribution. This observation points out that the analysis of Lo *et al.* in Ref. [100], based on the assumption of steady-state, is not valid for $R < 1$. Later, Hod provided insight on the behaviour of this MG model using time-dependent random walk arguments [102]. Also the observed temporal oscillations were further studied by Nakar and Hod in Ref. [103].

Another model that is in some ways related to the EMG model of Johnson *et al.* was presented by Reents *et al.* in Ref. [104]. They assumed that an agent i has a strategy that assigns an action $a_i(t)$ to that agent, so that if the agent is successful in a given turn, he/she will make the same decision next turn, $a_i(t+1) = a_i(t)$. Otherwise, the agent will change his/her action to the opposite one with a certain probability p such that $P(a_i(t+1) = -a_i(t)) = p$. This model is a stochastic one-step process, and it has a stationary distribution that can be solved analytically. One of the main results by Reents *et al.* is that σ^2/N can be decreased even to the optimal level for a suitably chosen p.

2.8.2 Models without probabilistic features

The models discussed in section 2.8.1 are quite different from the basic MG model. In this section we review studies that are in closer relation to the basic MG.

Li *et al.* [30, 31] carried out an extensive study of the effects of evolutionary mechanisms on the basic MG. Their evolutionary rules resemble those that Challet and Zhang mentioned in Refs. [28, 41]. In Ref. [30] Li *et al.* examine evolution in games in which the dimension of the strategy space is the same for all agents and fixed for all time. To make the MG evolving, they define a time τ, which is the lifetime of one generation. During τ time steps, the agents' strategies do not change. After τ time steps, the agents are ranked by wealth accumulated during that generation, and half of the so-called poor agents, those who belong to the lowest percentile w of agent wealth, are selected at random and removed. A removed agent is replaced with a new agent who inherits the wealth of the old agent but gets new randomly chosen strategies with scores set to zero. Li *et al.* conducted their simulations with real histories and they found that evolution results in a substantial improvement in overall system performance, though the best system performance still occurs (minimum of σ^2/N exists) at z_c, the same value as in the basic MG. They discovered that in the low-z region the evolutionary mechanism tends to select against high intra-agent HDs and against low inter-agent HDs. Thus, the strategies of an agent become more similar, whereas the active strategy of the agent moves away from the active strategies of the other agents. In the high-z region, only the latter tendency of evolutionary selection appears to take place. In Ref. [31] Li *et al.* continued their study of evolution in minority games by examining games in which agents with poorly performing strategies can trade their strategies for new ones from a different strategy space that consists of strategies with different length, M. Similarly to Ref. [30], they defined a generation of τ time steps, and replaced half of the poor agents, whose wealth is in the lowest percentile w with new ones. They applied two rules of replacement. In the first one, an agent that is chosen for replacement is given strategies of any memory $1 \leq M \leq 16$ with equal probability. In the second one, an agent with memory M chosen for replacement is given strategies with memory $M+1$ or $M-1$ with equal probability. It turns out that in the presence of such evolutionary mechanisms the most wealthy agents are those who have low values of M. For a given number of agents N, the wealth per agent is roughly a step function as a function of M with a transition occurring at $M = M_c - 1$, where M_c is the value of M at which σ^2/N attains a minimum value for fixed N. It also turned out that the numbers of agents who have strategies with length M are quite equal for $M < M_c - 1$, and close to zero for $M > M_c - 1$. This means that the system tends to evolve such that the effective size of the strategy space occupied by most of the agents is about 2^{M_c}.[10] Thus, the system evolves into a state where it performs the best, that is, z moves close to the critical value z_c where σ^2/N has its minimum. A prominent

[10]The effective size is approximatively 2^{M_c} since $2 + \dots 2^{M_c-1} = 2^{M_c} - 2$. Terms on the left hand side are the lengths of the strategies that most of the agents use.

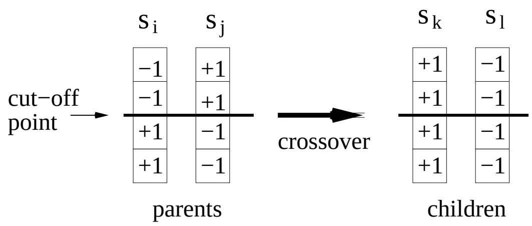

Figure 2.5: Schematic diagram to illustrate the mechanism of crossover for producing new strategies. The strategies s_i and s_j are the parents. We choose the cut-off point randomly and cross the entries

result both in Ref. [30] and in Ref. [31] is that such a minimum of σ^2/N occurs.

Sysi-Aho *et al.* (Publications I–V) applied genetic algorithms to the basic MG. Their models belong to the same class of evolutionary modifications as those studied by Li *et al.* in Ref. [30] in that the dimension of the strategy space is the same for all agents and fixed for all time. The genetic algorithm works such that two of an agent's S strategies are chosen to be parent strategies, say s_i and s_j, and then these two strategies are cut into two pieces at a randomly chosen cut-off point $k_c \in \{1, \dots, 2^M\}$ that is the same for both strategies. Then, two child strategies, s_k and s_l, are formed; the first one by catenating the first k_c components of strategy s_j with the last $2^M - k_c$ components of strategy s_i, and the second one by catenating the first k_c components of strategy s_i with the last $2^M - k_c$ components of strategy s_j. Schematic diagram of the crossover mechanism for $M = 2$ and $k_c = 2$ is shown in Fig. 2.5.

There are various alternatives for defining i) when the agents should apply the genetic-crossover, and ii) how the parent strategies and the children strategies should be chosen. For i) the authors adapted the convention of Li *et al.* [30] by defining a generation of τ time steps during which the strategies of the agents do not change. In (Publications I–IV), the agents are ranked by wealth accumulated during the entire game up to the current time step at the end of every τ time steps, and those agents who belong to the lowest percentile, w, of agent wealth are allowed to cross their strategies. In (Publication V) the agents are arranged on a scale-free network [105], and an agent compares his/her wealth to the wealth of his/her nearest neighbours after every τ time steps. Those agents who have a neighbour with a higher wealth, cross their strategies. For ii) several cases were studied:

1. Two parent strategies are selected at random and after the crossover the par-

ent strategies are substituted with the two new children.

2. Two parent strategies are selected at random, and after the crossover the two worst performing strategies are substituted with the two new children.

3. Two best performing strategies are selected as parents and after the crossover the parents are substituted with the two new children strategies.

4. Two best performing strategies are selected as parents and after the crossover two worst performing strategies are substituted with the two new children.

The mechanisms were found to differ from each other with respect to their efficiency in improving the system performance and increasing the wealth of individual agents. In these terms, the fourth mechanism outperformed the others. However, all the mechanisms were found to improve the system performance as well as the individual agent wealth in various circumstances: in (Publications I and II) the histories were real, and in (Publications III–V) they were random[11]; in (Publications I and II) the agents' strategies were drawn from a reduced set of strategies such that initially the Hamming distance, Eq. (2.6), $H_d(s_i, s_j) = 1/2$ for all i, j, and in (Publications III–V) they were drawn from the full set of strategies.

