Transfer matrix for the hexagonal self-avoiding walk

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ABSTRACT OF THE
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In the thesis I show that a self-avoiding walk in the hexagonal lattice can be defined as a sequence of configurations indexed by height. Using these configurations I introduce a transfer matrix formulation for self-avoiding walks in rectangular subdomains of the lattice with endpoints fixed to the top and bottom of the domain. The transfer matrix allows me to calculate visiting probabilities of the self-avoiding walk in an explicit form. The eigensystem of the transfer matrix makes it possible to calculate the same probabilities in an infinitely high rectangle or vertical strip. I map the infinitely high vertical strip to the half-plane and compare the edge visiting probabilities of the critical self-avoiding with the conjectured scaling limit, the conformally invariant stochastic Löwner evolution curve $SLE_{8/3}$. I also recall the proof of the connective constant of the hexagonal lattice that defines the critical self-avoiding walk needed in the thesis.

Keywords: Self-avoiding walk, SAW, hexagonal lattice, connective constant, transfer matrix, SLE, conformal invariance, scaling limit, statistical mechanics, probabilistic methods

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Osoitan diplomityössäni, että itseään välttävä kävely voidaan määritellä kuusikolmiohilassa korkeuden avulla indeksoituna konfiguraatiojonona. Esitän konfiguraatioita hyödyntäen siirtomatriisiformulaation suorakulmion muotoisen alueen pohjasta ylälaitaan kulkeville itseään välttäville kävelyille. Siirtomatriisin avulla pystyn laskemaan itseään välttävän kävelyn vierailutodennäköisyyksiä eksplisiittisesti. Siirtomatriisin ominaisavaruuden avulla pystyn laskemaan samat vierailutodennäköisyydet myös, kun suorakulmiota kasvatetaan äärettömän korkeaksi liuskaksi. Kuvaan kriittisen itseään välttävän kävelyn äärettömän korkeasta liuskasta konformisti puolitasolle ja vertaan reunavierailutodennäköisyyksiä konjekturoituun skaalausrajaan, konformi-invarianttiin stokastiseen Löwner-evoluutiokäyrään $SLE_{8/3}$. Kertaan myös todistuksen työssä tarvittavalle kriittisen itseään välttävän kävelyn määrittävälle hilavakiolle kuusikulmiohilassa.

Avainsanat: Itseään välttävä kävely, siirtomatriisi, todennäköisyys, kuusikulmiohila, hilavakio, tilastollinen fysiikka, SLE, skaalausraja, konformiinvarianssi

Preface

I want to express my gratitude to Prof. Kalle Kytölä for introducing this interesting topic to me and for his patient supervision. I also want to thank Univ. Lecturer Harri Varpanen for the mentoring and the countless cups of tea he has offered me during the recent years. Additionally Prof. Akira Sakai of Hokkaido University deserves a special mention for giving me a wider perspective to the topic during my half-year exchange period in Sapporo, Japan.

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Tommi M. S. Laine

Contents

1 Introduction

Recently there has been much interest in relating two-dimensional discrete models of statistical mechanics to conformally invariant Schramm-Löwner Evolution curves. Such results have been established for the loop-erased random walk [LSW04a], the critical percolation [Smi01] and the Ising model [Smi06]. It has been conjectured that in the half-plane the continuum, or scaling, limit of the critical self-avoiding walk is the chordal Schramm-Löwner curve $SLE_{8/3}$ [LSW04b]. A motivation for this thesis is to provide additional computational support for this conjecture.

To reach this aim I first introduce a probabilistic model called the self-avoiding walk, and define the critical version of it. The critical self-avoiding walk (SAW) depends on a lattice-specific connective constant, which is generally not known, however the hexagonal lattice provides an exception due to recent efforts. I consider the critical self-avoiding walk in vertical strip domains of the hexagonal lattice. In vertical strips, a self-avoiding walk γ conditioned to proceed from the bottom of the strip to the top can be bijectively associated to a sequence of configurations $(\gamma_h)_{h\in\mathbb{Z}}$ it assumes at heights $h \in \mathbb{Z}$. This allows the expression of generating functions for the SAW in terms of a transfer matrix defined for the basis of configurations. I prove that the probability of the walk hitting the right boundary of the strip at heights 0 and $h > 0$ can be calculated using the largest eigenvalue and the corresponding eigenvectors of the transfer matrix. These can be numerically computed, which, after mapping the strip to the half-plane, allows comparison with the two-point function of the chordal $SLE_{8/3}$. I conclude by presenting the results obtained for different heights by combining data for strips of various mesh sizes.

Simply put, self-avoiding walks are walks on adjacent vertices of a lattice that never return to points they have once visited. The scientific motivation for self-avoiding walks (SAWs for short) first came from physical chemistry as Paul Flory[Flo53] idealized the formation of linear polymers in solvents. It is natural to think of molecules as objects taking up volume in space, thus monomers in a polymer chain are unable to overlap. However it should be noted that the presence of a good solvent is a critical assumption in the model. Without a good solvent, intermolecular forces tend to coil up the polymer chains, making even the simple random walk a model that produces reasonable results. Self-avoiding walks have also been used to study protein folding. The development of fractures in materials, lightning bolts striking ground and the growth of vines along walls can also be thought of as self-avoiding walks.

The probabilistic model for self-avoiding walks assumes that walks have a fixed starting point, and that the probability of a walk is proportional to some variable z to the power of the length of the walk in steps. Walks of the same length are assumed to have equal probability. This model favours shorter walks when z is smaller than some critical value or subcritical, longer walks when z is greater than the critical value or supercritical, and gives almost uniform distribution of walk lengths when z equals the critical value z_c. The self-avoiding walk model with $z = z_c$ is called the critical self-avoiding walk.

The mathematical motivation for studying self-avoiding walks comes from the efforts to prove the conjectured conformal invariance of the scaling limit, and from the fact that SAWs are one of the simplest poorly understood models of statistical mechanics. For example even getting sharp estimates for the number c_n of n-step self-avoiding walks with a fixed starting point has proven to be a difficult question sparking a lot of mathematical interest. This number can be shown to behave roughly exponentially in terms of the lattice dependent constant $\mu_c = z_c^{-1}$, but the value of the constant z_c is usually unknown[Sla06]. Conformal invariance means that for every simply connected domain Ω and every lattice approximation Ω_δ of mesh size δ of Ω, the measure $\mathbb{P}_{z_c,\Omega_{delta}}$ of self-avoiding walks γ_δ on Ω_δ converges to some measure $\mathbb{P}'_\Omega$ of self-avoiding paths γ on the set Ω as δ goes to zero and if f is a conformal mapping, the probability measure of the paths $f(\gamma)$ on $f(\Omega)$ is $\mathbb{P}'_{f(\Omega)}$.

The critical self-avoiding walk requires knowledge of the lattice-dependent constant $\mu_c = \frac{1}{z_c}$, which was recently solved rigorously by Hugo Duminil-Copin and Stanislav Smirnov for the hexagonal, or honeycomb lattice [DCS12]. After a review of their proof in the third section the remainder of the thesis focuses on self-avoiding walks on the hexagonal lattice as the critical value z_c has been established.

To compare the behaviour of the critical self-avoiding walk (SAW) to the half-plane SLE curve, a domain for the SAW that is conformally mapped to be half-plane needs to be chosen. Since the scaling limit of the critical self-avoiding is conjectured to be conformally invariant, in principle any simply connected set in the hexagonal lattice that can be conformally mapped to the half-plane is suitable for analysis. The fourth section of the thesis deals with self-avoiding walks in vertical strip domains. The choice of a vertical strip domain of finite height for analysing self-avoiding walks with fixed end points at the bottom and top of the domain makes it to possible to bijectively associate to every SAW in the strip a sequence of configurations, indexed by height, that keep track of the trajectory of the SAW below that height. This then makes it possible to express generating functions as matrix products in the space of configurations, by using what is called a transfer matrix. It turns out that the transfer matrix is essentially aperiodic and irreducible. Self-avoiding walks in an infinitely tall strip can then be analyzed by numerically solving the largest eigenvalues and corresponding eigenvectors of the transfer matrix.

In the final section I show how to assemble the basis of configurations and the transfer matrix, and using the transfer matrix and the critical value z_c, I compute the probabilities of a self-avoiding walk visiting the right edge of the strip at height 0 and returning to the right edge at higher height, and compare these with the two-point function for $SLE_{8/3}$ given by Kytölä, Jokela and Järvinen[JJK15].

2 Preliminaries

Definition 2.1. A *lattice graph* is a graph whose tiling forms a regular tiling when embedded in an Euclidean space $\mathbb{R}^n$. A lattice graph is from now on referred to simply as a *lattice*.

Definition 2.2. Fix a lattice $\mathbb{L}$, e.g. hypercubic lattice, triangular lattice or hexagonal lattice: A *self-avoiding walk* is a self-avoiding sequence of adjacent vertices on the lattice. We denote adjacency of vertices by the symbol $\sim$. An n-step self-avoiding walk(SAW for short) starting at vertex x is a sequence of vertices $(\gamma(0), \gamma(1), \ldots, \gamma(n))$ with $\gamma(0) = x$, $\gamma(j+1) \sim \gamma(j)$, and $\gamma(i) \neq \gamma(j)$ for $i \neq j$. Denote the number of steps, or length, of γ by $l(\gamma) = n$. Define the set of n-step self-avoiding walks from x to y by

$$C_n(x, y) := \{\gamma = (\gamma(0), \gamma(1), \ldots, \gamma(n)) |\ \gamma\ SAW, \gamma(0) = x, \gamma(n) = y\},$$

and the number of n-step self-avoiding walks from x to y by

$$c_n(x, y) := |C_n(x, y)|,$$

with the convention that $c_0(x, y) = \delta_{x,y}$. If the lattice $\mathbb{L}$ is placed such that the origin is a vertex and $V(\mathbb{L})$ denotes the set of vertices of $\mathbb{L}$, by convention

$$c_n(0, x) := c_n(x), \sum_{x \in V(\mathbb{L})} c_n(x) := c_n.$$

A fundamental, yet difficult problem is the number of n-step walks c_n. Ever since Flory introduced the model in 1953 there has been interest in knowing the behaviour of c_n as a function of n. The answer depends on the dimension and the lattice. It is however not difficult to prove exponential growth for c_n by bounding it from above by the number of simple walks, allowed to self-intersect freely, and from below by walks with a fixed propagation direction.

Any SAW of length $n + m$ can be obtained by trying to attach a self-avoiding walk of length n to the end of a SAW of length m, but the result is not always self-avoiding. This implies that self-avoiding walks have the subadditivity property:

$$c_{m+n} \leq c_m c_n.$$

Lemma 2.3 (Fekete's lemma)**.** *If a sequence of real numbers $(a_n)_{n \in \mathbb{N}}$ is subadditive, i.e. $a_{m+n} \leq a_m + a_n$, there exists a limit*

$$\lim_{n \to \infty} \frac{a_n}{n} = \Phi \in [-\infty, +\infty)$$

and for all n we have $a_n \geq n\Phi$.

Subadditivity for the logarithm of the number of walks c_n implies by Fekete's lemma that there exists a limit

$$\mu_c := \lim_{n \to \infty} \sqrt[n]{c_n}.$$

Furthermore Fekete's lemma says that μ_c is the infimum of the sequence $(\sqrt[n]{c_n})_{n\in Z_{\geq 0}}$.

The constant μ_c is called the connective constant of the lattice. It follows that the number of n-step self-avoiding walks behaves as $c_n = \mu_c^{n+o(n)}$. For nontrivial lattices, the exact value of μ_c has only been established for the hexagonal lattice in the plane, for which Smirnov and Duminil-Copin showed that $\mu_c = \sqrt{2+\sqrt{2}}$ [DCS12].

Definition 2.4. The two important functions for the self-avoiding walk from the point of view of statistical mechanics are the *two-point function*

$$G_z(x,y) := \sum_{n=0}^{\infty} c_n(x,y)z^n, \quad G_z(x) = G_z(0,x)$$

and the *susceptibility*

$$\chi(z) := \sum_{x\in V(\mathbb{L})} G_z(x) = \sum_{n=0}^{\infty} c_n z^n.$$

Theorem 2.5. *The susceptibility χ has the radius of convergence $z_c = 1/\mu_c$.*

Proof. If $z \geq 1/\mu_c$, we have

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n \geq \sum_{n=0}^{\infty} \mu_c^n/\mu_c^n = \infty.$$

If $z < 1/\mu_c$, we first note that by Fekete's lemma and the definition of μ_c as the limit of the sequence $\left(c_n^{1/n}\right)$, for every $\varepsilon > 0$ there exists an index N_ε such that $c_n \leq (\mu+\varepsilon)^n$ whenever $n > N_\varepsilon$. Define

$$K_\varepsilon := \max\left[1, \max_{n\in\{1,\ldots,N_\varepsilon\}} \frac{c_n}{(\mu_c+\varepsilon)^n}\right]$$

to get that for every $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ such that $c_n \leq K_\varepsilon(\mu+\varepsilon)^n$ for every nonnegative n. Now we note that $z < 1/\mu_c = \frac{1}{\mu_c+\delta}$ for some $\delta > 0$. By choosing the constant ε to be smaller than δ, we then have

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n \leq K_\varepsilon \sum_{n=0}^{\infty} \left(\frac{\mu_c+\varepsilon}{\mu_c+\delta}\right)^n < \infty.$$

□

Remark 2.6. It can also be proven that for a fixed x the two-point function $G_z(x)$ has the same radius of convergence z_c in z as the susceptibility χ. A proof can be found in [MS93].

Definition 2.7. Fix a lattice $\mathbb{L}$ and the value of $z < \frac{1}{\mu_c}$. Then the two-point function G_z defines a probability measure between any points $x, y \in V(\mathbb{L})$ for self-avoiding paths γ between the points.

$$\mathbb{P}_{z,x,y}[\gamma] = \frac{z^{l(\gamma)}}{G_z(x,y)}.$$

Definition 2.8. The *mesh size* of the lattice means the (largest) Euclidean distance between two neighbouring vertices of the lattice.

There is a version of definition 2.7 more suited to our needs.

Definition 2.9. Let Ω be a simply connected domain in $\mathbb{R}^d$, with points x, y on the boundary. For $\delta > 0$ let Ω_δ be the largest connected component of $\delta\mathbb{L} \cap \Omega$ and let x_δ, y_δ be the two sites closest to x and y. The triplet $(\Omega_\delta, x_\delta, y_\delta)$ is then an approximation of (Ω, x, y).
Let $z > 0$ and define the two-point function for Ω_δ as

$$G_{z,\Omega_\delta}(x, y) = \sum_{\gamma \subset \Omega_\delta : x_\delta \to y_\delta} z^{l(\gamma)},$$

where the length of each walk γ is measured in the number of steps it takes in $\delta\mathbb{L}$. The associated probability measure is

$$\mathbb{P}_{z,\Omega_\delta,x,y}[\gamma] = \frac{z^{l(\gamma)}}{G_{z,\Omega_\delta}(x, y)}.$$

Conjecture 2.10. *The set of self-avoiding walks with probability distribution*

$$\mathbb{P}_{z_c,\Omega_\delta,x,y}[\gamma] = \frac{z_c^{l(\gamma)}}{G_{z_c,\Omega_\delta}(x, y)}$$

is called the critical self-avoiding walk. The critical SAW in the complex plane is conjectured to be conformally invariant as $\delta \to 0$. This means that there exists a probability measure $\mathbb{P}'$ on self-avoiding paths in $\mathbb{C}$ such that:

- *For every $\Omega \subsetneq \mathbb{C}$ and $x, y \in \partial\Omega$ the distribution $\mathbb{P}_{z_c,\Omega_\delta,x,y}[\gamma]$ converges to $\mathbb{P}'_{\Omega,x,y}[\gamma]$ as δ tends to zero.*
- *For every pair of open sets $\Omega, \Omega' \subsetneq \mathbb{C}$, boundary points $x, y \in \partial\Omega$ and conformal mapping $f : \Omega \to \Omega'$ we have that if γ has the law $\mathbb{P}'_{\Omega,x,y}$, then $f(\gamma)$ has the law $\mathbb{P}'_{f(\Omega),f(x),f(y)}$. This is called conformal invariance.*

The two propositions of the conjecture are illustrated in fig. 1.