Application of genetic algorithms to the agents' strategies was found to cause a locking effect in case of using real histories (Publication II). In the locking effect, one history starts to repeat over and over again, and consequently the agents divide into two groups, those who win every time step and those who lose every time step. Using randomly generated signals instead of real histories provides a mean of preventing the locking effect. Furthermore, the use of random histories eliminates the occurrence of histories in cyclic patterns, and consequently the agents cannot benefit from any prediction method based on such cycles. Thus, in order for the strategies of the agents to be 'good', the strategies should provide successful decisions for individual agents and result in small fluctuations at the system-level for an unpredictable occurrence of any history u_M. This means that the use of random histories sets the strategies of the agents into harder test of success than the use of real histories would do, and consequently the success of the genetic algorithms in modifying the strategies can be more generally judged. In regard to this, there is a remarkable result in (Publication V) which indicates that σ^2/N can obtain minimal values for a wide range of control parameter values z. This property is illustrated in Fig. 2.6, where σ^2/N approaches the optimal limit $1/N$ when S is increased. Thus, for sufficiently high S, σ^2/N is minimal all the time and there

[11]A real history comprises the last M minority groups, and a random history is just drawn from a uniform distribution of numbers $\{1, \ldots, 2^M\}$. For discussion about real and random histories see Refs. [53, 72], and section 2.6.

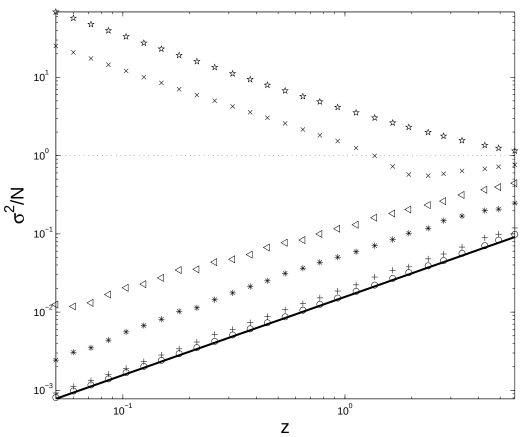

Figure 2.6: The per capita fluctuations σ^2/N versus the control parameter $z = 2^M/N$ for $S = 5$ (triangles), $S = 8$ (asterisks), $S = 13$ (plus-signs), and $S = 21$ (circles) in MG where the genetic-algorithm rule of (P5) is applied to the agents' strategies. For comparison, σ^2/N versus z in the basic MG is also plotted for $S = 5$ (crosses) and $S = 21$ (stars). We used $M = 6$, $\tau = 2P$, simulated for 64000 time steps, and calculated σ^2/N from the last 1000 time steps averaging over 200 ensemble runs. Compare with Fig. 2.3.

is no turning point at any $z = z_c$, a result that is different from several previous studies where the existence of such a point has proven to be highly robust under various modifications of the basic MG (see, e. g. , Refs. [30, 31, 53, 54, 57]). It was found also that in the course of the game, the intra-agent Hamming distances tend to converge toward small values, close to zero (Publications II–IV), whereas the inter-agent Hamming distances avoid clustering (Publication V). These findings are qualitatively consistent with those that Li *et al.* made in Ref.[30].

Yang *et al.* [106] applied genetic algorithms to the MG with variable length strategies by studying two variations. In the first one each agent has two strategies, $S = 2$, with equal length, but the strategies of different agents may vary in length. The evolution is realized such that after each generation of τ time steps, the agents are ranked by wealth, and the poorest fraction w of agents will adjust their strategies. The rule of adjustment is defined by a cutting operator and a splicing operator, from which one at a time is applied with probability $1/2$. For an agent adopting the cutting operator, each of his/her strategies will be cut into two halves, and only one half of them is kept as new strategies. For an agent adopting the splice operator, a random binary string of length equal to the length of the current strategy of that agent is generated, and catenated with the old strategy.

In the second variation of Yang *et al.* , an agent has four strategies, $S = 4$, of different length. After every generation part of the agents are selected for modifying their strategies as described above. An agent who is selected for crossover again uses either the cut operator or the splice operator. However, now an agent who adopts the cut operation will select his/her best strategy, and cut it into two halves, which are then used to replace the two worst strategies of the agent. In case an agent adopts the splice operator, he/she selects two of his/her best strategies, catenate them together and use the resulting strategy to replace the strategy with the worst strategy. The results of Yang *et al.* [106] indicate that their second variation leads the system into a more efficient state than the first variation, though both systems improve the system performance considerably compared to the basic MG. Furthermore, as the game evolves, the number of agents who posses strategies with length $P = 2^M$ resemble a step function as a function of M, a result that is qualitatively consistent with the result of Li *et al.* in Ref. [31].

2.9 Other modifications

A major part of the MG papers that have been published so far concern the topics described above. In this section we will take a look at few other variations of the game.

2.9.1 Multichoice minority game

One immediate question that arises in context of minority games is what happens if each agent has more than two options among which to choose? Extensions of the MG model to multiple choices try to answer this question. Ein-Dor *et al.* [107] introduced a multichoice MG, where each agent chooses one of the N_c states, aiming to choose the state with the smallest number of agents at each time step. Their model closely resembles neural network models and differs quite significantly from the basic MG. Each agent has one strategy which evolves according to its performance, and the reward of winning need not be the same as the fine from losing (for reasons of using asymmetric payoffs see Ref. [101]).

Another multichoice MG model that is in closer contact with the basic MG was introduced by Chau and Chow [108]. In their model each agent has to choose among N_c states according to a strategy which assigns a choice to each possible history of the past M time steps. Likewise in the basic MG, the history comprises knowledge of the past minority groups. For memory M, there are N_c^M possible histories, and a strategy is composed of N_c^M entries that assign a choice to each of these histories. Furthermore, like in the basic MG, each agent initially has S randomly chosen strategies that amass points during the course of the game. One point is assigned to each strategy that predicted the group which happened to have the least non-zero number of agents in a turn, and one point is subtracted from all the other strategies. Thus, the extended model of Chau and Chow [108] is strongly analogous to the basic MG of Challet and Zhang [28]. Results from numerical simulations also indicate that the multichoice MG behaviour is qualitatively similar to the behaviour of the basic MG. In particular, there is a control parameter, and a critical value of this control parameter at which σ^2/N attains a minimum value. Also the overall behaviour of σ^2/N as a function of this control parameter is reminiscent of the behaviour of σ^2/N in the basic MG as a function of its control parameter, z. Chow and Chau carried out further analysis of basically the same model in [109], and their results strengthen the finding that their multichoice MG behaves similarly to the basic MG.