Theorem 2.11. *[LSW04b] If the two propositions of conjecture 2.10 are true, then the scaling limit of the critical self-avoiding walk is the chordal $SLE_{8/3}$.*

Proposition 2.12. *The probability measure $\mathbb{P}_{z,\Omega_\delta,x,y}[\gamma]$ exhibits a phase transition as δ tends to 0. Here again the critical value z_c is the same μ_c^{-1} as before. When $z < z_c$ a walk is penalized by its length, and γ_δ converges to the geodesic between x and y as δ tends to 0 [Iof98]. When $z > z_c$, a walk is favored by its length, and the probability of γ_δ not intersecting any open set $U \subset \Omega$ tends to zero [DCKY14]. Finally when $z = z_c$, γ_δ converges to a simple random curve, and if the scaling limit of the SAW exists, this curve is the Schramm-Löwner evolution $SLE_{8/3}$ [LSW04b].*

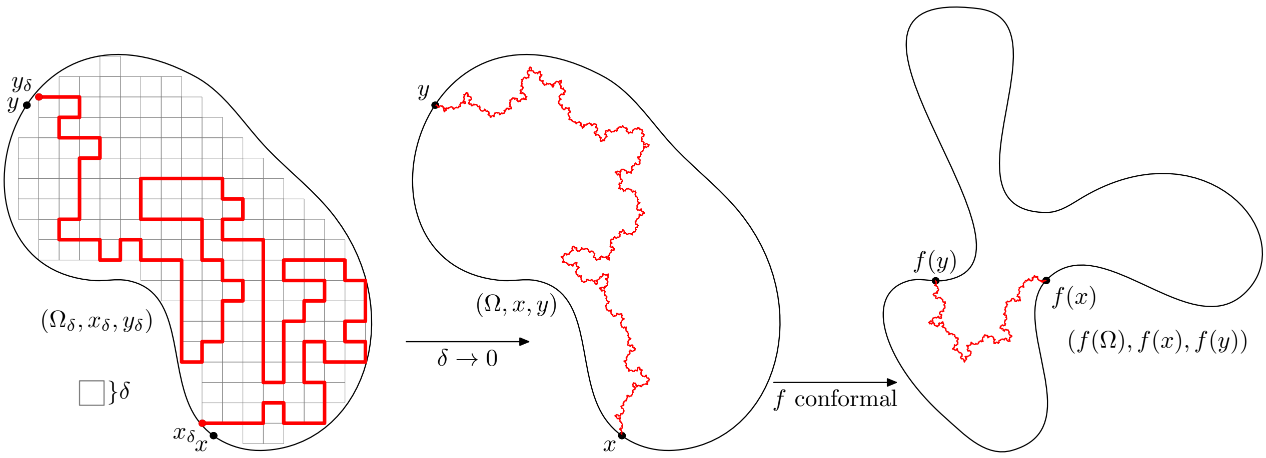

Figure 1: Conjectured conformal invariance: $\gamma_\delta \sim \mathbb{P}_{z_c,\Omega_\delta,x_\delta,y_\delta} \xrightarrow[\delta\to 0]{} \gamma \sim \mathbb{P}'_{\Omega,x,y}$ and if f is conformal, then the images $f(\gamma)$ of the self-avoiding paths γ have the distribution that one gets by taking the limit $\delta \to 0$ of $\mathbb{P}_{z_c,(f(\Omega))_\delta,(f(x))_\delta,(f(y))_\delta}$: $f(\gamma) \sim \mathbb{P}'_{f(\Omega),f(x),f(y)}$.

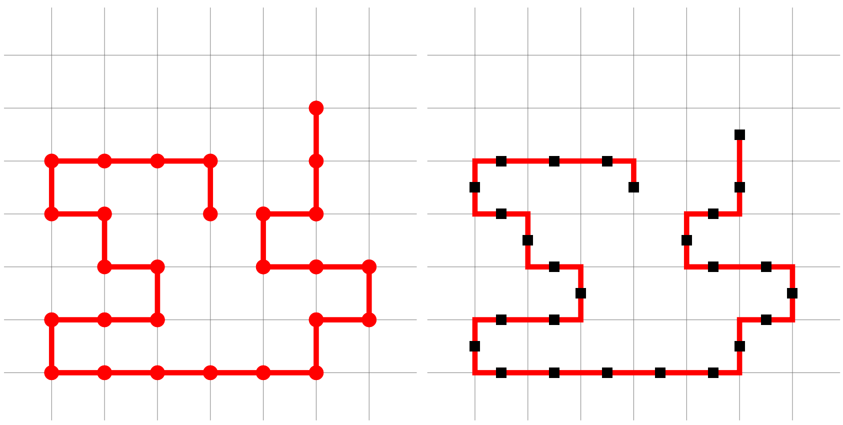

Figure 2: A self-avoiding walk γ on the edges of the square lattice and a self-avoiding walk $\tilde{\gamma}$ on the midedges of the lattice.

Definition 2.13. A *midedge* is the point halfway between two adjacent vertices on the lattice, in the middle of the edge connecting these vertices.

Remark 2.14. To each N-step self-avoiding walk γ defined as above, it is possible to associate an $N-1$-step self-avoiding walk $\tilde{\gamma}$ on midedges with

$$\tilde{\gamma}(i) = \frac{\gamma(i) + \gamma(i+1)}{2}, \quad i = 0, 1, \ldots, N-1.$$

Between these points, the walk $\tilde{\gamma}$ is drawn such that the walks γ and $\tilde{\gamma}$ have the same trajectory between points $\tilde{\gamma}(0)$ and $\tilde{\gamma}(N-1)$. An example is shown in fig 2. This thesis deals with self-avoiding walks on midedges.

Remark 2.15. Let $\widetilde{c}_n$ denote the number of n-step midedge-to-midedge self-avoiding walks and let e be the number of edges adjacent to a vertex. By splitting the midedge-to-midedge walks into the first half edge, the last half edge and the $n-1$-step vertex-to-vertex midsection, one gets the bounds

$$2 \cdot \frac{e-1}{e} c_{n-1} \leq \widetilde{c}_n \leq 2 \cdot \frac{e-1}{e} c_{n-1} \cdot (e-1).$$

In particular this implies that the connective constant is the same for midedge-to-midedge and vertex-to-vertex self-avoiding walks.

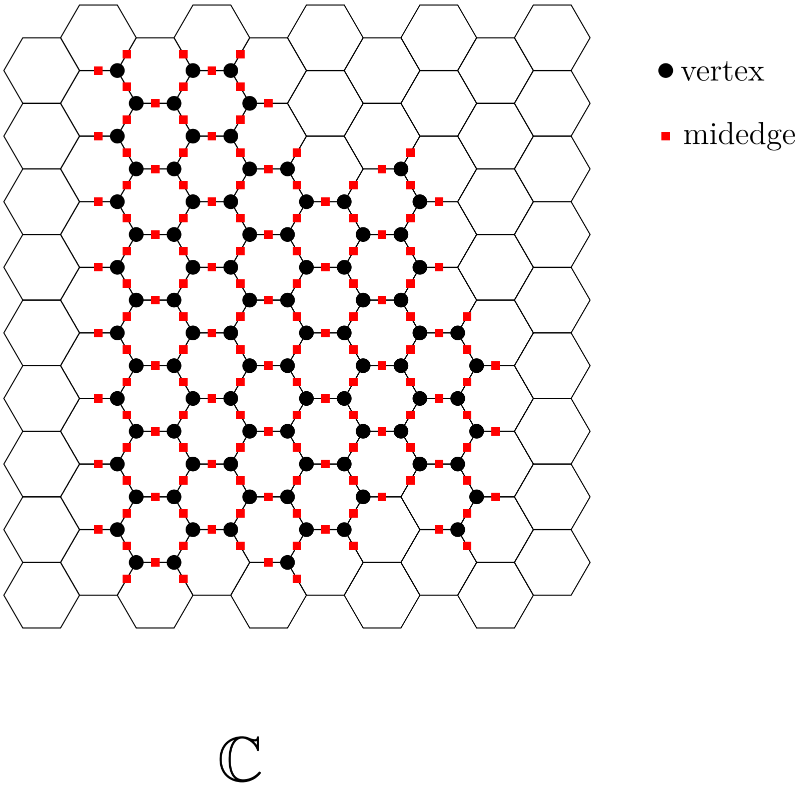

Figure 3: A set of vertices V and emanating midedges

3 Complex analysis in the honeycomb lattice

In this section we introduce a parafermionic observable F, which is a weighted version of the two-point function $G_{z,\Omega}$ of the self-avoiding walk, also called the partition function or the generation function for an enumeration of walk lengths, in a simply connected region of a hexagonal discretization of the complex plane. It turns out that discrete integrals of this observable F along the elementary contours of the dual lattice vanish. This implies that the integral of F along any closed contour in the dual lattice vanishes. By choosing an appropriate contour to integrate along, we review the proof of Nienhuis' prediction[Nie82] that the connective constant μ_c for the hexagonal lattice equals $\frac{1}{\sqrt{2+\sqrt{2}}}$ by Smirnov and Duminil-Copin [DCS12].

3.1 Parafermionic observable

Consider the complex plane $\mathbb{C}$ and the hexagonal lattice $\mathbb{H}$ with edge length 1 embedded on it. Choose a set of vertices V from $\mathbb{H}$ such that the vertices and the midedges of the edges emanating from V form a simply connected graph $\Omega(V)$.

Define the set Ω of midedges as follows: The midedge x belongs to Ω if at least one of its adjacent vertices is in the set V and to the boundary $\partial\Omega$ of Ω if only one of its adjacent vertices is in V. This makes the boundary $\partial\Omega$ a subset of Ω, meaning

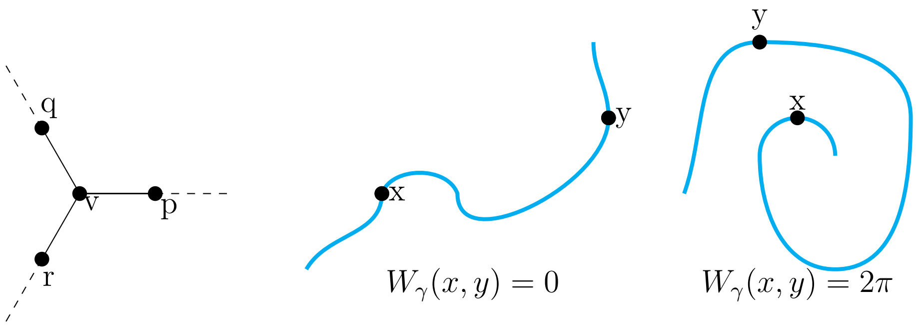

Figure 4: Ordering of the midedges p, q, r around vertex v, and an example of the winding $W_\gamma(x, y)$ for two different walks.

that the set Ω is closed.

Definition 3.1. For a self-avoiding walk γ traversing through midedges x and y we define the *winding* of γ, denoted by $W_\gamma(x, y)$, as the cumulative angle of turns the walk makes between x and y in radians. See also figure 4. The function W_γ has the following properties for $x, x', x'' \in \gamma$:

$$\begin{aligned} W_\gamma(x, x) &= 0 \\ W_\gamma(x, x') &= -W_\gamma(x', x) \\ W_\gamma(x, x') + W_\gamma(x', x'') &= W_\gamma(x, x'') \end{aligned}$$

Definition 3.2. Fix $x \in \partial\Omega$, $z > 0$ and $\sigma \in \mathbb{R}$. For a midedge $y \in \Omega$ we define the *parafermionic observable* F as

$$F(y) = F_\Omega(x, y, z, \sigma) = \sum_{\gamma \subset \Omega : x \to y} e^{-i\sigma W_\gamma(x,y)} z^{l(\gamma)}.$$

In the hexagonal lattice every turn counterclockwise contributes $+\pi/3$ to the winding and every turn clockwise contributes $-\pi/3$ to the winding. Therefore, the complex weight $e^{-i\sigma W_\gamma(x,y)}$ is a product of terms $e^{-i\sigma\pi/3}$ and $e^{+i\sigma\pi/3}$, where the former corresponds to a single turn in the positive or counterclockwise direction and the latter to a clockwise turn.

Lemma 3.3. *If $z = z_c = 1/\sqrt{2 + \sqrt{2}}$ and $\sigma = 5/8$, the parafermionic observable F satisfies for every $v \in V$:*

$$(p - v)F(p) + (q - v)F(q) + (r - v)F(r) = 0, \tag{3.1}$$

where p, q and r are the three midedges in Ω adjacent to the vertex v, ordered counterclockwise around v. In particular, we have $q - v = e^{+i2\pi/3}(p - v)$ and $r - v = e^{-i2\pi/3}(p - v)$.

Remark 3.4. The identity can be interpreted as a discrete integral along an elementary contour on the dual lattice or a Riemann sum approximation of complex

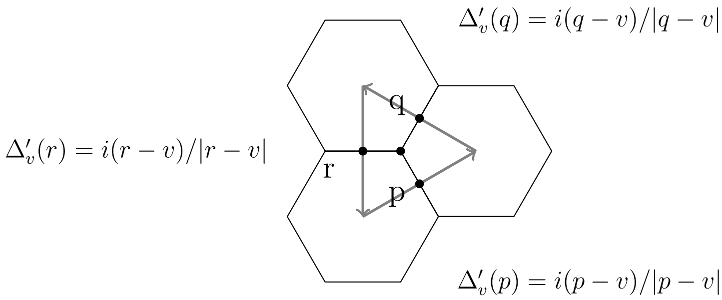

Figure 5: The definition of an elementary contour Δ_v of the dual lattice

integral of the function F along an elementary contour. An elementary contour Δ_v around vertex v in the dual lattice visits the centers of the three hexagons that v is part of lattice connects these with straight line segments. Thus the contour Δ_v has the parametrization

$$\Delta_v(t) = \begin{cases} i\frac{(p-v)}{|p-v|}t + y_1, \; 0 \le t \le T_1, \\ i\frac{(q-v)}{|q-v|}t + y_2, \; T_1 < t \le T_2, \\ i\frac{(r-v)}{|r-v|}t + y_3, \; T_2 < t \le T_3. \end{cases}$$

By defining the discrete integral over the elementary contour Δ_v for functions defined on midedges as the sum

$$\begin{aligned} \oint_{\Delta_v} f(y)dy &= f(p)\Delta'_v(\Delta_v^{-1}(p)) + f(q)\Delta'_v(\Delta_v^{-1}(q)) + f(r)\Delta'_v(\Delta_v^{-1}(r)) \\ &= f(p)i\frac{(p-v)}{|p-v|} + f(q)i\frac{(q-v)}{|q-v|} + f(r)i\frac{(r-v)}{|r-v|}, \end{aligned}$$

we have that the lemma 3.3 can be alternatively stated as $\oint_{\Delta_v} = 0$ for all v in V. As a result of the elementary contour integrals vanishing, all closed contour integrals of F in the dual of Ω vanish. This can be interpreted as discrete holomorphicity of F. As mesh size tends to zero the above relation would establish F as holomorphic function; however F also depends on the mesh, therefore the lemma 3.3 alone is insufficient to guarantee existence of a holomorphic limit function.

Proof of lemma 3.3. We first note that the parafermionic observable of definition 3.2 can be expanded into the contribution of individual walks. For a walk ending at q, its contribution to the identity $(p-v)F(p) + (q-v)F(q) + (r-v)F(r)$ is

$$c(\gamma) = (q-v)e^{-i\sigma W_\gamma(x,q)}z^{l(\gamma)}.$$

Next we note that one can partition walks ending at p, q or r into pairs and triplets as follows:

- For a walk γ_1 going through all three midedges we get a loop by adding one more midedge. If we now reverse the direction this loop is traversed and omit the last midedge of the reversed loop, we associate to γ_1 a walk γ_2 that has the same trajectory up to v and then goes through the loop from v to v in the other direction. Thus one can group the walks visiting all three midedges in pairs.

- If a walk γ_1 visits only one of the midedges, we get walks γ_2 and γ_3 by prolonging the walk with one step. The reverse is also true: a walk visiting two midedges is naturally associated to a walk visiting only one midedge by erasing its last step. Hence, walks visiting one or two midedges can be grouped in triplets.

The following step is to show that the contribution of every pair and triplet to equation (3.1) equals zero, which completes the proof of the lemma. For both pairs and triplets we can without loss of generality assume that the walk γ_1 first visits the midedge p.

In the case of pairs: Let γ_1 end at q and γ_2 end at r. The walks γ_1 and γ_2 agree until p after which they go through an almost complete loop in opposite directions. This implies that $l(\gamma_1) = l(\gamma_2)$ and

$$\begin{cases} W_{\gamma_1}(x,q) & = W_{\gamma_1}(x,p) - 4\pi/3 \\ W_{\gamma_2}(x,r) & = W_{\gamma_1}(x,p) + 4\pi/3 \end{cases}.$$

For the windings of γ_1 and γ_2 we have used the fact that x is on the boundary of the simply connected set Ω, making it impossible for the walk to wind around x.

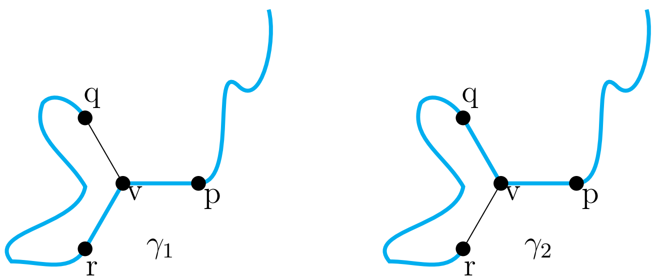

Figure 6: An illustration of a pair of walks visiting p, q and r.

The total contribution of the pair to relation (3.1) is

$$\begin{aligned} c(\gamma_1) + c(\gamma_2) &= (q-v)e^{-i\sigma W_{\gamma_1}(x,q)} z^{l(\gamma_1)} + (r-v)e^{-i\sigma W_{\gamma_2}(x,r)} z^{l(\gamma_2)} \\ &= (p-v) z^{l(\gamma_1)} e^{-i\sigma W_{\gamma_1}(x,p)} \left(e^{i2\pi/3} \left(e^{i\sigma\pi/3}\right)^4 + e^{-i2\pi/3} \left(e^{-i\sigma\pi/3}\right)^4 \right) \\ &= C \left(e^{i2\pi/3} e^{i\sigma 4\pi/3} + e^{-i2\pi/3} e^{-i\sigma 4\pi/3} \right). \end{aligned}$$

The sum of a complex number ξ and and its conjugate $\overline{\xi}$ vanishes exactly when ξ has no real component. In order to guarantee that the sum cancels out we must choose

σ such that

$$\begin{aligned} e^{i2\pi/3}e^{i\sigma 4\pi/3} &= \pm i \\ 2\pi/3 + 4\pi\sigma/3 &= \pi/2 + n\pi, n \in \mathbb{Z} \\ \sigma &= \frac{6n-1}{8}, n \in \mathbb{Z}. \end{aligned}$$

We note that $\sigma = 5/8$ is one solution. We then prove that with this σ there exist a z such that the contribution of every triplet of walks to the equation (3.1) vanishes.

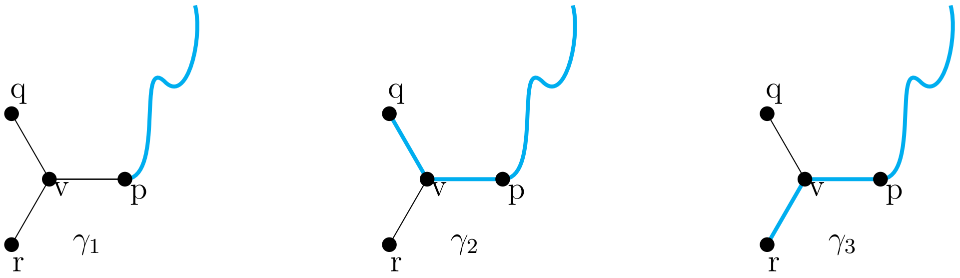

Figure 7: A triplet of walks visiting one or two of p, q and r.

In the case of triplets: Without loss of generality assume that the walk γ_1 ends at p, the walk γ_2 ends at q and that the walk γ_3 ends at r. It follows that

$$l(\gamma_2) = l(\gamma_3) = l(\gamma_3) + 1$$

and that

$$\begin{cases} W_{\gamma_2}(x, q) &= W_{\gamma_1}(x, p) - \pi/3 \\ W_{\gamma_3}(x, r) &= W_{\gamma_1}(x, p) + \pi/3 \end{cases}.$$

The total contribution of the triplet is

$$c(\gamma_1) + c(\gamma_2) + c(\gamma_3) = (p - v)z^{l(\gamma_1)}e^{-i\sigma W_{\gamma_1}(x,p)}\left(1 + ze^{i2\pi/3}e^{i\sigma\pi/3} + ze^{-i2\pi/3}e^{-i\sigma\pi/3}\right).$$

Requiring this to vanish, we have

$$\begin{aligned} z\left(e^{i2\pi/3}e^{i\sigma\pi/3} + e^{-i2\pi/3}e^{-i\sigma\pi/3}\right) &= -1 \\ z &= \frac{-1}{2\cos(2\pi/3 + \sigma\pi/3)}. \end{aligned}$$

Plugging in $\sigma = \frac{6n-1}{8}, n \in \mathbb{Z}$, we get a family of solutions

$$\sigma = \frac{6n-1}{8}, z = \begin{cases} \frac{1}{\sqrt{2-\sqrt{2}}} & , n \equiv 0 \mod 8 \\ \frac{1}{\sqrt{2+\sqrt{2}}} & , n \equiv 1 \mod 8 \\ \frac{1}{\sqrt{2+\sqrt{2}}} & , n \equiv 2 \mod 8 \\ \frac{1}{\sqrt{2-\sqrt{2}}} & , n \equiv 3 \mod 8 \\ -\frac{1}{\sqrt{2-\sqrt{2}}} & , n \equiv 4 \mod 8 \\ -\frac{1}{\sqrt{2+\sqrt{2}}} & , n \equiv 5 \mod 8 \\ -\frac{1}{\sqrt{2+\sqrt{2}}} & , n \equiv 6 \mod 8 \\ -\frac{1}{\sqrt{2-\sqrt{2}}} & , n \equiv 7 \mod 8 \end{cases}. \tag{3.2}$$

$z = 1/\sqrt{2+\sqrt{2}}$, $\sigma = 5/8$ is what we get when n equals 1, which completes the proof. □

3.2 Connective constant of the hexagonal lattice

Having established equation (3.1), we can sum the relation over all of the vertices V in the simply connected set Ω. There are two benefits in doing this. The first one is that midedges p not on the boundary of Ω do not contribute to this sum, as their contributions $F(p)$ enter the sum twice with coefficients that cancel each other. The second benefit is that on the boundary $\partial\Omega$ we know the relative orientations of the midedges y with respect to the starting point x and also the winding $W_\gamma(x, y)$ is fixed once the start and end points x, y are known. This can be exploited in an area $S_{T,L}$ of the shape of an isosceles trapezoid when we position the lattice so that there is a horizontal edge associated to the mid-edge x located at the origin. In addition we position the trapezoid $S_{T,L}$ so that the real axis acts as its axis of symmetry. We assume the trapezoid to be $2L$ hexagons high and T hexagons wide. As we let L tend to infinity, the trapezoid $S_{T,L}$ converges to the infinite vertical strip S_T in the lattice. For reference, see figure 8.

Denote by α the left boundary of the trapezoid, by β the right boundary of the trapezoid and by ε and $\overline{\varepsilon}$ the bottom and top boundaries of the trapezoid. Introduce the partition functions:

$$A^z_{T,L} := \sum_{\gamma: x \to \alpha \setminus \{x\}} z^{l(\gamma)}, \ B^z_{T,L} := \sum_{\gamma: x \to \beta} z^{l(\gamma)}, \ E^z_{T,L} := \sum_{\gamma: x \to \varepsilon \cup \overline{\varepsilon}} z^{l(\gamma)}.$$

Lemma 3.5. *For $z = z_c = \frac{1}{\sqrt{2+\sqrt{2}}}$, it holds that*

$$1 = c_\alpha A^{z_c}_{T,L} + B^{z_c}_{T,L} + c_\varepsilon E^{z_c}_{T,L}, \tag{3.3}$$

with $c_\alpha = \cos(\frac{3\pi}{8}) = \frac{1}{2}\sqrt{2-\sqrt{2}}$ and $c_\varepsilon = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

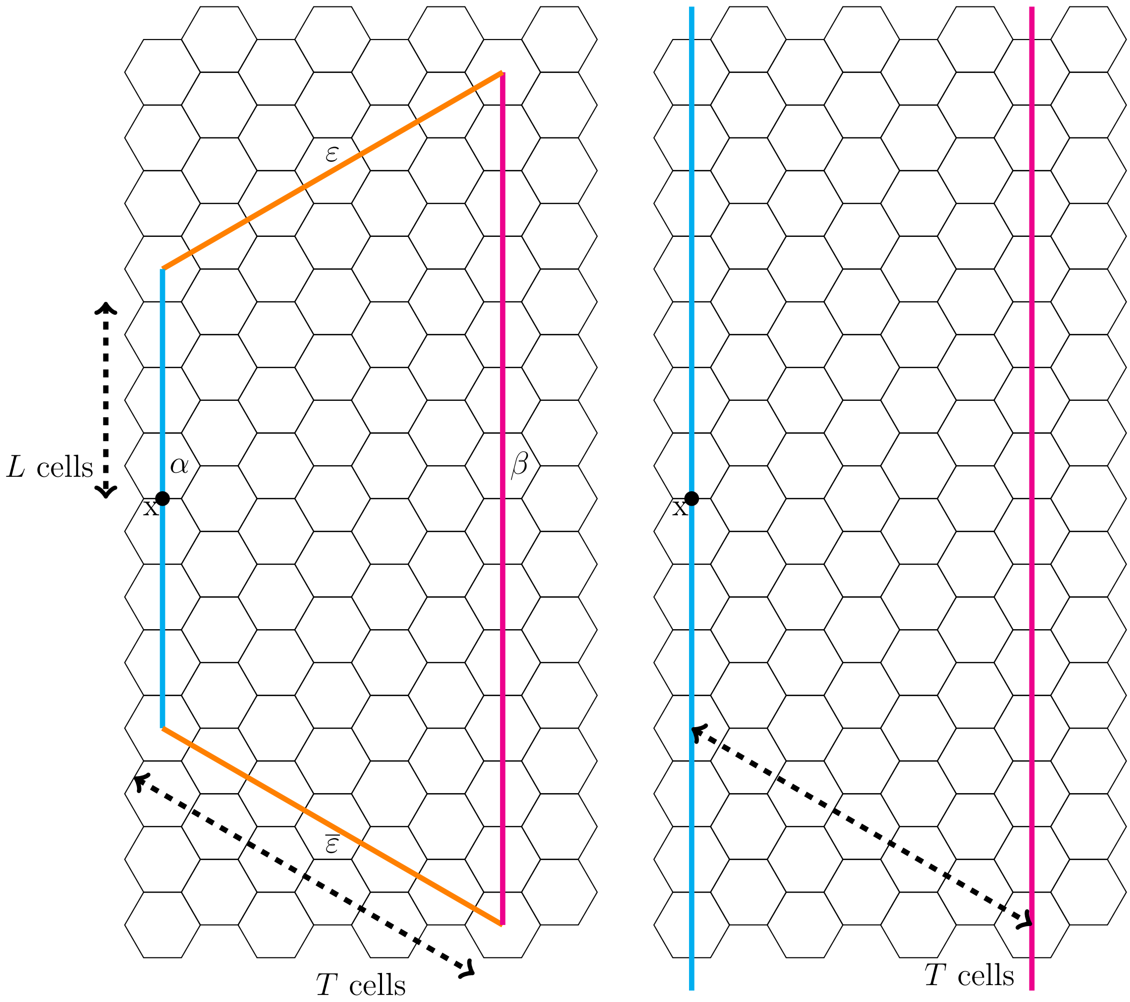

Figure 8: The trapezoidal domain $S_{T,L}$ and the infinite strip S_T.

Proof. Summing the identity (3.1) over all of the vertices $V(S_{T,L})$ and normalizing yields

$$0 = -\sum_{y\in\alpha} F(y) + \sum_{y\in\beta} F(y) + e^{i2\pi/3}\sum_{y\in\varepsilon} F(y) + e^{-i2\pi/3}\sum_{y\in\bar{\varepsilon}} F(y).$$

The symmetry of the domain implies that walks from x to the top part of α and walks from x to the bottom part of α contribute equally to $A^{z_c}_{T,L}$. The winding of any self-avoiding walk from x to the top part of α is π while the winding to the bottom part is $-\pi$. Thus

$$\begin{aligned}\sum_{y\in\alpha} F(y) = F(x) + \sum_{y\in\alpha\setminus\{x\}} F(y) &= 1 + \frac{e^{-i\sigma\pi}}{2}A^{z_c}_{T,L} + \frac{e^{i\sigma\pi}}{2}A^{z_c}_{T,L}\\ &= 1 + \cos(\sigma\pi)A^{z_c}_{T,L}.\end{aligned}$$

Above we have used the fact that the only self-avoiding walk from x to x is the trivial one of length 0, hence $F(x) = 1$. Similarly to α, the winding from x to any mid-edge in β(resp. ε and $\overline{\varepsilon}$) is 0(resp. $\frac{2\pi}{3}$ and $\frac{-2\pi}{3}$), therefore

$$\sum_{y\in\beta} F(y) = B^{z_c}_{T,L}$$

and, since by symmetry ε and $\overline{\varepsilon}$ must contribute equally to $E^{z_c}_{T,L}$,

$$\begin{aligned} e^{i2\pi/3}\sum_{y\in\varepsilon} F(y) + e^{-i2\pi/3}\sum_{y\in\overline{\varepsilon}} F(y) &= \frac{e^{-i(1-\sigma)2\pi/3}}{2} E^{z_c}_{T,L} + \frac{e^{i(1-\sigma)2\pi/3}}{2} E^{z_c}_{T,L} \\ &= \cos((1-\sigma)2\pi/3)E^{z_c}_{T,L}. \end{aligned}$$

Thus, the identity (3.1) leads us to the equation

$$1 = \cos\left((1-\sigma)\pi\right) A^{z_c}_{T,L} + B^{z_c}_{T,L} + \cos\left((1-\sigma)2\pi/3\right) E^{z_c}_{T,L},$$

where σ and z_c belong to the solution family (3.2). In particular $\sigma = 5/8$, $z_c = 1/\sqrt{2+\sqrt{2}}$ gives the values c_α, c_ε in the statement of the lemma. □

Remark 3.6. The proof of the connective constant will rely on z_c and c_α being positive. These two conditions imply that only the solutions of (3.1) where $n \equiv 1, 2$ mod 8 in (3.2) are possible. Looking at these solutions, we see that $z_c = \sqrt{2+\sqrt{2}}^{-1}$ is uniquely determined.