2.9.2 Networks, local neighbourhoods and changes in information

One branch of MG modifications concentrates on a network arrangement where the agents are placed on the nodes of a network, and some type of information flows through edges connecting the nodes. Often the information to an agent consists of knowledge of choices made by agents who are connected to that agent, or knowledge of the minority group sign in the local neighbourhood of that agent. Typically, the agents are placed on one of the following network types:

Type 1 – Circle. A regular one dimensional chain, where each agent is connected to his/her K nearest neighbours that are on his/her right and on his/her left. In addition, boundaries are chosen to be periodic such that the chain forms a circle.

Type 2 – Two dimensional lattice. A regular two dimensional lattice, where each agent is connected either to his/her four nearest neighbours ('North', 'East', 'South', 'West'), or to his/her eight nearest neighbours (NE, SE, SW, NW in addition). Boundaries are usually chosen to be periodic.

Type 3 – Random network. A random network where each agent is connected to K other agents that are selected at random among the other agents such that each agent can be selected only once.

Paczuski *et al.* studied an MG model on networks of Type 3 [110]. In their model each agent has one strategy that determines an action, 1 or -1, for each possible state of the neighbouring agents.[12] Thus, the information provided for each agent consists of the states of the neighbouring agents, as opposed to the sequence of past M minority groups. The performance of an individual agent is measured by counting the number of times each agent has been in the majority. After a period of time, defining an epoch (similar to generation in Refs. [30, 31]), the worst performer who was in the majority most often changes his/her strategy. The new strategy is chosen at random. As a result of their model, Paczuski *et al.* [110] observed that irrespective of the initial conditions, the network ultimately self-organizes into an intermittent steady state at the borderline between two dynamical phases. Furthermore, their numerical simulations indicate that the distribution of attractor lengths in the self-organized state is broad and consistent with power-law behaviour for large enough attractor lengths.

Slightly later than Paczuski *et al.* , Kalinowski *et al.* published a paper on MG with local information [55]. In their model, the agents are arranged on Type 1 network, and the input signal to each agent consists of previous decisions of his/her $K = M$ nearest neighbours. Similarly to the basic MG, each agent has S strategies that select an action, -1 or 1, as the response to each possible state of the neighbouring agents. These states are strings of length M of 1s and -1s that correspond to the actions of the neighbouring agents (and to the action of the agent itself if M is odd). It turns out that the behaviour of σ^2/N in this local MG model is qualitatively similar to the behaviour of σ^2/N in the basic MG. Kalinowski *et al.* also studied whether the system performance could be optimized in their local MG by applying two particular evolutionary mechanisms. In the first mechanism, which they called the global evolutionary mechanism, the agent who performs worst is removed after every τ time steps and replaced with a new agent that is a variant of the removed agent. The new variant gets the values of the parameters M and S of the old agent, and these parameters can be increased or decreased by one with a certain probability. The other evolutionary mechanism is local: after τ time steps each agent looks at his/her two neighbours, and if the best neighbour has at least one percent more points than the agent, the agent copies the properties of this neighbour as described above. The results of Kalinowski *et al.* indicate that if the global evolutionary mechanism is applied to the game, the game behaviour does not change significantly. By contrast, if the local evolutionary mechanism is applied to the game, some substantial changes will emerge. For instance, the values of M and S evolve such that the degree of cooperation is the best, a feature that is reminiscent of the basic MG behaviour under evolution as reported by Li *et al.* in Ref. [31]. An interesting result of their model is that σ^2/N attains a minimum

[12] This setting is the so-called Kauffman network [111]. The name is due to Stuart A. Kauffman who extensively studied the properties of these networks.

value at $M = 3$ irrespective of the value of N.

Moelbert *et al.* [112] studied minority games on Type 1 and Type 2 network settings. After every round each agent will be announced the minority group of his/her local neighbourhood, and points are awarded to those agents who belong to the minority group in his/her local neighbourhood. Moelbert *et al.* made use of the formalism presented in Refs. [40, 58] in order to describe their local MG in terms of spin glasses. In this each agent has two strategies which they use with probabilities of Eq. (2.8). Unlike the basic MG where the maximum number of winners is $(N-1)/2$, their local MG has configurations where each individual can win, depending on the nature of the regions. For instance, such a situation occurs in Type 2 network with four neighbours if each agent playing 1 (-1) has at most one other agent in his/her neighbourhood who plays also 1 (-1). Then each agent belongs to the minority group in his/her local neighbourhood and win. Results of Moelbert *et al.* indicate that the system shows global, collective behaviour, because it benefits from the spatial arrangement of the individuals. For individual agents it is highly efficient to possess anticorrelated strategies, a situation that is different from the basic MG (see, e. .g. , Manuca *et al.* [39]). Furthermore, it is disadvantageous for the agents to shuffle randomly between their strategies (large Γ in Eq. (2.8)). By contrast, in the basic MG it is advantageous for an agent to shuffle between his/her strategies in the low-z region [67, 74].

Agents in the models of Paczuski *et al.* [110] and Kalinowski *et al.* [55] make use of global information, because the success of strategies depends on the minority group with respect to the whole population. By contrast, the agents in the model of Moelbert *et al.* [112] make use of local information, since the success of agents and their strategies depends only on each agent's local neighbourhood. However, in the real-world people can consider both local level information, such as rumours from peers, and global-level information, like news from the media, when they make decisions. To study the interplay between global and local information in a minority game, Chau *et al.* [113] constructed a model where the agents use both the global and local information in their decisions. The basis of their model is the multichoice MG of Chau and Chow [108, 109]. The global information given to the agents is the typical time series of the past M_g minority groups of the entire population. The local information for an agent, in turn, is the time series of the past M_l minority groups in the agent's local neighbourhood. The local neighbourhood is defined by arranging the agents on a Type 1 network with $K = 2$. The history that an agent receives is a catenated sequence composed of the time series of the past M_g global and the M_l local minority groups. Each agent's strategies are rules that define an action to each possible history, just like in the basic MG or in its multichoice extension [108, 109]. The results of this composite model indicate that cooperation among agents is better than in the basic MG due to the local information which is disseminated through Type 1 network. In addition, with no global information (cf. Ref. [112]), many players on the network get frozen, i. e. , they play the same action at all times.

Caridi and Ceva presented an MG with local information too [57]. In their model the agents are arranged on a Type 2 network, and a fraction p of the agents are chosen to be interacting. The positions of the pN interacting agents on the lattice are selected at random at the outset of the game. At each time step of the game the agents follow the rules of the basic MG, except that the interacting agents are given the extra opportunity to modify their actions, after knowing what other interacting agents in their nearest neighbourhood will do in the same step. Because the interacting agents are distributed randomly, every one of them can have between zero and four other interacting agents in their nearest neighbourhood. The interacting agents try to be in the minority of the group formed by their interacting nearest neighbours and themselves. Results of Caridi and Ceva show that the efficiency of the system improves if the fraction p is small, being worst when $p \to 1$. It also turns out that there exists a critical value $p_c(z)$ that depends on the control parameter $z = 2^M/N$. When $p < p_c(z)$ all possible $P = 2^M$ histories occur in the system, whereas when $p > p_c(z)$ the histories tend to appear in a cyclic pattern with period shorter than P.