Note that the sequences $\left(A^{z}_{T,L}\right)_{L>0}$ and $\left(B^{z}_{T,L}\right)_{L>0}$ are increasing in L and are bounded for $z \le z_c$ thanks to their monotonicity in z and the identity (3.3). Therefore they have the limits

$$A^{z}_{T} := \lim_{L\to\infty} A^{z}_{T,L}, \ B^{z}_{T} := \lim_{L\to\infty} B^{z}_{T,L}.$$

By identity (3.3) we can deduce that $\left(E^{z_c}_{T,L}\right)_{L>0}$ decreases and converges to a limit $E^{z_c}_{T} := \lim_{L\to\infty} E^{z_c}_{T,L}$. Letting L tend to infinity in the identity (3.3), we arrive at

$$1 = c_\alpha A^{z_c}_{T} + B^{z_c}_{T} + c_\varepsilon E^{z_c}_{T}. \tag{3.4}$$

Proof of $\mu = \sqrt{2+\sqrt{2}}$. We prove first that $\chi(z_c) = \infty$, and hence $\mu \ge \sqrt{2+\sqrt{2}}$. Suppose first that for some T, $E^{z_c}_{T} > 0$. As noted before, $E^{z_c}_{T,L}$ decreases in L and so

$$\chi(z_c) \ge \sum_{L>0} E^{z_c}_{T,L} \ge \sum_{L>0} E^{z_c}_{T} = +\infty.$$

Assuming on the contrary that $E^{z_c}_{T} = 0$ for all T, the equation (3.4) renders to

$$1 = c_\alpha A^{z_c}_{T} + B^{z_c}_{T}.$$

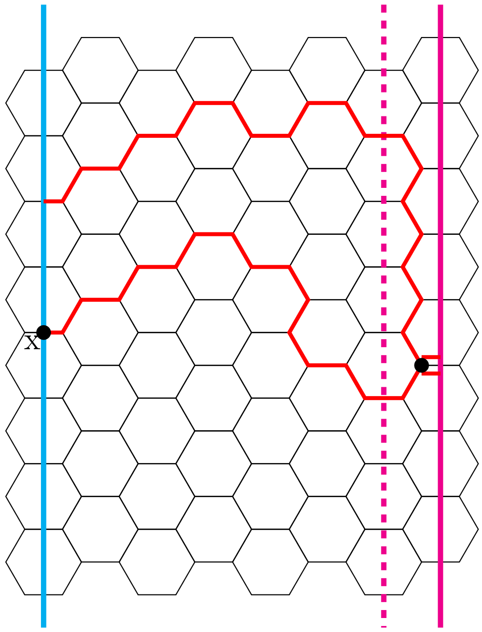

Figure 9: An example of a walk γ counted to A^{z}_{T+1} but not A^{z}_{T}.

A walk γ that is counted to $A^{z_c}_{T+1}$ but not to $A^{z_c}_{T}$ has to visit some vertex adjacent to the right edge of S_{T+1}. Cutting γ at the first such point, we uniquely decompose it into two walks crossing S_{T+1} (these walks are usually called bridges), which together are one step longer than γ. This gives us the estimate

$$A^{z_c}_{T+1} - A^{z_c}_{T} \leq \frac{1}{z_c}\left(B^{z_c}_{T+1}\right)^2. \tag{3.5}$$

Combining the equations (3.4) and (3.5) for two consecutive values of T we get

$$\begin{aligned} 0 = 1 - 1 &= \left(c_\alpha A^{z_c}_{T+1} + B^{z_c}_{T+1}\right) - \left(c_\alpha A^{z_c}_{T} + B^{z_c}_{T}\right) \\ &= c_\alpha\left(A^{z_c}_{T+1} - A^{z_c}_{T}\right) + B^{z_c}_{T+1} - B^{z_c}_{T} \leq \frac{c_\alpha}{z_c}\left(B^{z_c}_{T+1}\right)^2 + B^{z_c}_{T+1} - B^{z_c}_{T}, \end{aligned}$$

therefore

$$B^{z_c}_{T} - B^{z_c}_{T+1} \leq \frac{c_\alpha}{z_c}\left(B^{z_c}_{T+1}\right)^2. \tag{3.6}$$

By replacing the inequality with an equality sign, the identity (3.6) can be thought of as a discretization of the differential equation $-B'(t) = cB(t)^2$, a solution of which is $B(t) = 1/(ct)$. This gives intuition to the following claim.

Claim 3.7. It follows from inequality (3.6) by induction that

$$B^{z_c}_{T} \geq \frac{\min\left[B^{z_c}_{1}, z_c/c_\alpha\right]}{T}$$

for every $T \geq 1$.

Proof of claim 3.7 Clearly $B_1^{z_c} \geq \min\left[B_1^{z_c}, \frac{z_c}{c_\alpha}\right]$ so the identity holds for $T = 1$. Denote now $\min\left[B_1^{z_c}, \frac{z_c}{c_\alpha}\right]$ by m and assume that the identity $B_T^{z_c} \geq \frac{m}{T}$ holds for some $T \geq 1$. Substituting m and the induction assumption into the inequality (3.6) we have

$$\frac{m}{T} - B_{T+1}^{z_c} \leq \frac{\left(B_{T+1}^{z_c}\right)^2}{m}$$

Solving this with respect to $B_{T+1}^{z_c}$ yields

$$B_{T+1}^{z_c} \geq m\left(\sqrt{\frac{1}{T} + \frac{1}{4}} - \frac{1}{2}\right).$$

Finally note that $\sqrt{\frac{1}{T} + \frac{1}{4}} - \frac{1}{2} \geq \frac{1}{T+1}$ holds for T strictly greater than 0, thus

$$B_{T+1}^{z_c} \geq \frac{m}{T+1},$$

proving the claim.

The claim 3.7 gives the estimate

$$\chi(z_c) \geq m \sum_{T>0} \frac{1}{T} = +\infty.$$

This completes the proof for the estimate $\mu \geq z_c^{-1} = \sqrt{2 + \sqrt{2}}$.
It remains to prove the opposite inequality $\mu \leq z_c^{-1}$. To estimate the partition function from above, we will decompose self-avoiding walks into bridges. A *bridge* of width T is a self-avoiding walk in S_T from one side to the opposite side, defined up to vertical translation. The partition function of bridges of width T is B_T^z, which is at most 1 by 3.4. Noting that a bridge of width T has length at least T, we obtain for $z < z_c$

$$B_T^z \leq \left(\frac{z}{z_c}\right)^T B_T^{z_c} \leq \left(\frac{z}{z_c}\right)^T.$$

Thus for $z < z_c$ the sum $\sum_{T>0} B_T^z$ converges and so does the product $\prod_{T>0}\left(1 + B_T^z\right) < \prod_{T>0} e^{B_T^z}$. We will next use the fact that any self-avoiding walk can be canonically decomposed into a sequence of bridges of widths $T_{-i} < \ldots < T_{-1}$ and $T_0 > \ldots > T_j$. In addition if one fixes the first midedge, the first vertex and the last midedge visited by the walk, the decomposition uniquely determines the walk. Noting that in the hexagonal lattice the walk will take one step between the bridges of the decomposition and that $z < 1$ we have

$$\begin{aligned}\chi(z) &\leq 4 \sum_{\substack{T_{-i}<\ldots<T_{-1}\\ T_0>\ldots>T_j}} z^{i+j} \left(\prod_{k=-i}^{j} B_{T_k}^z \right) \\ &\leq 4 \sum_{\substack{T_{-i}<\ldots<T_{-1}\\ T_0>\ldots>T_j}} \left(\prod_{k=-i}^{j} B_{T_k}^z \right) \\ &= 4 \left(\prod_{T>0} 1 + B_T^z \right)^2 < \infty.\end{aligned}$$

The procedure for decomposing the walk into bridges goes as follows. First assume the walk γ is a half-plane self-avoiding walk, i.e. the starting point has extremal real part. Without loss of generality assume the start has minimal real part. To get the first bridge γ_0 of width T_0, take the one visited last of the vertices with maximal real part in γ. If this vertex is visited after N_0 steps, the bridge γ_0 consists of γ up to the N_0^{th} vertex and the midedge horizontally adjacent to it. To get the second bridge γ_1 of width $T_1 < T_0$, start from the $N_0 + 1^t h$ step $\gamma(N_0+1)$ and consider the last vertex in γ after that with a minimal real part, say the N_1^{th} vertex. The bridge γ_1 will then be trajectory from $\gamma(N_0+1)$ to the N_1^{th} vertex in γ with a half-step extension in the negative direction in the end. Now the part of γ starting from the point $\gamma(N_1+1)$ is a half-plane self-avoiding walk and we can repeat the steps performed before. Using this algorithm recursively yields a sequence of bridges of widths $T_0 > T_1 > \ldots > T_j$ that characterizes the half-plane walk up to the last step. It should be noted that the total length of the bridges will be j steps less than the length of the original walk γ.

Finally note that any self-avoiding walk in the plane can be divided into two half-plane walks. Let the first vertex with the maximal real part in a walk γ in the plane be the N^{th} one. This means that the walk γ up to the N^{th} vertex extended by one half-step in the positive direction is a half-plane walk and the part of γ from the $\gamma(N+1)$ to the end is a half-plane walk. The procedure for decomposing half-plane walks into bridges can then be applied for both of these to get sequences of bridges of widths $T_{-i} < T_{i-1} < \ldots < T_{-1}$ and $T_0 > T_1 > \ldots > T_j$. The factor 4 in equation (3.2) is a result of the two options for the first vertex of the walk and the two options for the last midedge of the walk, left undefined by the bridge decomposition. For an example, see figure 10. □

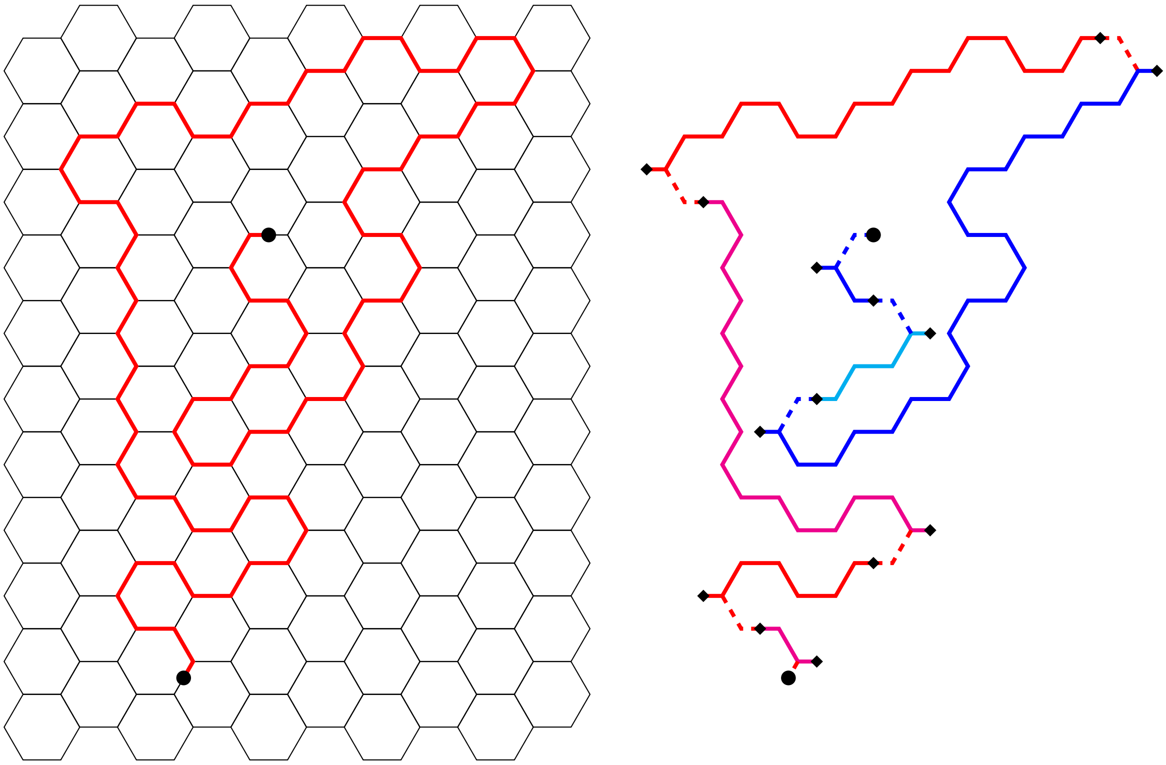

Figure 10: The decomposition of the walk on the left into bridges, on the right the blue bridges correspond to negative indices and the red ones to non-negative indices. In the example $T_{-1} = 7 > T_{-2} = 2 > T_{-3} = 1 > T_{-4} = 0$ and $T_0 = 8 > T_1 = 4 > T_2 = 3 > T_3 = 1$.

4 Transfer matrices

The idea of this chapter is to show that self-avoiding walks in vertical cylinder domains can be canonically described by the sequence of configurations indexed by height that the walk assumes at different layers of the cylinder. We then express the generating function G_z similar to the one in the first chapter in terms of a transfer matrix for the set of configurations. Using the probability measure associated to the generating function allows the calculation of edge visiting probabities in the cylinder sets. In particular the probability of the walk returning to the boundary of the cylinder can be calculated using the generating matrix, and as the height of the cylinder tends to infinity this probability converges to a value that can also be calculated. This provides one way to approximate the scaling limit of the self-avoiding walk. We also show that the limit measure on the set of walks conditioned to progress upward on the infinite cylinder is Markovian. The idea of using matrices to express the partition functions of SA walks in strips is not entirely new, for example Alm and Janson used a similar approach in 1990 [AJ90].

4.1 The self-avoiding walk in strip domains

Consider vertical strip domains of type $S = \{z \in \mathbb{C} |\ a \leq Re(z) \leq b\}$ in the complex plane, constructed as follows:

1. Embed the hexagonal lattice $\mathbb{H}_\delta$ of mesh size δ onto the complex plane so that the origin is a midedge of a vertical edge.
2. Place the left boundary a of the strip S onto the lattice so that there are no midedges on the negative real line.
3. Place the right boundary b so that there are L vertical edges on the real line inside the strip S.

Two natural ways to define the strip domain S for the hexagonal lattice are illustrated in fig. 11.

It is convenient to choose the mesh size $\delta = \frac{2}{3}$ for the hexagonal lattice, as it renders the height difference of two consecutive layers of hexagons in the strip to unity. In terms of analytic geometry, the choice $\delta = \frac{2}{3}$ for domains of fig. 11 leads to definitions

$$S_{w,L} = \left\{ a + ib \in \mathbb{H}_{2/3} \mid -\frac{1}{\sqrt{3}} \leq a \leq \frac{2}{\sqrt{3}}(L-1) + \frac{1}{\sqrt{3}} \right\}$$

and

$$S_{n,L} = \left\{ a + ib \in \mathbb{H}_{2/3} \mid 0 \leq a \leq \frac{2}{\sqrt{3}}(L-1) \right\},$$

where w and n stand for wide and narrow, respectively. We also consider the height-restricted subdomains of the strips:

$$S_{w,L,H} = \{ a + ib \ \in S_{w,L} \mid -H \leq b \leq H \}$$

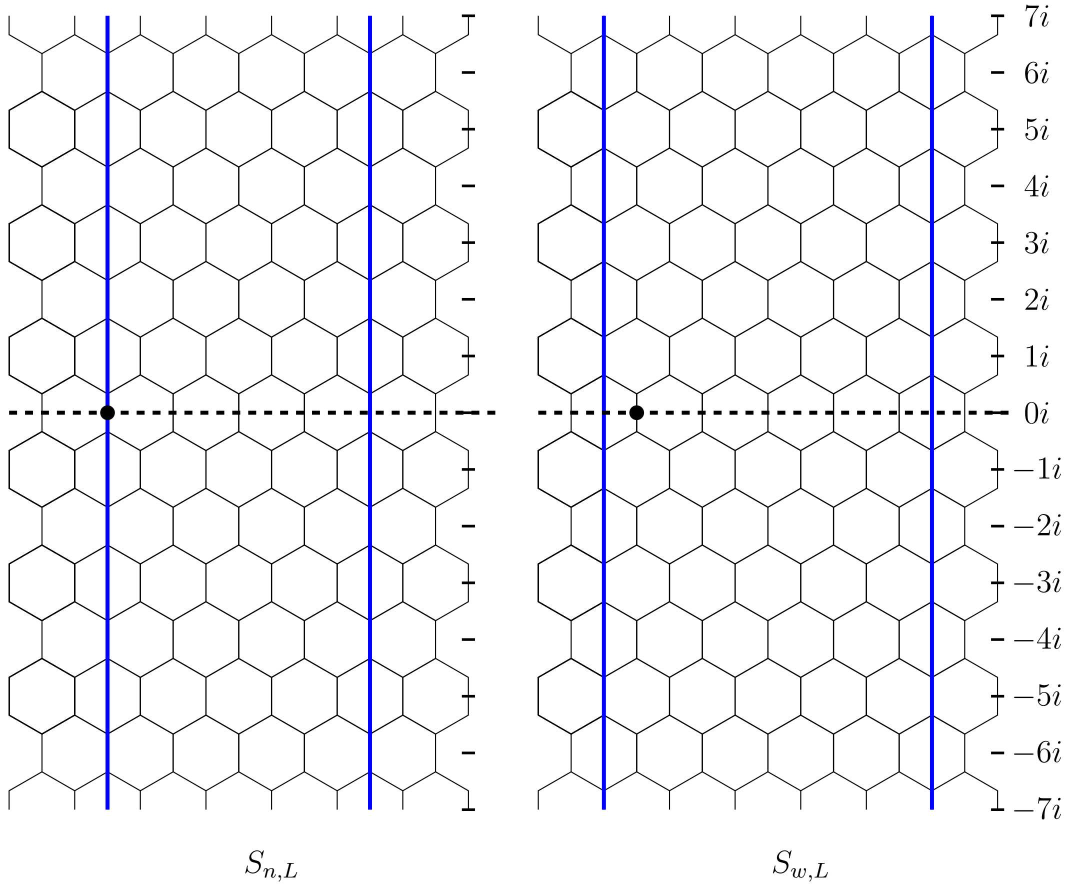

Figure 11: Examples of the strip domains $S_{n,L}$ and $S_{w,L}$ for $L = 5$.

and

$$S_{n,L,H} = \{a + ib \ \in S_{n,L} \ | H \leq b \leq H\} .$$

The results of this chapter will apply to both the pair $(S_{w,L,H}, S_{w,L})$ and $(S_{n,L,H}, S_{n,L})$. We will therefore make no preference between w and n and just refer to the infinite strip and its restriction as $(S_L, S_{L,H})$.

Definition 4.1. Consider self-avoiding walks $\gamma : x \to y$ in the domain $S_{L,H}$ with fixed endpoints $x = x_R - iH$ at the bottom of the domain and $y = y_R + iH$ at the top of the domain. An example is shown in figure 12. Analogously to the definition 2.9, the generating function for an enumeration of such walks is
$G_{z,H}(x, y) = G_{z,S_{L,H}}(x, y)$, where

$$G_{z,H}(x, y) = \sum_{\gamma : x \to y} z^{l(\gamma)},$$

and the sum is taken over all self-avoiding walks in the height-restricted strip $S_{L,H}$ starting at x and ending at y.

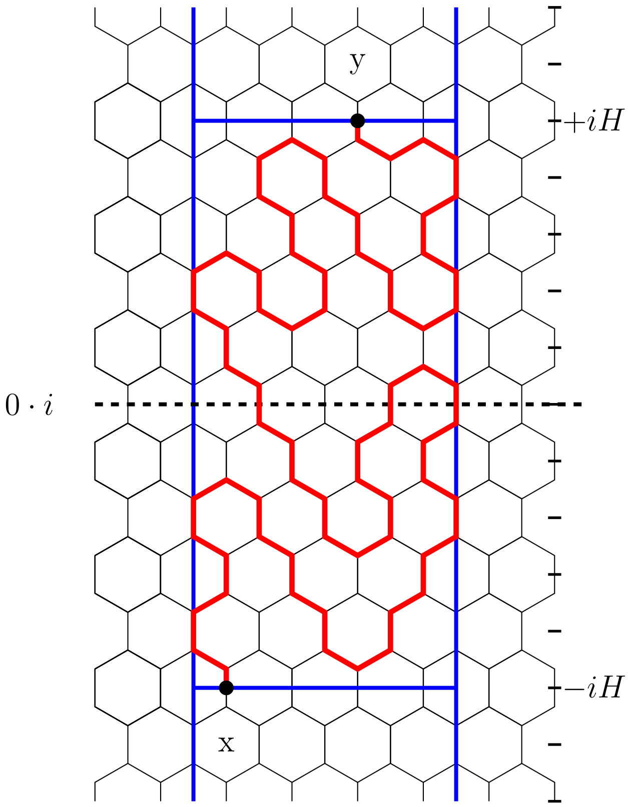

Figure 12: An example of a self-avoiding walk $\gamma : x \to y$ on the set $S_{L,H}$

Definition 4.2. Analogously to definitions 2.7 and 2.9, define the probability measure $\mathbb{P}_{z,H} = \mathbb{P}_{\{z,L,H,x,y\}}$ for self-avoiding walks from x to y in $S_{L,H}$ as

$$\mathbb{P}_{z,H}[\gamma] = \frac{z^{l(\gamma)}}{G_{z,H}(x, y)}.$$

The main result of this chapter is:

Theorem 4.3. *Fix $z > 0$ and $L \in \mathbb{Z}_{>0}$. Then there exists a vector space V, a linear operator $T = T_z : V \to V$ and vectors $v_x, v_y \in V$ such that*

1. *The generating function $G_{z,H}$ can be expressed as*
$$G_{z,H}(x, y) = v_y^T T^H v_x.$$

2. *For each midedge $e = a + ib$, where $b \in [-H, -H]$ and a is such that $a + ib$ is a midedge in $S_{L,H}$, there exists a linear operator $P(a)$ such that:*
$$\mathbb{P}_{z,H}\left[e \in \gamma\right] = \frac{v_y^T T^{\frac{H-y}{2}} P(a) T^{\frac{H+y}{2}} v_x}{v_y^T T^H v_x}.$$

3. *The sequence $(\mathbb{P}_{z,H})_{H\in\mathbb{N}}$ converges weakly towards a limit measure $\mathbb{P}_z$ as H tends to infinity.*

4. *The limit measure $\mathbb{P}_z$ is Markovian in the sense specified in section 4.5.*

By utilizing the results of this chapter, chapter 5 seeks to provide computational support for conjecture 2.10, first formulated by physicists in the 1980s and refined by theorem 2.11 by Lawler, Schramm and Werner in [LSW04b].

4.2 A fundamental vector space

Definition 4.4. A *level* is a horizontal line with an integer imaginary coordinate, i.e. $\{\cup_{a\in\mathbb{R}}\ a + ni \mid n \in \mathbb{Z}\}$. By the definitions of sets $S_{L,H}$ and S_L, each level halves a layer of hexagons in the lattice.

Definition 4.5. Consider self-avoiding walks in the set $S_{L,H}$. At a level in $S_{L,H}$ keep track of

- the real coordinates of the edges the walk uses.
- the real coordinates of the pairs of edges joined by the trajectory of the walk below the level.

This results in an *arc-thread configuration* consisting of an edge and a possibly empty set of pairs of edges, where the single edge, or *thread*, tells the place where the walk first intersected the level and the pairs of edges, or *arcs*, correspond to U-shaped loops in the trajectory of the walk below the level. An example of how to form an arc-thread configuration is illustrated in fig. 13. Figure 15 shows an example of this procedure applied to all levels of the set $S_{L,H}$.

Proposition 4.6. *The trajectory of a self-avoiding walk between two consecutive levels in the vertical strip is uniquely determined by the arc-thread configurations of the levels.*
Proof. This results from the properties of the hexagonal lattice. The only possible way of joining the configurations can be algorithmically expressed as follows:

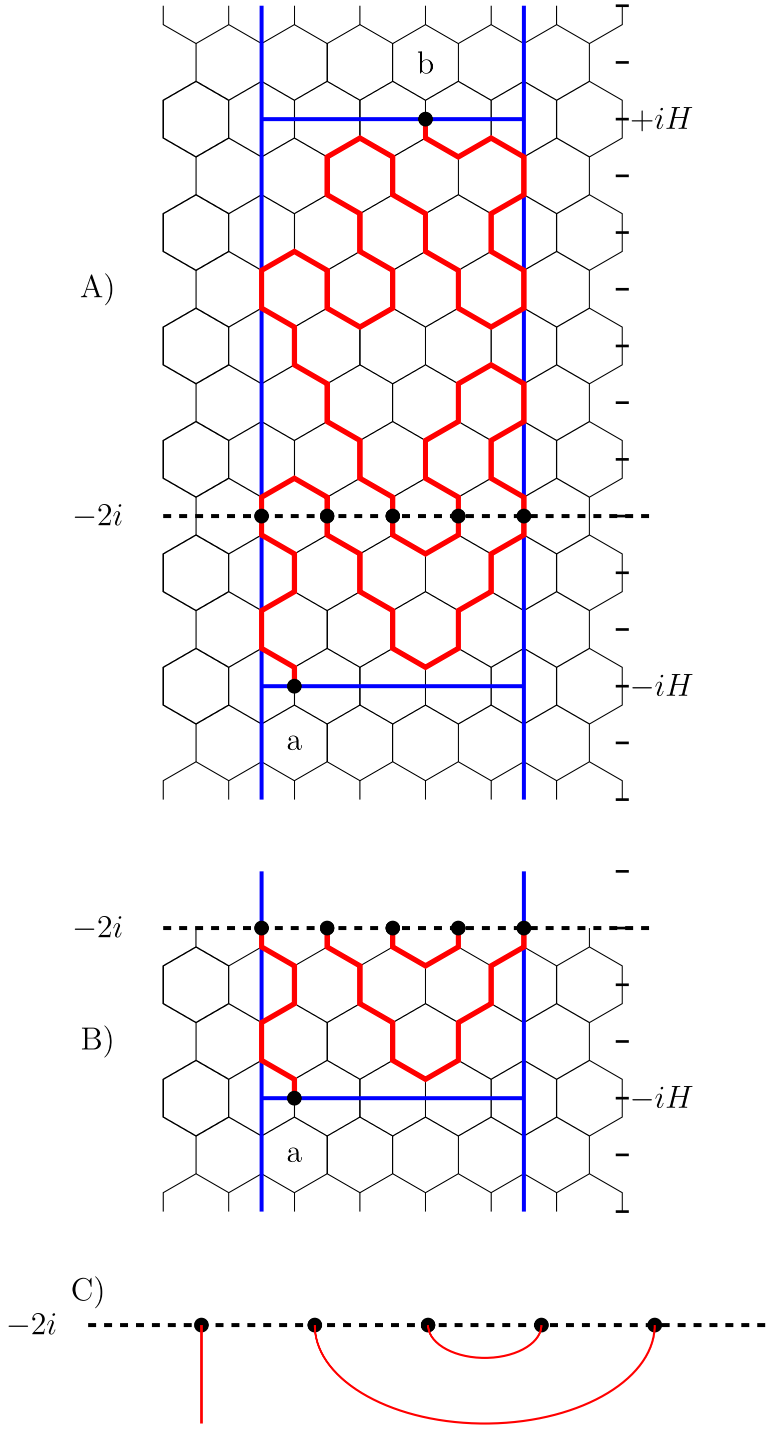

Figure 13: A)a SAW B)the same SAW with only the part below a level visible C)the arc-thread configuration of the SAW at that level

1. *Order the edges of the two levels with a single coordinate system by their real coordinates. Mark the edges used by the SAW by dots.*

2. *Connect the dot with the smallest real coordinate to the dot with the second smallest real coordinate, the third dot to the fourth one etc. until all dots have been connected. This is the only way to join the two configurations in a self-avoiding way. For example, if one tries to connect the first unconnected dot to the third unconnected dot, the dot between these can no more be connected anywhere in a way that avoids the first bridge.*

3. *Check that the connections indeed result in given arc-thread configurations. If they do not, the transition from the lower configuration to the upper one is impossible. The conditions that need to be checked are presented in section 5.*

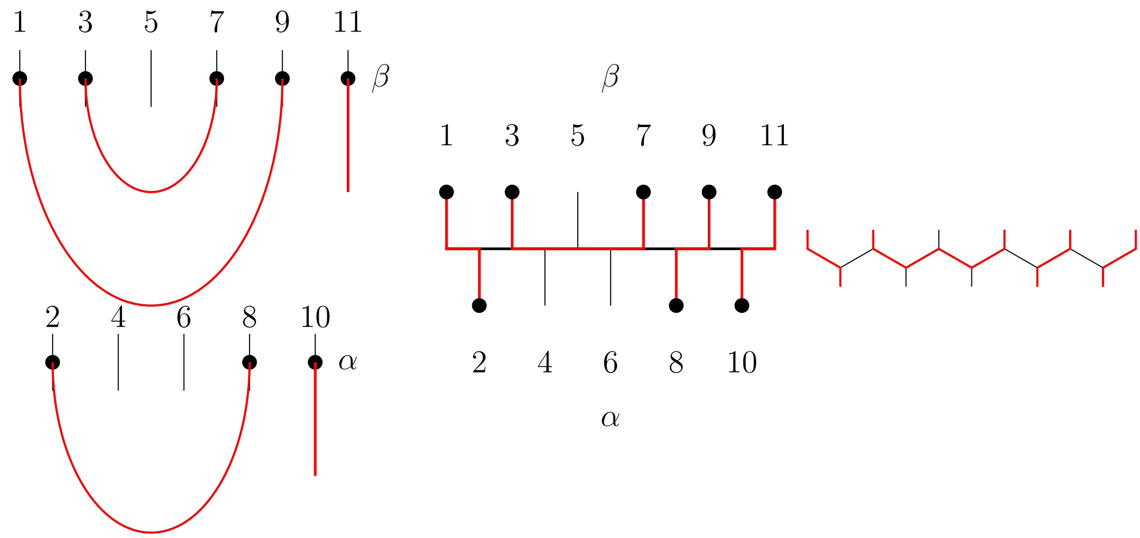

Figure 14: Construction of the walk segments between configurations α and β by connecting the edges 1 and 2, 3 and 7, 8 and 9, 10 and 11. The total length of these walk segments is 8 half-edges and 7 edges, thus $l(\alpha \to \beta) = 11$.

Definition 4.7. Let α, β be arc-thread configurations at two consecutive levels. If it is possible to draw non-overlapping walk segments between the levels such that the configuration the SAW at the lower level is α and the configuration of the SAW at the upper level is β, we define the length $l(\alpha \to \beta)$ to be the total length of these segments. If the transition from α to β is not possible, we set $z^{l(\alpha \to \beta)} = 0$ for all z.

Remark 4.8. It should be noted that in the domains $S_{L,H}$ there are only two sets of possible arc-thread configurations, C_E for levels with an even imaginary coordinate and C_O for levels with an odd imaginary coordinate.

Definition 4.9. We are now ready to define the transfer matrix T. Without loss of generality, consider only transitions between two consecutive even levels. Let $\alpha, \beta \in C_E$ and v_α, v_β be their vector forms in the set V_E of formal linear combinations of C_E. The transfer matrix $T = T_z$ is defined elementwise as the sum over all configurations in C_O:

$$v_\beta^T T v_\alpha = T_{(\beta,\alpha)} = \sum_{\delta \in C_O} z^{l(\alpha \to \delta)} z^{l(\delta \to \beta)}.$$

Note that the transfer matrix element $T_{(\beta,\alpha)}$ is the generating, or partition, function for an enumeration of sets of walk segments connecting lower configuration α to configuration β two levels above α.

Definition 4.10. By swapping C_E and C_O in the above definition one can also construct a transfer matrix for two consecutive odd levels. For more detailed analysis, we define the transfer matrices between two consecutive levels. Let $\alpha, \beta \in C_E, \delta \in C_O$ and v_α, v_β correspond to α, β in the set V_E, v_δ correspond to δ in the set V_O of formal linear combinations of C_O. Set the even-to-odd transfer matrix elements to be

$$T(e \to o)_{(\delta,\alpha)} = T^\star_{(\delta,\alpha)} = v_\delta^T T^\star v_\alpha = z^{l(\alpha \to \delta)}$$

and the odd-to-even transfer matrix elements to be

$$T(o \to e)_{(\beta,\delta)} = T^\circ_{(\beta,\delta)} = v_\delta^T T^\circ v_\beta = z^{l(\delta \to \beta)}.$$

By the definition of matrix product, this yields

$$T_{(\beta,\alpha)} = (T^\circ T^\star)_{(\beta,\alpha)}.$$

Now we are ready to prove the first part of theorem 4.3.