The idea of providing local information to the agents has also been applied to the evolutionary MG model introduced by Johnson *et al.* [68] (see section 2.8.1). Quan *et al.* [114] studied effects of local information transmission and imitation among agents on the evolutionary MG by setting the agents on Type 1 network. Agents in their model follow the same rules as agents in the Johnson's model except that each agent knows the wealth and the p-values of his/her two nearest neighbours, and if the agent has a wealthier neighbour, the agent adopts a new p-value that is centered around that of the neighbour. This modification leads to an enhanced cooperation with the number of winners per turn being larger in the modified model than in the basic MG or in the standard evolutionary MG. Burgos *et al.* [115] studied another type of modification to the evolutionary MG. In their model the agents are set on Type 1 – Type 3 networks, and the agents follow rules of the EMG except that the success of each agent is considered only in his/her local neighbourhood. The size of the local neighbourhood can be varied. A prominent result from this model is that the agents coordinate their actions well when they base their actions on local information and disregard the global trend in the self-segregation process.

Cara *et al.* [116] studied a variation of the basic MG in which information to the agents is personal, as opposed to the global information, u_M, provided to the agents in the basic MG. In their model agents are not set on a network, but each agent's personal information is defined to be the agent's knowledge of his/her own choices of the previous M time steps. It has been discovered that coordination among the agents can be improved with this change, that is, values of σ^2/N tend to be lower when the agents use personal information than values of σ^2/N are if the agents use the global information. Also Li *et al.* [56] studied MGs that were similar to the basic MG, except for the information provided to the agents. They explored three cases. In the first case, each agent uses his/her own personal history of successes during the past M time steps instead of the public history, like agents in the model of Cara *et al.* . In the second case, the agents are segregated into $P = 2^M$ groups of nearly equal size. Agents within the same group will use the same information that is generated randomly at each time step. Groups are fixed during the course of the game. In the third case, the arrangement is similar to that in the second case, except that each agent forms an own group. Thus, each agent randomly and independently chooses one of the P values of information. Consequently, the groups that share the same information are highly fluid, and there is only random persistence in their identity. In addition to these three cases, Li *et al.* [56] illustrated an interpolated game in which all agents use private information with probability r or public information with probability $1 - r$. In all the cases, the agents and their strategies are rewarded according to what the real minority group is in the entire population. Results from these models show that games with private information share a general structural similarity to the basic MG with public information in that there are two phases; one that is reminiscent of the basic MG behaviour in the low-z region and one that is reminiscent of the basic MG behaviour in the high-z region. However, the microscopic behaviour of the agents differs from case to case.

2.10 Experiments on human players

If the MG is to be considered a model for bounded human rationality, it is sensible to ask whether the agents in the game actually have something in common with real human beings. For example, in some financial MG models (section 2.7) it is assumed that the attendance time series $A(t)$ of Eq. (2.2) is closely related to the price of an asset under trade, and it would be interesting to know whether real people produce attendance time series with similar properties to those that have been produced by the MG agents. Also, it would be interesting to know how real people make use of the information available to the agents in the MG. Studies that aim at answering these questions have been carried out only recently.

Bottazzi and Devetag conducted laboratory experiments in which fixed groups of $N = 5$ human players play the MG for 100 periods [117]. They compared the system efficiency of a group of people to that of the random choice game. In order to find out whether humans can coordinate their actions better if the amount of aggregate information provided to them increases, they varied the memory M that players have regarding the game history, i. e. , the number of past outcomes of the game announced to the players. They also varied the amount of information that players had regarding the game history and the past actions of the other players. Their results show that the level of coordination between human players is higher than random. However, the amount of information that human players receive does not significantly alter the results. Thus, one can think that human players need only small amounts of information to coordinate efficiently, or from another perspective, human players are unable to make use of complicated information.

Laureti *et al.* investigated human speculative trading behaviour and information capacity by web-based experiments in which individual human beings played the MG against computer-modelled agents [118]. Their data consists of records of large number of plays in which a single person played the MG with many computerized agents. The total number of agents, including the human player, was $N = 95$. The human player is presented with a price history, $P(t+1) = P(t) + A(t)$ where $A(t)$ is the attendance of Eq. (2.2), of the past 50 time steps. The performance of the human player is compared to the performance of computer agents in different 'markets', which differ by the memory M of the computerized agents. Allegorically, the larger the M, the more 'complex' the market can be thought to be. Results from these experiments indicate that humans typically outperform the computer agents in 'easy' markets, with small M, but if M increases, humans do much worse. Moreover, it appeared that while human decisions are correlated with the long-term trend of the market, this correlation decreases as the markets become more difficult, being close to zero for large M. This means that when the information to the human beings is complicated enough, the players seem to ignore all aspects of the information presented by the market.

Also Platowski and Ramsza conducted an experiment where human beings played the MG [119]. In their experiment, a group of $N = 15$ persons played the game for 200 time steps. At each time step the only information displayed to the players was a historical record of the past M winning sides. Results from their experiment support the results of Bottazzi and Devetag about human coordination; it was found that humans do coordinate, but the length M of public information seems to have no influence on the game behaviour. Furthermore, Platowski and Ramsza pointed out that people can coordinate their actions better than the RCG agents in case they are provided real information (histories) about the winning sides, or even if they are not provided any information at all, that is, in case they choose their actions blindly, without any feedback. But most strikingly, it turned out that humans coordinate badly, worse than the RCG agents, if they are provided a fake signal, that is, a randomly generated signal of the past winning groups. This behaviour is substantially different from the behaviour of computerized MG agents whose coordination does not change significantly if the real history is replaced with the randomly generated signal, as reported by Cavagna in [53]. Thereupon, one can call into question whether the learning procedure adopted in the basic MG is an appropriate description of actual human behaviour in this game.

2.11 Discussions

The MG is a model that depicts a population of individuals with limited capabilities competing for scarce resources. Several existing or hypothetical phenomena occurring in social, biological, economic, and technological systems have been studied with the model, for instance, coordination among individuals, emergence of stylized facts in financial markets, or the influence of evolution and natural selection on the population of individuals. The MG model is theoretical, like much of the research related to it, and many questions concerning the model can be answered rather rigorously.

However, if the MG is to be considered as a model depicting real-world phenomena, in particular the behaviour of human beings, one should be cautious when interpreting the model and its results in the absence of empirical facts. For instance, assumptions made for financial MG models (section 2.7) about different agent groups with different functions in financial markets, or the assumption that financial agents do compare different strategies when they try to make profits, or that stylized facts in price time series emerge due to the properties introduced in the constructed models are speculative. Apart from few exceptions, such as, the knowledge of how the price of a stock is realized from the order book that lists the buy and sell orders of the stock [83], the proposals that are introduced in the financial MG models lack empirical basis. Mostly, the rules of behaviour of the agents in the game seem to be in contrast with human behaviour.