Proof for $G_{z,H}(x, y) = v_y^T T^H v_x$:
Assume without loss of generality that H is even. Recall the proposition 4.6 and note that it implies a walk in the strip $S_{L,H}$ is uniquely characterized by fixing its arc-thread configurations at every level. Further note that length of the walk equals the sum of lengths of bridges the walk makes between consecutive levels. By using the short-hand x for the starting configuration with only a thread of real coordinate x_R and y for the final configuration with only a thread of real coordinate y_R, the function $G_{z,H} = G_{z,H}(x, y)$ can be re-expressed as

$$\begin{aligned} G_{z,H} &= \sum_{\gamma: x \to y} z^{l(\gamma)} \\ &= \sum_{\substack{\alpha_1, \alpha_2, \ldots \alpha_{H-1} \in C_E \\ \delta_1, \delta_2, \ldots, \delta_H \in C_O}} z^{l(x \to \delta_1)} z^{l(\delta_1 \to \alpha_1)} z^{l(\alpha_1 \to \delta_2)} \ldots z^{l(\delta_{H-1} \to \alpha_{H-1})} z^{l(\alpha_{H-1} \to \delta_H)} z^{l(\delta_H \to y)} \\ &= \sum_{\alpha_1, \alpha_2, \ldots \alpha_{H-1} \in C_E} T_{(y,\alpha_{H-1})} T_{(\alpha_{H-1},\alpha_{H-2})} \ldots T_{(\alpha_2,\alpha_1)} T_{(\alpha_1,x)} \\ &= T^H_{(y,x)} = v_y^T T^H v_x. \end{aligned} \tag{4.1}$$

□

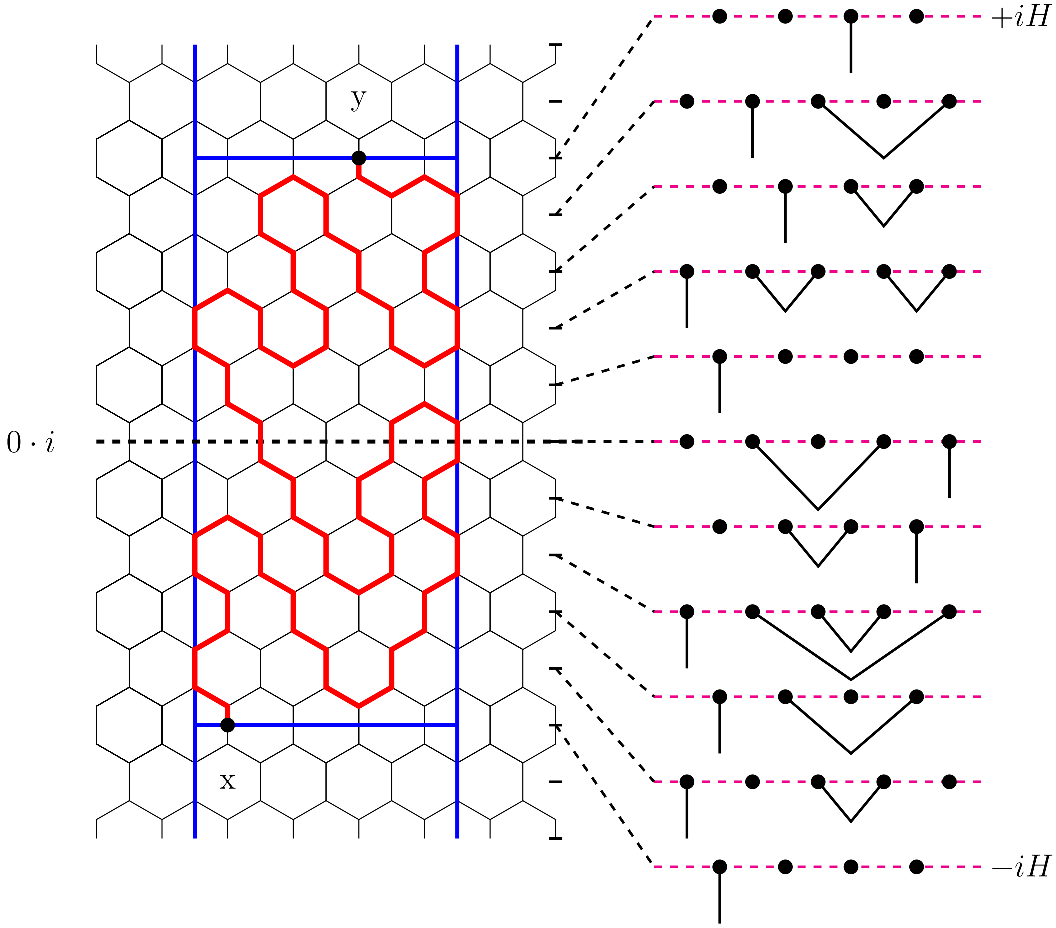

Figure 15: A characterization of a walk by arc-thread configurations

4.3 Calculation of edge visit probabilities

In this section we apply the fact that the generating function can be expressed with the transfer matrix to reduce the calculation of probabilities of the self-avoiding visiting vertical edges in the vertical strip to linear algebra. Using the probability measure $\mathbb{P}_{z,H}$ of definition 4.2, the probability of a self-avoiding walk in $S_{L,H}$ visiting the midedge

$$e = a + ib, \ -H + 1 \leq b \leq H - 1$$

is

$$\mathbb{P}_{z,H}[e \in \gamma] = \frac{\sum_{\gamma: x \to y} \mathbb{1}_{[e \in \gamma]} z^{l(\gamma)}}{G_{z,H}(x, y)}. \tag{4.2}$$

The results of the previous section reduce the proof of the second proposition of theorem 4.3 to proving that

$$\sum_{\gamma: x \to y} \mathbb{1}_{[e \in \gamma]} z^{l(\gamma)}$$

can be expressed with the transfer matrix T and a projection matrix $P(a)$.

Proof of Thm. 4.3.2. For simplicity, assume that b and H are both even. We make three remarks. The first remark is that according to definition 4.9 the arguments for the function $G_{z,H}(\cdot,\cdot)$ do not need to be points, they can be any arc-thread configurations. The second remark is that the generating function $G_{z,H}$ is translation invariant, if we move the cylinder domain $S_{L,H}$ an even number of levels up or down along the vertical strip S_L. The third remark is that the walk γ can be split into two parts, a self-avoiding set of walk segments $\gamma^{(1)}$ starting from x and ending in some arc-thread configuration, say α, below the level b and a self-avoiding set of walk segments $\gamma^{(2)}$ above the level b, starting from the configuration α and ending in y. By translating $\gamma^{(1)}$ and $\gamma^{(2)}$ to the sets $S_{L,(H+b)/2}$ and $S_{L,(H-b)/2}$, respectively, and applying the arguments of subsection 4.2, it follows that $\gamma^{(1)}$ has the generating function $T^{\frac{H+b}{2}}_{(\alpha,x)}$ and $\gamma^{(2)}$ has the generating function $T^{\frac{H-b}{2}}_{(y,\alpha)}$. The numerator on the right-hand side of the equation (4.2) can be expressed by summing the product of partition functions for walks $\gamma^{(1)}$ and $\gamma^{(2)}$ over all the configurations α at level b:

$$\sum_{\gamma:x\to y} \mathbb{1}_{[e\in\gamma]} z^{l(\gamma)} = \sum_{(\alpha:a\in\alpha)} T^{\frac{H-b}{2}}_{(y,\alpha)} T^{\frac{H+b}{2}}_{(\alpha,x)}.$$

By introducing projection matrices $P(a)$ for the set of configurations, defined entrywise by

$$v_\beta^T P(a) v_\alpha = P(a)_{(\beta,\alpha)} = \begin{cases} 1, & \text{if } \beta=\alpha \text{ and } a\in\alpha \\ 0, & \text{else} \end{cases}$$

the condition $a\in\alpha$ can be omitted from the sum:

$$\begin{aligned} \sum_{\gamma:x\to y} \mathbb{1}_{[e\in\gamma]} z^{l(\gamma)} &= \sum_{\alpha} T^{\frac{H-b}{2}}_{(y,\alpha)} P(a)_{(\alpha,\alpha)} T^{\frac{H+b}{2}}_{(\alpha,x)} \\ &= \sum_{\alpha,\beta} T^{\frac{H-b}{2}}_{(y,\beta)} P(a)_{(\beta,\alpha)} T^{\frac{H+b}{2}}_{(\alpha,x)}. \end{aligned}$$

Finally using the definition of matrix product yields the result of theorem 4.3

$$\begin{aligned} \mathbb{P}_{z,H}[e\in\gamma] &= \frac{\left(T^{\frac{H-b}{2}} P(a) T^{\frac{H+b}{2}}\right)_{(y,x)}}{T^H_{(y,x)}} \\ &= \frac{v_y^T T^{\frac{H-b}{2}} P(a) T^{\frac{H+b}{2}} v_x}{v_y^T T^H v_x}. \end{aligned}$$

□

It is straightforward to generalize the probability formula for a pair of edges

$$e_1 = a_1 + ib_1,\ e_2 = a_2 + ib_2,\ b_1 \le b_2.$$

For simplicity assume that H, b_1 and b_2 are all even. The walk can now be divided into three sets of walk segments, $\gamma^{(1)}$ below the level b_1, $\gamma^{(2)}$ between the levels b_1

and b_2 and $\gamma^{(3)}$ above the level b_2. Using the same arguments as in the proof for the probability of visiting single edge yields

$$\mathbb{P}_{z,H}[e_1, e_2 \in \gamma] = \frac{v_y^T T^{\frac{H-b_2}{2}} P(a_2) T^{\frac{b_2-b_1}{2}} P(a_1) T^{\frac{H+b_1}{2}} v_x}{v_y^T T^H v_x}.$$

Remark 4.11. The same technique works for expressing the probability of visiting any set vertical of edges $(e_j)_{j=1}^n, e_j = a_j + ib_j$ with the rule $b_j \leq b_{j+1}$.

For the next section, we also need to consider the equation (4.2) with odd b. Recall the notation from definition 4.10 where $T^\star$ denotes the transfer matrix from an even level to the following odd one and T° denotes the transfer matrix from an odd level to the following even one. By splitting the walk to the segments $\gamma^{(1)}$ below level $b-1$, $\gamma^{(2)}$ between levels $b-1$ to b, $\gamma^{(3)}$ from b to $b+1$ and $\gamma^{(4)}$ above level $b+1$, the same reasoning as before yields for $e = a + ib$

$$\mathbb{P}_{z,H}[e \in \gamma] = \frac{v_y^T T^{\frac{H-1-b}{2}} T^\circ P(a) T^\star T^{\frac{H-1+b}{2}} v_x}{v_y^T T^H v_x}.$$

Finally we give the same formulas for $e = a + ib$ when H is odd. First when b is even,

$$\mathbb{P}_{z,H}[e \in \gamma] = \frac{v_y^T T^\circ T^{\frac{H-1-b}{2}} P(a) T^{\frac{H-1+b}{2}} T^\star v_x}{v_y^T T^\circ T^{H-1} T^\star v_x},$$

and similarly when both b and H are odd:

$$\mathbb{P}_{z,H}[e \in \gamma] = \frac{v_y^T T^\circ T^{\frac{H-2-b}{2}} T^\star P(a) T^\circ T^{\frac{H-2+b}{2}} T^\star v_x}{v_y^T T^\circ T^{H-1} T^\star v_x}.$$

4.4 Weak convergence to a limit measure

In this section we first present a criterion that guarantees weak convergence for a sequence of probability measures on a metric space X given that the probabilities converge for a certain family that belongs to the Borel σ-algebra of X. We show that the spaces S^I with a finite set of values S and a countable number of indices I are metric spaces, and in addition that they are separable and compact. We then apply the first criterion to cylinder events and use Prokhorov's theorem to prove that the convergence of probabilities of cylinder events in the sets S^I characterizes weak convergence. Finally we prove that the sequence $(\mathbb{P}_{z,H})_{H\in\mathbb{N}}$ converges to a limit measure $\mathbb{P}_z$ by showing that the probabilities of cylinder events converge.

Proposition 4.12. *Let (X, ρ) be a metric space and $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra. Suppose $\mathcal{E} \subset \mathcal{B}$ is a collection such that*

- *$\mathcal{E}$ is stable under finite intersections, i.e. $E_1, E_2 \in \mathcal{E}$ implies $E_1 \cap E_2 \in \mathcal{E}$*
- *Any open set $U \subset X$ is a countable union of sets from $\mathcal{E}$:*

$$U = \bigcup_{i\in\mathbb{N}} E_i, \text{ where } E_i \in \mathcal{E}.$$

Then a sequence of probability measures $(\nu_h)_{h\in\mathbb{N}}$ of probability measures on X converges weakly to a probability measure ν if for all $E \in \mathcal{E}$ we have $\nu_h[E] \to \nu[E]$.

Proof. If $E_1, E_2, \dots E_m \in \mathcal{E}$, then their intersection $\bigcap_{i=1}^m E_i$ is also in $\mathcal{E}$. By the inclusion-exclusion formula

$$\nu_h[\bigcup_{i=1}^m E_i] = \sum_{\substack{J\subset\{1,2,\dots,m\}\\ J\neq\emptyset}} (-1)^{|J|}\nu_h[\bigcap_{j\in J} E_j].$$

The finite intersections of sets in $\mathcal{E}$ belong to the family $\mathcal{E}$ and thus the inclusion-exclusion formula converges to

$$\sum_{\substack{J\subset\{1,2,\dots,m\}\\ J\neq\emptyset}} (-1)^{|J|}\nu[\bigcap_{j\in J} E_j] = \nu[\bigcup_{i=1}^m E_i].$$

If a set $U \subset X$ is open, by the second condition there is a countable union of sets in E_i in the collection $\mathcal{E}$ such that $U = \bigcup_{i=1}^\infty E_i$. Convergence of the sequence of probability measures for sets $E \in \mathcal{E}$ gives

$$\nu[\bigcup_{i=1}^m E_i] = \lim_{h\to\infty} \nu_h[\bigcup_{i=1}^m E_i] \leq \liminf_{h\to\infty} \nu_h[U].$$

On the other hand, $(\bigcup_{i=1}^m E_i)_{m\in\mathbb{N}}$ is an increasing sequence in the sense that $\bigcup_{i=1}^m E_i \subset \bigcup_{i=1}^{m+1} E_i$ with limit U so the left hand side tends to $\nu[U]$ by monotone approximation of measures. The equation is now in a form that characterizes weak convergence of ν_h to ν by the Portmanteau theorem. □

Theorem 4.13 (Prokhorov's theorem)**.** *[Shi95, p. 318] Let (S,ρ) be a separable metric space. Then for every sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on (S,ρ) there exists a weakly converging subsequence if and only if the set of probability measures on (S,ρ) is* tight*.*

Remark 4.14. The set of probability measures on a metric separable space (S,ρ) is tight if the space is compact.

Lemma 4.15. *Let S be a finite set of values and I a countable set of indices. Define the metric ρ on S^I as:*

$$\rho(\omega,\omega') = \sum_{j\in I} 2^{-j}\mathbb{1}_{[\omega_j\neq\omega'_j]} = \sum_{j:\omega_j\neq\omega'_j} 2^{-j}.$$

This leads to a complete, separable and compact metric space $\left(S^I,\rho\right)$.

Proof. It can be assumed without loss of generality that $I = \mathbb{N}$.