Conducting a sufficient number of experiments with human players in controlled laboratory settings could provide valuable information to remedy such contradictions. However, knowledge that could be obtained from the experiments might still be too indeterminate for mathematical modelling purposes. As an example, one can think of the very basic observation mentioned at the beginning of this chapter; that we, human beings, carry out localized deductions based on our current hypotheses and act on them, and as the feedback from the environment comes in, we may strengthen or weaken our beliefs. This observation is based on psychological experiments, and indeed the El Farol BP and later the MG were tailored for modelling such behaviour. However, recent experiments show that the model realized with computerized MG agents does not produce results that would be in line with the results obtained from human players who play the MG in the same circumstances as the MG agents (section 2.10). Thus, one's suspicions of the ways how human information processing abilities are modelled or how human deductive thinking is modelled, arise. In particular, the finding that a group of human players coordinate badly if they are provided a random signal instead of the real history comprising the past minority groups in the MG shows a stark contrast to the behaviour of a group of computerized agents that is not sensitive to such changes (see section 2.6).

All in all, the connections between the world of MGs and the ambient real-world of us are mostly metaphoric. Instead of trying to match the model and all its details with the real-world, one should be open-minded in one's view, and take the model as a suggestive tool that makes qualitative and quantitative speculation possible about things that are far-reaching and still largely beyond our current understanding. Also, the questions arising from the MG models can inspire new experiments on human beings or animal societies that may further contribute to our understanding of their behaviour, and maybe allow more precise mathematical modelling of these systems one day.

Chapter 3

Spatial games and cooperation

In the previous chapter we have seen that agents following simple rules in an MG can give rise to phenomena that are reminiscent of those observed in animal and human societies, which are composed of considerably more complex actors than the MG agents. In particular, agents in an MG were found to coordinate their actions such that they make good use of the resources that are available to their society. In this chapter we shall continue with agent-based models, now focusing on the problem of persistence of cooperative behaviour in animal and human societies. Our treatment of the current subject will be more limited than our treatment of the MG.

The outline of this chapter is as follows. In section 3.1 we introduce the problems related to the existence of cooperative behaviour in animal and human societies, and discuss theoretical frameworks that have been utilized for modelling this problem. We will give a brief description of evolutionary game theory and its key concepts. We will ponder whether time scales associated with evolutionary games are appropriate for explaining behaviour of adaptive individuals who can act on much shorter time scales than those that are typical in genetic evolution. Our primary focus will be on spatially structured populations that are composed of individuals who can repeatedly interact with each other. In section 3.2 we describe a spatially structured snowdrift game, and apply a simple adaptive mechanism to determine the players' decisions on time scales that are shorter than time scales associated with 'genetic' evolution of the players. In section 3.3 we give a short summary of the results of our model. Finally, in section 3.4 we draw conclusions, and compare our results with the results of Hauert *et al.* who studied the same model using the so-called replicator dynamics [37].

3.1 Background

Understanding the emergence and persistence of cooperation is one of the central problems in evolutionary biology and socioeconomics [32, 120, 121, 122, 123]. The difficulty of explaining the existence of cooperative behaviour arises from the fact that selfish individuals can reap the benefits of cooperation without bearing the costs of cooperating themselves. For instance, in a bring-a-dish party, cooperative individuals contribute to the common epicurean offering and bear the costs, whereas non-cooperative individuals only take benefit without costs. Thus, one could think that non-cooperative individuals would have a fitness advantage over cooperative individuals, and natural selection would lead to the extinction of cooperators in the long-term.

In investigating this problem, the standard framework utilized is evolutionary game theory [15, 34, 51]. In evolutionary game theory one usually considers a population of individuals, henceforth called agents, associated with fitness values that depict the reproductive capabilities of the agents. Typically, the population consists of different types of agents with different characteristics, or patterns of strategic behaviour; for example, some agents may be cooperative whilst others are defective. In a evolutionary game, attention is often focused on the abundance of different types of agents in consecutive generations. The way of modelling strategic behaviour with evolutionary game theory is advantageous compared to the way of modelling strategic behaviour with traditional game theory [22, 33], where each agent is assumed to behave rationally by calculating the best response to the current state of the game at all times. In contrary, in evolutionary games one usually introduces a evolutionary mechanism that is responsible for preventing the reproduction of some agents and favouring the reproduction of some other agents. Typically, agents with low fitness reproduce poorly, whereas agents with high fitness reproduce successfully. Finally, in the long-term, the population of agents may equilibrate into a state in which only a certain type of agent remains. Furthermore, these remaining agents who passed the evolutionary sieve may act strategically similarly to the highly rational agents of traditional game theory [15].

In order to specify what type of an agent would likely survive in a evolutionary game, it is customary to utilize the concept of evolutionary stable strategy (ESS) introduced by Maynard Smith and Price [32]. This concept can be illustrated as follows: assume that we can divide the agent population into two groups, the A type agents and the B type agents, both of which behave according to different patterns, A and B. Furthermore, assume that types A and B are in some sort of relationship, such that the expected payoff to an individual of type A from an interaction with an individual of type B is $E_B(A)$, and the expected payoff to B is $E_A(B)$. Then A is an ESS if $E_A(A) > E_A(B)$, with the additional requirement that if $E_A(A) =$

$E_A(B)$, then $E_B(A) > E_B(B)$. In other words, agents of type A playing against other agents of type A get at least as good payoff as any other type playing against A. Then type A is stable against 'invasion' by any type B, and in the case of a tie ($E_A(A) = E_A(B)$) A can defeat B on 'its own ground', i. e. , A gets better payoff from playing against B than B does from playing against itself. Thus, under selective pressure the population tends to become occupied by those agents who play according to the ESS.

The standard metaphor for the problem of cooperation is the prisoner's dilemma (PD) game [121, 124, 125, 126, 127, 128]. The PD is a two-player game that illustrates well the paradox of the evolution of cooperation. Each agent in the PD game can behave either cooperatively (C) or defectively (D) in an interaction with another agent. If both players choose C, both get a payoff of magnitude R; if one defects while the other cooperates, D gets the biggest payoff of the game, T, while C gets the smallest payoff, S. If both defect, both get P. The payoffs to the agents are illustrated in Table 3.1, where the first number in each entry indicates the payoff to the row agent and the second number indicates the payoff to the column agent. With

$$T > R > P > S \tag{3.1}$$

the paradox is evident: the strategy D is unbeatable, because independent of the choice of the other agent, playing D is the 'safe choice' that always yields a higher payoff than playing C ($P > S, T > R$). However, if both players play C, they get higher payoff than if both would play D.

The PD game can be illustrated with a situation in which two players decide whether or not to award the other player a price of worth $c > 0$. If a player awards the price to the other player, the donor will suffer a loss of c, whereas the recipient will benefit $b > c$ due to a third party who has increased the awarded amount for a token of altruistic behaviour. Mutual cooperation thus pays a net benefit of $R = b - c$, whereas mutual defection results in payoff $P = 0$. With unilateral cooperation, defection yields the highest payoff, $T = b$, at the expense of the cooperator bearing the cost $S = -c$. In fully mixed populations, i. e. , in populations where each agent can interact with any other agent, the only ESS outcome is to defect [34].

Another game that has been widely used in studies of cooperative behaviour is the snowdrift (SD) game, which is also known as the hawk-dove or the chicken game [32, 37, 129, 130]. The SD game is similar to the PD game, except for the order of payoffs in Table 3.1. In the SD game

$$T > R > S > P. \tag{3.2}$$

The SD game can be illustrated with a situation in which two cars are caught in a blizzard and there is a snowdrift blocking their way. The cars are equipped with

	D	C
D	P, P	T, S
C	S, T	R, R

Table 3.1: Payoffs to the agents in a two-player game. Agent 1 chooses an action from the rows and agent 2 from the columns. By convention, the payoff to the row agent is the first payoff given, followed by the payoff of the column agent.

shovels, and the drivers have two choices: either start shoveling the road open (C) or remain in the car (D). If the road is cleared, both drivers gain the benefit b of getting home. On the other hand, clearing the road requires some work, and a cost c can be assigned to it ($b > c > 0$). If both drivers are cooperative and willing to shovel, this workload is shared between them, and both of them gain total benefit of $R = c - b/2$. If both choose to defect, i.e. remain in their cars, neither one gets home and thus both obtain zero benefit $P = 0$. If only one of the drivers shovels, both get home, but the defector avoids the cost and gains the benefit $T = b$, whereas the cooperator's benefit is reduced by the workload, i.e. $S = b - c$. The best action depends on the action of the co-player: defect if the other player cooperates and cooperate if the other defects. A simple analysis shows that the game does not have an ESS [15], if the agents use only pure strategies.[1] This leads to stable existence of cooperators and defectors in well-mixed populations [37].

Other models that have been applied for modelling cooperative behaviour include the ultimatum game [131] and the public goods game [132]. In the ultimatum game, two players are offered a gift, provided they manage to share it. One of the players—the proposer—suggests how to split the offer, the other player—the responder—can either agree or reject the deal. In each case the decision is final. A rational player should accept the smallest positive offer, because the alternative is getting nothing, and correspondingly a rational proposer who believes that his/her opponent is rational should claim almost the whole sum. On the other hand, if the responder rejects, both players get nothing. In the public goods game, N players can either contribute or refuse to contribute to the common collection, by say one unit of money. Then these individual contributions are multiplied by a factor $1 < r < N$, and divided between all the players independent of their contribution. Thus, every individual player is better off defecting than cooperating, no matter what the other players will do. If $N = 2$, the public goods game reduces to the PD

[1]If an agent plays a pure strategy he/she can choose either to cooperate or to defect with probability one, but he/she is not allowed to use a strategy which mixes either of these actions with some probability $q \in (0, 1)$. See section 2.3.

game.

A common feature of all these models of cooperative behaviour is that for a rational agent it pays off to be defective rather than cooperative. By contrast, experiments on human players, for example, have proved that in real social situations people show cooperative behaviour [24, 122]. Thus, the theoretical models as such fail to describe the reality appropriately. However, if one adds some features to these theoretical models, they can more successfully account for the existence of cooperative behaviour. For instance, if the PD game is iterated such that the same agents have opportunities for repeated interactions, an agent who cooperates only with those agents who reciprocate cooperatively will be favoured by natural selection [121, 133]. The reciprocity can also be indirect, in which case cooperative strategies directed towards recipients that have helped others in the past are rewarded [134, 133]. The deficiency of these models of reciprocal interaction is that they assume that individuals can adopt more or less complex strategies that take into account the past history of their interactions with other individuals, or that the individuals are capable of recognizing other individuals in some depth to recognize their characteristics.

Another approach for modifying these theoretical models is to introduce some structure for the agent population. Several studies have revealed that especially a spatial structure in agent population usually helps sustain cooperative bevaviour. Typically, a spatially structured population consists of agents that are set on a two dimensional lattice. Then, the agents on the lattice interact with their nearest neighbours, and gain payoff from each pairwise interaction according to the payoffs of Table 3.1. Nowak *et al.* pointed out that if the payoffs correspond to the PD game, Eq. (3.1), spatial structure can sustain cooperation at considerable levels depending on the entries of the payoff matrix of Table 3.1 [36, 124]. The spatial structure has also proven to sustain cooperation in the ultimatum game [135] and in the public goods game [136, 137]. Even in studies where non-spatial factors have been believed of being responsible for the persistence of cooperative behaviour, it has turned out that without spatial structure, cooperation can not be maintained. For instance, one could think that if the players in a PD game contributed by a non-fixed amount, starting from a small contribution, and if the amount of investment could evolve in time, increased levels of cooperation would emerge. This feature was investigated by Doebeli and Knowlton in [138], and by Killingback *et al.* in Refs. [125, 126]. Results from these studies suggested that with variable investments, increased levels of cooperation can be obtained. However, Scheuring recently pointed out that the maintenance of cooperation in variable investment models is connected to the spatial structure of the agents in these models, and that in unstructured populations cooperative behaviour disappears [128]. In relation to the SD game, Hauert *et al.* lately found that spatial structure tends to inhibit cooperative behaviour instead of boosting it [37]. This result was surprising, as intermediate levels of cooperation persist in unstructured SD games, and the common tendency has appeared to be that spatial structure is usually beneficial for sustained levels of cooperation.

Hauert *et al.* obtained their result by applying the usual dynamical rule of evolutionary game theory, the so-called replicator dynamics [15], in a slightly varied form to the agent population. This mechanism can be viewed as depicting Darwinian evolution, where the fittest have the largest chance of survival and reproduction, the success being the better, the better an individual agent's fitness is compared to the average fitness of the whole population of agents:

$$\frac{1}{F_i}\frac{dF_i}{dt} = U_i(F) - \sum_{i=1}^{N} F_i U_i(F), \quad i = 1, \ldots, N. \tag{3.3}$$

Eq. (3.3) describes time evolution of the population densities of N types of agents, when F_i is the proportion of agents of type i, $\sum_{i=1}^{N} F_i = 1$, and $U_i(F)$ is the expected fitness of agents of type i, when the composition of the population is $F = (F_1, \ldots, F_N)$.

The use of replicator dynamics is justified if one considers time scales that are much longer than the lifetime of an individual agent. However, living organisms act on time scales that are shorter than their own lifetime. In relation to this, one can ask whether cooperative behaviour could arise as a consequence of the adaptive behaviour of the individual agents when they try to succeed in their environment. Existence of cooperative behaviour could then be a manifestation of self-organization in a complex adaptive system, possibly realized through some sort of learning, rather than a consequence of natural selection that has favoured 'cooperative genes'.