- ρ is a metric on S^I:
  By the definition of ρ, we have

$$0 \leq \rho(\omega,\omega') \leq \sum_{i\in\mathbb{N}} 2^{-i} = 1.$$

Thus ρ satisfies the non-negativity axiom of metrics and the space $(S^{\mathbb{N}}, \rho)$ is bounded with diameter 1.
To prove the identity of indiscernibles for ρ, note that by definition of $\rho(\omega, \omega') = 0$ is equivalent with $\omega_i = \omega'_i$ for all i in $\mathbb{N}$, which means that $\omega = \omega'$.
It is obvious that ρ satisfies the symmetry axiom. If $\omega_i \neq \omega'_i$, then $\omega'_i \neq \omega_i$, i.e.

$$\rho(\omega, \omega') = \rho(\omega', \omega).$$

To prove the triangle inequality, let $\omega, \omega', \omega''$ belong to $S^{\mathbb{N}}$. By simple reasoning

$$\rho(\omega, \omega'') = \sum_{\{i\in\mathbb{N}:\omega_i\neq\omega''_i\}} 2^{-i} \leq \sum_{\{i\in\mathbb{N}:\omega_i\neq\omega'_i\}} 2^{-i} + \sum_{\{i\in\mathbb{N}:\omega'_i\neq\omega''_i\}} 2^{-i} = \rho(\omega, \omega') + \rho(\omega', \omega'').$$

This establishes the function ρ as a metric, as it satisfies the required axioms. Next we prove the properties of the space S^I using the metric ρ.

- The projections $\pi_i : S^I \to S$, $\pi_i(\omega) = \omega_i$ are continuous:
  Let $\omega, \omega' \in S^{\mathbb{N}}$ be such that $\rho(\omega, \omega') < \sum_{i=k}^{\infty} 2^{-i} = 2^{-(k-1)}, k \in \mathbb{N}$. This directly implies that
  $$\pi_i(\omega) = \omega_i = \omega'_i = \pi_i(\omega') \text{ for all } i \leq k.$$
  Let us write this in the form of Weierstrass continuity
  $$\text{If } \rho(\omega, \omega') < \frac{1}{2^{k-1}}, \text{ then } \pi_k(\omega) = \pi_k(\omega').$$
  The above holds for all $k \in \mathbb{N}$, which means that the projections π are all continuous.

- The space $\left(S^I, \rho\right)$ is complete:
  Define $\varepsilon_k = 2^{-k+1}$ and let m_k be the first index of the Cauchy sequence of sequences $\left(\omega^{(n)}\right)$ in $S^{\mathbb{N}}$ subject to $\rho(\omega^{(n)}, \omega^{(m)}) < \varepsilon_k$ for all $m, n \geq m_k$. Then by the continuity of projections π_i it holds that $\omega_k^{(m)} = \omega_k^{(n)}$ for all $m, n \geq m_k$. Setting now elementwise $\omega_k = \omega_k^{(m_k)}$ results in a sequence ω such that $\omega_j^{(n)} \longrightarrow \omega_j$ for every j. Hence Cauchy sequences converge in $\left(S^{\mathbb{N}}, \rho\right)$, meaning that the space is complete.

- The space $\left(S^I, \rho\right)$ is separable:
  Assume S is non-empty and pick any $s^* \in S$. Define
  $$\mathcal{D} = \left\{\omega \in S^{\mathbb{N}} | \exists\ N \text{ s.t. } \omega_j = s^* \text{ for all } j \geq N\right\}.$$
  Then
  $$\mathcal{D} = \bigcup_{N\in\mathbb{N}} S^N \times \{s^*\}^{\mathbb{N}-N}$$
  $$|\mathcal{D}| = |\bigcup_{N\in\mathbb{N}} S^N| = |\mathbb{N}|,$$

since $\bigcup_{N\in\mathbb{N}} S^N$ is a countable union of countable sets. Moreover, for every $\omega \in S^{\mathbb{N}}$, we can define

$$\omega_j^{(n)} = \begin{cases} \omega_j, & j < n \\ s^*, & j \geq n \end{cases}$$

when $\omega^{(n)} \in \mathcal{D}$ and $\omega^{(n)} \to \omega$. Since there is a countable subset of $S^{\mathbb{N}}$ where we can have sequences converging to any sequence in $S^{\mathbb{N}}$, the space $(S^{\mathbb{N}}, \rho)$ is separable.

- The space $\left(S^I, \rho\right)$ is compact:
  Let $(\omega^{(n)})_{n\in\mathbb{N}}$ be a sequence with elements $(\omega_i^{(n)})_{i\in I} \in S^I$. The finite set S is compact, so for any $i \in I$ we find subsequences $(\omega^{(n_k)})_{k\in\mathbb{N}}$ such that $\omega_i^{(n_k)}$ converges. Since I is countable, by diagonal extraction we find a subsequence such that $\omega_i^{(n_k)}$ converges for all $i \in I$. A componentwise limit is a limit.

□

Definition 4.16. Events $C = \{\omega \in S^I | \omega_{i_1} \in S_1, ..., \omega_{i_n} \in S_n\}$, where $S_1, ..., S_n$ are subsets of the finite set of values S, are called *cylinder events*. Cylinder events are both open and closed.

Lemma 4.17. *A sequence of probability measures $(\nu_h)_{h\in\mathbb{N}}$ on S^I converges weakly if and only if for every cylinder event C the limit $\lim_{h\to\infty} \nu_h[C]$ exists.*

Proof of lemma 4.17.

- Weak convergence implies convergence for cylinder events:
  Suppose that the sequence (ν_h) of probability measures converges weakly to ν and let C be a cylinder event. Then C is both open and closed, in particular $\partial C = \emptyset$. Thus by the Portmanteau theorem the sequence $\nu_h[C]$ converges to a limit $\nu[C]$ as h tends to infinity.

- Convergence for cylinder events implies weak convergence:
  The collection of cylinder events is stable under finite intersections, and any open set is a countable union of cylinder sets. By the proposition (4.12), it is sufficient for weak convergence that $\nu_h[C] \to \nu[C]$ for all cylinder sets C, where ν is a probability measure on S^I. Assume that $\alpha[C] = \lim_h \nu_h[C]$ exists for all cylinders C, whereafter all that needs to be done is to show that α is a probability measure.

  Recall that S^I is compact and therefore the sequence of probability measures $(\nu_h)_{h\in\mathbb{N}}$ is automatically tight. By Prokhorov's theorem for separable metric spaces there exists a subsequence $(\nu_{h_k})_{k\in\mathbb{N}}$ such that ν_{h_k} converges weakly to a probability measure ν as k tends to infinity. Clearly $\nu[C] = \alpha[C]$. This shows that α is a probability measure.

□

In the sets $S_{L,H}$ we take the set of values to be $S = \{0, 1\}$ and the set of indices I to be an enumeration $e_1, e_2, e_3, ...$ of vertical edges in the hexagonal lattice. We further interpret that $\gamma(e_n) = 1$ if the self-avoiding walk γ visits the edge e_n and $\gamma(e_n) = 0$ if the walk does not visit the edge e_n.
Next we use the inclusion-exclusion principle to show that the probabilities of cylinder events converge. A cylinder event in our case is a collection of edges E that the walk visits and a collection of edges D that the walk does not visit, yielding the indicator function

$$\begin{aligned}
\mathbb{1}_{[(\bigcap_{e\in E}[e\in\gamma])\cap(\bigcap_{d\in D}[d\notin\gamma])]} &= \prod_{e\in E}\mathbb{1}_{[e\in\gamma]}\prod_{d\in D}\mathbb{1}_{[d\notin\gamma]} \\
&= \prod_{e\in E}\mathbb{1}_{[e\in\gamma]}\prod_{d\in D}\left(1-\mathbb{1}_{[d\in\gamma]}\right) \\
&= \prod_{e\in E}\mathbb{1}_{[e\in\gamma]}\sum_{D'\subset D}\prod_{d\in D'}\left(-\mathbb{1}_{[d\in\gamma]}\right).
\end{aligned}$$

Using the probability measure 4.2, the probability of this event is

$$\begin{aligned}
&\mathbb{P}_{z,H}\left[\left(\bigcap_{e\in E}[e\in\gamma]\right)\cap\left(\bigcap_{d\in D}[d\notin\gamma]\right)\right] \\
&= \frac{\sum_{\gamma:x\to y}\prod_{e\in E}\mathbb{1}_{[e\in\gamma]}\sum_{D'\subset D}\prod_{d\in D'}\left(-\mathbb{1}_{[d\in\gamma]}\right)z^{l(\gamma)}}{G_{z,H}(x,y)}.
\end{aligned}$$

Assume that the sets D and E are contained between levels $[-N, N]$ for some even N and $H > N$. Split the walk into three parts: a lower part starting with configuration x at level $-H$ and ending at an arbitrary configuration at level $-N$, an arbitrary middle part between levels $-N$ and $+N$, and an upper part starting with an arbitrary configuration at level $+N$ and ending in configuration y at level $+H$. By treating all the edges in the sets D' and E of the sum analogously to how the probability of visiting given two edges (4.3) was calculated in the previous section, one can define $2N$-step combined transfer and projection matrices $P(D, E)$ for the middle part. For even H one then has:

$$\begin{aligned}
&\mathbb{P}_{z,H}\left[\left(\bigcap_{e\in E}[e\in\gamma]\right)\cap\left(\bigcap_{d\in D}[d\notin\gamma]\right)\right] \\
&= \frac{\sum_{\gamma:x\to y}\prod_{e\in E}\mathbb{1}_{[e\in\gamma]}\sum_{D'\subset D}\prod_{d\in D'}\left(-\mathbb{1}_{[d\in\gamma]}\right)z^{l(\gamma)}}{G_{z,H}(x,y)} \\
&= \frac{v_y^T T^{\frac{H-N}{2}} P(D,E) T^{\frac{H-N}{2}} v_x}{v_y^T T^H v_x},
\end{aligned} \tag{4.3}$$

while for odd H the same probability is given by

$$\begin{aligned}
&\mathbb{P}_{z,H}\left[\left(\bigcap_{e\in E}[e\in\gamma]\right)\cap\left(\bigcap_{d\in D}[d\notin\gamma]\right)\right] \\
&= \frac{v_y^T T^{\circ} T^{\frac{H-1-N}{2}} P(D,E) T^{\frac{H-1-N}{2}} T^{\star} v_x}{v_y^T T^{\circ} T^{H-1} T^{\star} v_x},
\end{aligned}$$

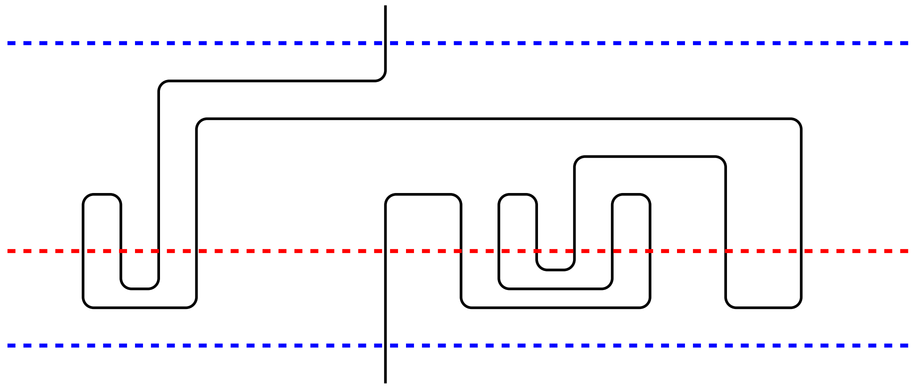

Figure 16: Transition from a configuration of no arcs to arbitrarily many arcs and back

where $T^{\star}$ again denotes the odd-to-even transfer matrix and T° the even-to-odd transfer matrix.

Proposition 4.18. *In domains where the set of arc-thread configurations at level h is the same as the one at level $h+n$ for some $n \in \mathbb{N}$ and all h, all reachable configurations belong to the same communication class and the configurations are aperiodic. Therefore the Perron-Frobenius theorem applies to submatrices of reachable configurations of the corresponding n-step transfer matrices.*

Proof. Let us start with a configuration of one thread and no arcs. To see that all configurations belong to the same connection class, consider a transition to a configuration with m sets of arcs inside each other to added the right of the thread and n sets of arcs inside each other added to the left of the thread. To make the transition to this system and back, we can make the walk traverse first through the arcs on the right, going each set from the outermost arc to the innermost arc and then continuing to the next set. After all arcs on the right have been traversed, the walk goes straight to the left to the arc set closest to where the thread is and what happened on the right hand side is repeated. Finally the walk exits to the center, proving that also the transition back from the configuration with arcs to the configuration with no arcs is possible. This is shown in fig. 16. It is trivial that for a configuration with no arcs, the thread can stay where it was with a single transition, and the thread can be moved anywhere with a finite number of transitions, which completes the proof.

□

In particular the proposition implies that there exists a triplet (λ, w^T, v) such that

- λ is a unique positive eigenvalue of T
- v is a unique non-negative right eigenvector of T with positive entries v_i for every i corresponding to a reachable configuration α_i.

- w^T is a unique non-negative left eigenvector of T with positive entries w_i for every i corresponding to a reachable configuration α_i.
- For every non-negative column vector $u \neq 0$ there are positive constants A_u, B_u such that
$$\lim_{k\to\infty} \frac{T^k u}{\lambda^k} = A_u v, \ \lim_{k\to\infty} \frac{u^T T^k}{\lambda^k} = B_u w^T.$$

Applying these identities to equation 4.3 for fixed D, E and any sequence of endpoints $(x_H, y_H)_{\{H>N\}}$, we have

$$\begin{aligned}
&\lim_{H\to\infty} \mathbb{P}_{z,H}\left[\left(\bigcap_{e\in E}[e\in\gamma]\right)\cap\left(\bigcap_{d\in D}[d\notin\gamma]\right)\right] \\
&= \lim_{H\to\infty} \frac{w^T B \lambda^{\lfloor \frac{H-N}{2}\rfloor} P(D,E) \lambda^{\lfloor \frac{H-N}{2}\rfloor} A v}{w^T B \lambda^{\lfloor \frac{H-N}{2}\rfloor} T^N \lambda^{\lfloor \frac{H-N}{2}\rfloor} A v} \\
&= \frac{w^T P(D,E) v}{w^T \lambda^N v}
\end{aligned}$$

This means that the probabilities of cylinder events converge as H tends to infinity, which in turn implies that there exists a limit measure $\mathbb{P}_z$ that the sequence $(\mathbb{P}_{z,H})_{H\in\mathbb{N}}$ converges to.
For the limit measure $\mathbb{P}_z$ we get the edge visiting probabilities by taking the limit and using properties of the matrix T for equations (4.3) and (4.3). Here we can without loss of generality assume that H is even. For the edge $e = a + ib$

$$\begin{aligned}
&\mathbb{P}_z[e\in\gamma] \\
&= \lim_{H\to\infty} \mathbb{P}_{z,H}[e\in\gamma] \\
&= \lim_{H\to\infty} \frac{v_y^T T^{\lfloor \frac{H-b}{2}\rfloor} (T^\circ)^{\mathbb{1}_{[b\ odd]}} P(a) (T^\star)^{\mathbb{1}_{[b\ odd]}} T^{\lfloor \frac{H+b}{2}\rfloor} v_x}{v_y^T T^H v_x}.
\end{aligned}$$

This is simplified similarly to the probability of a cylinder event, resulting in

$$\begin{aligned}
&\mathbb{P}_z[e\in\gamma] \\
&= \lim_{H\to\infty} \frac{v_y^T T^{\lfloor \frac{H-b}{2}\rfloor} (T^\circ)^{\mathbb{1}_{[b\ odd]}} P(a) (T^\star)^{\mathbb{1}_{[b\ odd]}} T^{\lfloor \frac{H+b}{2}\rfloor} v_x}{v_y^T T^{\lfloor \frac{H-b}{2}\rfloor} T^{\mathbb{1}_{[b\ odd]}} T^{\lfloor \frac{H+b}{2}\rfloor} v_x} \\
&= \lim_{H\to\infty} \frac{ABw^T \lambda^{\lfloor \frac{H-b}{2}\rfloor} (T^\circ)^{\mathbb{1}_{[b\ odd]}} P(a) (T^\star)^{\mathbb{1}_{[b\ odd]}} \lambda^{\lfloor \frac{H+b}{2}\rfloor} v}{ABw \lambda^{\lfloor \frac{H-b}{2}\rfloor} \lambda^{\mathbb{1}_{[b\ odd]}} \lambda^{\lfloor \frac{H+b}{2}\rfloor} v} \\
&= \frac{w^T (T^\circ)^{\mathbb{1}_{[b\ odd]}} P(a) (T^\star)^{\mathbb{1}_{[b\ odd]}} v}{w \lambda^{\mathbb{1}_{[y\ odd]}} v}.
\end{aligned}$$