In order to investigate whether the adaptive responses of agents on relatively short time scales can give rise to cooperative behaviour—and if yes, to what extent—we studied a spatial snowdrift game in (Publication VI). We chose to consider the spatial SD, because the results of Hauert *et al.* in [37] provided a ready basis for comparative study. Our approach for modelling the effect of spatial structure to the SD was similar to that of Hauert *et al.* , except that we replaced replicator dynamics with an adaptive rule which models short-term responses to the environment. Our adaptive rule governs the decisions of the agents at consecutive time steps, instead of the reproductive success of different strategy genotypes in consecutive agent generations. One can think that the adaptive agents in our model are endowed with primitive intelligence—they consider the state of their environment fixed, and decide what their strategy should be to get the most payoff. The adaptive rule that we have applied to the spatial SD is similar to the rule that was applied to the MG by Reents *et al.* [104] (see last paragraph of section 2.8.1). Our numerical results and analytic inference indicate that application of such adaptive rule results in cooperation levels which differ to a large extent from those obtained using the replicator dynamics. Next, we present our version of spatially structured snowdrift game and summarize results obtained from its analysis.

3.2 Spatial snowdrift game with myopic agents

In order to study the effect of spatial structure on the SD game, we set the agents on a regular two-dimensional square lattice consisting of m cells. We identify each cell by an index $i = 1, \ldots, m$ which also refers to its spatial position. Each cell, representing an agent, is characterized by its strategy s_i, which can be either to cooperate ($s_i = 1$) or to defect ($s_i = 0$). The spatio-temporal distribution of the agents is then described by $S(t) = (s_1(t), \ldots, s_m(t))$. Then each agent interacts with his/her n nearest neighbours. We use either a Moore neighbourhood, in which case each agent has $n = 8$ neighbours, or a von Neumann neighbourhood in which case each agent has $n = 4$ neighbours. We require that an agent plays simultaneously with all his/her n neighbours, and define the payoffs for this $(n+1)$–player game such that an agent i who interacts with n_c^i cooperators and n_d^i defectors, $n_c^i + n_d^i = n$ gains a benefit of

$$\begin{aligned} u_i(s_i = 0) &= n_c^i T + n_d^i P \quad (3.4) \\ u_i(s_i = 1) &= n_c^i R + n_d^i S \quad (3.5) \end{aligned}$$

from defecting or cooperating, respectively. The letters in Eqs. (3.4) and (3.5) refer to the payoffs in Table 3.1 with the ordering of Eq. (3.2).

For determining their strategies, the agents are endowed with primitive decision-making capabilities. The agents retain no memory of the past, and are not able to predict how the strategies of the neighboring agents will change. Every agent simply assumes that strategies of other agents within his/her neighborhood remain fixed, and chooses an action that maximizes his own payoff. In this sense the agents are myopic. The payoff is maximized, if an agent (a) defects when $u_i(0) > u_i(1)$, and (b) cooperates when $u_i(1) > u_i(0)$. If (c) $u_i(0) = u_i(1)$ the situation is indifferent. Using Eqs. (3.4) and (3.5) we can connect the preferable choice of an agent and the payoffs of the game. Let us denote

$$\frac{1}{r} = 1 + \frac{S - P}{T - R}. \quad (3.6)$$

Then, if

$$\frac{n_c^i}{n} > 1 - r \text{ defecting is profitable, or if} \quad (3.7)$$

$$\frac{n_c^i}{n} < 1 - r \text{ cooperating is profitable, or if} \quad (3.8)$$

$$\frac{n_c^i}{n} = 1 - r \text{ choices are indifferent}. \quad (3.9)$$

Thus, for each individual agent, the ratio r determines the following decision boundary:

$$\theta = n(1 - r), \quad (3.10)$$

which depends on the neighbourhood size n and the "temptation" parameter r. Because r is determined only by the differences $T - R$ and $S - P$, we can fix two of the payoff values, say $R = 1$ and $P = 0$. Based on the above, we define the following rules for the agents:

1. If an agent i plays at time t a strategy $s_i(t) \in \{0, 1\}$ for which $u_i(s_i) \geq u_i(1 - s_i)$, then at time $t + 1$ the agent plays $s_i(t + 1) = s_i(t)$.

2. If an agent i plays at time t a strategy $s_i(t) \in \{0, 1\}$ for which $u_i(s_i) < u_i(1 - s_i)$, then at time $t + 1$ the agent plays $s_i(t + 1) = 1 - s_i(t)$ with probability p, and $s_i(t + 1) = s_i(t)$ with probability $1 - p$.

Hence, the strategy evolution of an individual agent is determined by the current strategies of the other agents within his/her neighborhood, with the parameter p acting as a "regulator" which moderates the rate of changes.

3.3 Summary of results

In (Publication VI) we have studied the above described spatial snowdrift model. We have specifically analyzed the behavior of the cooperator density F_c, and equilibrium lattice configurations both by analytic reasoning and by numerical simulations.

We have shown analytically that

$$\frac{1 - r}{2 - r} \leq F_c \leq \frac{1}{r + 1}. \quad (3.11)$$

This result holds for any lattice or network that has a fixed number of nearest neighbours per lattice site or node. Moreover, by studying local equilibrium configurations of elementary blocks, like those in Fig. (3.1), on an infinite lattice, we obtained the stricter limits of Table 3.2 for F_c.

i	r_l	r_u	$N_{c\|d} \geq$	$N_{c\|c} \leq$	$F_{c,L}$	$F_{c,U}$
1	0	$1/8$	8	7	$3/4$	$8/9$
2	$1/8$	$2/8$	7	6	$2/3$	$4/5$
3	$2/8$	$3/8$	6	5	$1/2$	$2/3$
4	$3/8$	$4/8$	5	4	$1/2$	$2/3$
5	$4/8$	$5/8$	4	3	$4/9$	$1/2$
6	$5/8$	$6/8$	3	2	$1/3$	$1/2$
7	$6/8$	$7/8$	2	1	$2/9$	$1/3$
8	$7/8$	$8/8$	1	0	$1/9$	$1/4$

Table 3.2: Limits for the equilibrium fraction of cooperators based on repeating elementary configuration blocks. When $r_l < r < r_u$, the number of cooperators in each defector's neighborhood $N_{c|d}$ must be at least $9 - i$ and the number of cooperators in each cooperator's neighborhood $N_{c|c}$ at most $8 - i$. Considering possible repeating configuration blocks which fulfill these conditions, we obtain lower limits $F_{c,L}$ and upper limits $F_{c,U}$ for the density of cooperators.

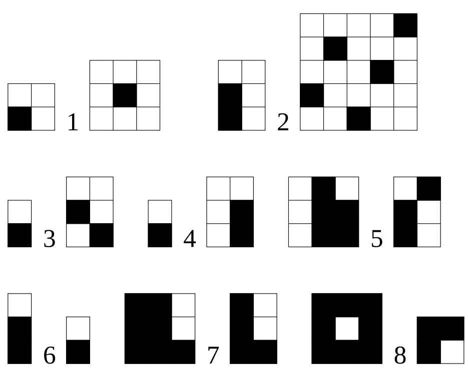

Figure 3.1: Examples of elementary configuration blocks which can be repeated without overlap to fill an infinite lattice, for various values of r. The numbering refers to i in Table 3.2. A black cell denotes a defector while an empty cell denotes a cooperator. For a particular number the lower limit of density is obtained by filling the lattice with the blocks on the left, and the upper by using the blocks on the right.

In numerical simulations, we have used random sequential updating so that each simulation round, the strategies of all agents are updated in random order. Each strategy is updated using the above rules, which assume that the strategies in the neigbourhood of an agent remain fixed. One simulation round consists of going through all the agents in this fashion. In the following, the time scale is defined in terms of these simulation rounds.

Fig. (3.2) shows results from numerical simulations with $m = 100 \times 100$-lattice with periodic boundary conditions for a Moore neighborhood ($n = 8$). In the middle panel we show F_c as a function of r of Eq. (3.6). The dotted lines in the middle panel indicate the upper and lower limits of Eq. (3.11), the solid lines indicate the upper and lower limits of Table 3.2, and the dashed diagonal line is $F_c = 1 - r$, corresponding to the fraction of cooperators in the fully mixed case [25, 37, 15]. The fraction of cooperators $\langle F_c \rangle$ is seen to follow a stepped curve, with steps corresponding to $r = i/n$, where $i = 0, \ldots, n$. This is a natural consequence of Eqs. (3.7)-(3.8), where the decision boundary $\theta = n(1 - r)$ can take only discrete values.

The peripheral panels of Fig. (3.2) depict the central part of the 100×100-lattice after 1000 simulation rounds using the Moore neighborhood and $p = 0.1$, with white pixels corresponding to cooperators and black pixels to defectors. The values of r have been selected so that the equilibrium situation corresponds to each plateau of F_c illustrated in the central panel. The observed configurations are rather polymorphic, and repeating elementary patterns like those in Fig. (3.1) are not seen. This reflects the fact that the local equilibrium conditions can be satisfied by various configurations; the random initial configuration and the asynchronous update then lead to irregular-looking equilibrium patterns, which vary between simulation runs. The patterns seem to be most irregular when r is around 0.5; this is because then the equilibrium numbers of cooperators and defectors are close to each other, and the ways to assign strategies within local neighborhoods are the most numerous. To be more exact, there are $\binom{8}{i}$ ways to distribute i cooperators in the 8-neighborhood, and if e.g. $3/8 < r < 4/8$, i is at least 4 and at most 5, maximizing the value of the binomial coefficient. Hence, the ways of filling the lattice with these neighborhoods in such a way that the equilibrium conditions are satisfied everywhere are most numerous as well.

Results obtained with a von Neumann neighbourhood ($n = 4$) are qualitatively similar to those we have reported for a Moore neighbourhood ($n = 8$). Further discussion can be found in (Publication VI).

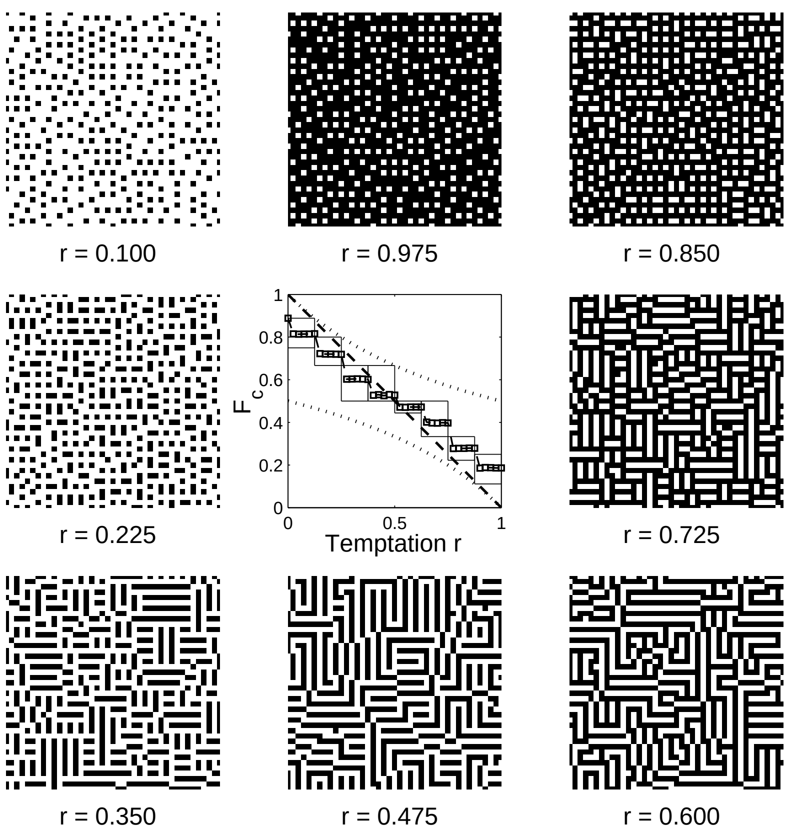

Figure 3.2: Example equilibrium configurations of defectors and cooperators on a $m = 100 \times 100$ lattice for various values of r when a Moore neighborhood is used. The configurations were recorded after $T = 1000$ simulation rounds. Only the central part of the lattice is shown for the sake of clarity. The middle panel depicts the average fraction of cooperators F_c in the whole population as a function of the temptation r (squares), together with the upper and lower limits of Eq. (3.11) (dotted lines) and the limits of Table 3.2 (solid lines). The values of F_c are averages over the last 500 simulation rounds and the dashed diagonal line is $F_c = 1 - r$, corresponding to the fraction of cooperators in the fully mixed case.

3.4 Conclusions

The equilibrium densities of cooperators that we have observed differ largely from those resulting from applying the replicator dynamics [37]. With our strategy evolution rules, cooperation persists through the whole temptation parameter range as can be seen from Fig. (3.2). By contrast, with the replicator dynamics based approach the fraction of cooperators in spatially structured populations is below the fraction of cooperators in fully mixed populations for a wide range of temptation parameter values. Furthermore, with replicator dynamics cooperators totally vanish from spatially structured populations when r is larger than a critical r_c [37]. Hence, we argue that no conclusions on the effect of spatiality on the snowdrift game can be drawn without taking into consideration the strategy evolution mechanism; local decision-making in a restricted neighborhood yields results which are different from those resulting from the evolutionary replicator dynamics. This should, in principle, apply for other spatial games as well. Care should especially be taken when interpreting the results of investigations on such games: the utilized strategy evolution mechanism should reflect the system under study. We argue that especially when modeling social or economic systems, there is no *a priori* reason to assume that generalized conclusions can be drawn based on results using the evolution inspired replicator dynamics approach, where high-payoff strategies get copied and "breed" in proportion to their fitness. As we have shown here, local decision-making with limited information (neighbor strategies are known payoffs are not) can result in different outcome.

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