Likewise the limit of the probability 4.3 of visiting two edges

$$e_1 = a_1 + ib_1, e_2 = a_2 + ib_2$$

at heights $b_1 \le b_2$ can be calculated

$$\begin{aligned}
&\mathbb{P}_z[e_1, e_2 \in \gamma] \\
&= \lim_{H\to\infty} \mathbb{P}_{z,H}[e_1, e_2 \in \gamma] \\
&= \lim_{H\to\infty} \frac{v_y^T T^{\lfloor \frac{H-b_2}{2} \rfloor} (T^\circ)^{\mathbb{1}[b_2\ odd]} P(a_2) (T^\star)^{\mathbb{1}[b_2\ odd]} T^{\lfloor \frac{b_2}{2} \rfloor - \lceil \frac{b_1}{2} \rceil} (T^\circ)^{\mathbb{1}[b_1\ odd]} P(a_1) (T^\star)^{\mathbb{1}[b_1\ odd]} T^{\lfloor \frac{H+b_1}{2} \rfloor} v_x}{v_y^T T^H v_x}. \\
&= \frac{w^T (T^\circ)^{\mathbb{1}[b_2\ odd]} P(a_2) (T^\star)^{\mathbb{1}[b_2\ odd]} T^{\lfloor \frac{b_2}{2} \rfloor - \lceil \frac{b_1}{2} \rceil} (T^\circ)^{\mathbb{1}[b_1\ odd]} P(a_1) (T^\star)^{\mathbb{1}[b_1\ odd]} v}{w^T \lambda^{\mathbb{1}[b_1\ odd] + \mathbb{1}[b_2\ odd] + \lfloor \frac{b_2}{2} \rfloor - \lceil \frac{b_1}{2} \rceil} v}.
\end{aligned} \tag{4.4}$$

4.5 The Markov property of the limit measure

Next we show that the limit measure $\mathbb{P}_z$ defines a Markovian process with respect to levels. The probability measure $\mathbb{P}_z$ (respectively $\mathbb{P}_{z,H}$) is a distribution on the set of walks γ. By proposition4.6 a walk γ can be characterized by the infinite (resp. finite) sequence of configurations $(\alpha_h)_{h\in\mathbb{Z}}$ indexed by height. For a random self-avoiding walk γ, the sequence (α_h) can be thought of as a configuration-valued stochastic process indexed by h. This process turns out to be Markovian.

Start with the two-edge visit probability (4.4) defined above, substitute $b_2 = b_1 + 1$ and replace the projection matrices $P(a_1), P(a_2)$ by matrices $P(\alpha), P(\beta)$ projecting onto configurations α and β, respectively. This renders the probability of the walk reaching configuration β at level $h+1$ and configuration α at level h to:

$$\begin{aligned}
&\mathbb{P}_{z,H}[\gamma_{h+1} = \beta, \gamma_h = \alpha] \\
&= \frac{v_y^T T^{\lfloor \frac{H-h-1}{2} \rfloor} (T^\circ)^{\mathbb{1}[h+1\ odd]} P(\beta) (T^\star)^{\mathbb{1}[h+1\ odd]} (T^\circ)^{\mathbb{1}[h\ odd]} P(\alpha) (T^\star)^{\mathbb{1}[h\ odd]} T^{\lfloor \frac{H+h}{2} \rfloor} v_x}{v_y^T T^H v_x}.
\end{aligned}$$

Using the same technique on the single edge visit probability yields the probability of the walk reaching configuration α at level h.

$$\mathbb{P}_{z,H}[\gamma_h = \alpha] = \frac{v_y^T T^{\lfloor \frac{H-h}{2} \rfloor} (T^\circ)^{\mathbb{1}[h\ odd]} P(\alpha) (T^\star)^{\mathbb{1}[h\ odd]} T^{\lfloor \frac{H+h}{2} \rfloor} v_x}{v_y^T T^H v_x}.$$

Combining these two, the conditional probability for the event $\gamma_{h+1} = \beta$ with the condition $\gamma_h = \alpha$ is

$$\begin{aligned}
&\mathbb{P}_{z,H}[\gamma_{h+1} = \beta | \gamma_h = \alpha] = \frac{\mathbb{P}_{z,H}[\gamma_{h+1} = \beta, \gamma_h = \alpha]}{\mathbb{P}_{z,H}[\gamma_h = \alpha]} \\
&= \frac{v_y^T T^{\lfloor \frac{H-h-1}{2} \rfloor} (T^\circ)^{\mathbb{1}[h+1\ odd]} P(\beta) (T^\star)^{\mathbb{1}[h+1\ odd]} (T^\circ)^{\mathbb{1}[h\ odd]} P(\alpha) (T^\star)^{\mathbb{1}[h\ odd]} T^{\lfloor \frac{H+h}{2} \rfloor} v_x}{v_y^T T^{\lfloor \frac{H-h}{2} \rfloor} (T^\circ)^{\mathbb{1}[h\ odd]} P(\alpha) (T^\star)^{\mathbb{1}[h\ odd]} T^{\lfloor \frac{H+h}{2} \rfloor} v_x}
\end{aligned}$$

To simplify the equation, we will treat the cases h even, h odd separately. When h is even

$$\begin{aligned}
&\mathbb{P}_{z,H}[\gamma_{h+1} = \beta | \gamma_h = \alpha, h \text{ even}] \\
&= \frac{v_y^T T^{\frac{H-h-2}{2}} T^\circ P(\beta) T^\star P(\alpha) T^{\frac{H+h}{2}} v_x}{v_y^T T^{\frac{H-h}{2}} P(\alpha) T^{\frac{H+h}{2}} v_x}.
\end{aligned}$$

Taking the limit $H \to \infty$ yields

$$\mathbb{P}_z[\gamma_{h+1} = \beta | \gamma_h = \alpha, h \text{ even}] = \frac{wT^\circ P(\beta) T^\star P(\alpha) v}{w\lambda P(\alpha) v}.$$

Doing the same calculations for odd h yields

$$\mathbb{P}_z[\gamma_{h+1} = \beta | \gamma_h = \alpha, h \text{ odd}] = \frac{wP(\beta) T^\circ P(\alpha) T^\star v}{wT^\circ P(\alpha) T^\star v}.$$

In neither of these cases does the conditional probability depend on the trajectory of the walk below level h, thus we conclude that the limit measure $\mathbb{P}_z$ for infinitely long self-avoiding walks on the strip S_L is Markovian with respect to the configurations at integer levels. It should be noted however, that the measures $\mathbb{P}_{z,H}$ for walks on the finite strip domains $S_{L,H}$ do not share this property.

5 Computations with the transfer matrix

The goal of this section is to provide computational support for the conjecture that the scaling limit of the critical self-avoiding walk is conformally invariant. We consider the SAW in a vertical strip. Fixing the width of the strip and the fugacity z, we present a way to construct the sets of possible configurations C_O, C_E for both definitions of S_L and assemble the transfer matrices $T^\circ, T^\star$. By taking the matrix product of these, we have the square matrix T, eigenvalues and eigenvectors of which determine the limit measure $\mathbb{P}_L$. After computationally solving the largest eigenvalue and the associated eigenvectors of the matrix T, the idea is to compare the probability of a self-avoiding walk in the infinite strip returning to the edge of strip with the probability of the conformally invariant half-plane curve chordal $SLE_{8/3}$ returning to the real line. When the strip is conformally mapped to the half-plane the probabilities of the SAW and the SLE returning to the real line should converge as the mesh size of the strip approaches zero.
There are five steps to be done in this process:

- Constructing the sets of possible configurations C_O, C_E
- Assembling the transfer matrices $T^\circ, T^\star, T$
- Solving the largest eigenvalue and corresponding left and right eigenvectors of the square matrix T
- Calculating the probabilities of a walk in the infinite strip returning to the right boundary of the strip
- Comparing these probabilities with the two-point function of chordal $SLE_{8/3}$.

5.1 Constructing the sets of possible configurations

Recalling definition 4.5, the idea is to consider all ways to place the thread in the system of L edges. For all thread placements we want to calculate all possible ways to place arcs to the left of the edge and all ways to place arcs to the right of the edge. We then get the basis of arc-thread configurations by taking the union over all thread locations for the cartesian product of the location of the thread, all ways to place arcs to the left of the thread and all ways to place arcs to the right of the thread .
The task of constructing the basis for a level that is $L-1$ hexagons, or L edges, wide can be reduced to

- Creating a function that calculates all possible ways fill an even number $2n$ of edges $[1, 2, \ldots, 2n]$ with n arcs.
- Saving the full arc configurations on the interval $[1, 2, \ldots, 2n]$.
- Creating a function that returns all subsets with even number of members from a given interval

- Creating a function that replaces the indices $[1, 2, \ldots, 2n]$ of a precomputed full arc configuration with the indices $[i_1, i_2, \ldots, i_{2n}]$ of an arbitrary set of $2n$ edges
- Creating a function that places the thread in edge k in the system of l edges indexed as $1, 2, \ldots, L$ and calculates the set L_k of legitimate arc configurations in the interval $[1, \ldots, k-1]$ and the set R_k of legitimate arc configurations in the interval $[k+1, \ldots, L]$
- Collecting the configurations associated to thread placement k by taking the cartesian product $\{k\} \times L_k \times R_k$.
- Finally we get all configurations by looping k over $1, 2, ..., L$ and taking the union
$$\bigcup_{k=1}^{L} \{\{k\} \times L_k \times R_k\}.$$

```
cartesianProd[a_List, b_List] :=
  Map[{a[[#[[1]]]], b[[#[[2]]]]} &,
   Flatten[Table[{i, j}, {i, 1, Length[a]}, {j, 1, Length[b]}], 1]];
```

As a preliminary function for the main loop we define a Cartesian product for two sets with lists of elements a and b as the collection of all pairs where first part of the pair comes from the first set, and the second part of the pair comes from the second set.

```
arcsList[0] = {{}};
combineArcConfigs[p_, n_] :=
  Map[Union[{{1, p}}, #] &,
   Map[Apply[Union, #] &,
    cartesianProd[arcsList[(p - 2)/2] + 1,
     arcsList[(2*n - p)/2] + p]]];

arcsList[n_Integer /; n > 0] :=
  Apply[Union, Map[combineArcConfigs[#, n] &, Range[2, 2*n, 2]]];
```

The function `arcsList[n]` gives all ways to fill edges $[1, 2, \ldots, 2n]$ with arcs. `arcsList[0]` is defined as the one-element set containing only the empty set. For arguments $n > 0$, we use recursion. `combineArcConfigs[p_ ,n_ ]` is a function that gives union of the arc $\{1, p\}$ with the cartesian product of the possible ways to fill the interval $[2, p-1]$ with arcs and the possible ways to fill the interval $[p+1, 2n]$ with arcs. Letting p run from 2 to $2n$ in steps of 2 and collecting the results, we get all ways to fill the the interval $[1, 2n]$ with arcs. This is what the function `arcsList[n_Integer /; n > 0]` does.

```
evenSubsets[a_List] :=
  Apply[Union, Map[Subsets[a, {2*#}] &, Range[0, Length[a]/2]]];
```

The function evenSubsets does the second step in the algorithm of constructing the basis, it gives all sublists of even parity for a list of edges.

```
confSubset[a_List] :=
  Module[{conf = preCompArcs[[(Length[a]/2) + 1]], n},
   For[n = Length[a], n >=  1, n--, conf = conf /. {n -> a[[n]]}];
   conf];
```

The function confSubset does the third step in the algorithm. It takes in as argument a list $i_1, ..., i_{2m}$ and starting from $2m$ replaces all instances of k by i_k in the set of precomputed full arc configurations for the interval $1, 2, ..., 2m$. In other words it gives all ways to fill an arbitrary list of even parity with arcs.

Using functions evenSubsets and confSubset it is now possible to create the sets of admissible arc configurations L_k and R_k left and right to a thread placed in edge k. An example of the whole program to construct the basis is presented below.

```
basis[l_Integer, nl_Integer] :=
  Module[ {basisv = {}},
   (* l := number of edges*)
    (*nl := number of threads *)
   preCompArcs =
    Table[arcsList[k], {k, 0, Floor[(l - nl)/2]}];(*Precompute and save
     full arc configurations for the intervals [1,2m], m=0,1,2,...
   l-nl is the largest possible length for the interval*)

   grid = Range[l]; (* the set of edges 1,2,...,l*)

   threadloc =
    Subsets[grid, {nl}]; (*all possible ways to place the nl thread(s)*)

   For[i = 1, i <= Length[threadloc], i++,
    (* index the possible ways to place the thread(s) with i *)
    config[i] = {};(* config[i] := configurations associated to thread location i*)
    threadloc[[i]] = Union[{0}, threadloc[[i]], {l + 1}];

    For[j = 1, j <= nl + 1, j++,

     (*numerate the intervals between the threads by j*)

     interval[i][j] =
      Range[threadloc[[i, j]] + 1, threadloc[[i, j + 1]] - 1];
     (*Form the intervals*)

     arcs[i][j] =
      Apply[Union, Map[confSubset[#] &, evenSubsets[interval[i][j]]]];
     (*arcs[i][j] the arc configurations of the jth interval in the thread placement i*)

     If[arcs[i][j] !=  {{}} && Length[config[i]] != 0,
       config[i] = cartesianProd[config[i], arcs[i][j]]
        ;]
      If[Length[config[i]] == 0,
       config[i] = arcs[i][j];]
     (*collect to config[i]:
```

```
all arc possibilities of the intervals j*)
 ];
threadloc[[i]] = Most[Rest[threadloc[[i]]]];

config[i] = cartesianProd[{threadloc[[i]]}, config[i]];

(*add the information about thread placement*)
basisv = Union[basisv, config[i]]
(*add the configurations associated to the thread placement i to the union*)
];

basisv];
```

The following lemma can be used to check that the program gives the correct number of configurations.

Lemma 5.1. *Let L denote the number of edges in the given level, and k the number of edges not connected to the thread or arcs. Then the number of configurations $|C(L)|$ in the level can be calculated from the formula*

$$|C(L)| = \sum_{k=0}^{L-1} \mathbb{1}_{[L-k \text{ is odd}]} \binom{L}{k} \left(\binom{L-k}{\frac{L-k+1}{2}} - \binom{L-k}{\frac{L-k-3}{2}} \right).$$

Proof. For simplicity consider first the case where all edges are occupied. Each configuration can be seen as a discrete walk

$$(X_t)_{t=0}^{L}, X_0 = 0, X_{t+1} = X_t \pm 1,$$

with the interpretation that an arc starting from the n^{th} edge corresponds to a step in the positive direction at time $t = n$, a closing arc corresponds to a step in the negative direction and the thread is an arc that starts but does not close. Then there are $\binom{L}{n}$ walks reaching $X_L = +1$ in w steps, where m is the number of steps in positive direction and is given by $m = \frac{L+1}{2}$. To have a valid interpretation as configurations, we demand that $X_t > -1$ for all t. The number of L-step walks that end in $X_L = 1$ and intersect the line $y = -1$ at some time is by symmetry the same as the number of l-step walks that end in $X_L = -3$ and cross the line $y = -1$ at some time. However, every walk that ends in $X_L = -3$ crosses the line $y = -1$. Hence the number of w-step walks that do not cross the line $y = -1$ and end in $y = 1$ is

$$\left(\binom{L}{\frac{L+1}{2}} - \binom{L}{\frac{L-3}{2}} \right).$$

Allowing unoccupied edges, these correspond to steps where the walk stays where it is, and there are $\binom{L}{k}$ ways to place the unoccupied edges in the level. Combining these two observations, the lemma readily follows. □

number of edges L	the number of configurations $\|C(L)\|$
1	1
2	2
3	5
4	12
5	30
6	76
7	196
8	512
9	1353
10	3610
11	9713
12	26 324
13	71 799
14	196 938
15	542 895
20	91 695 540
30	3 162 376 205 180
40	125 769 718 187 920 320
50	5 432 932 054 880 789 103 450
60	247 713 018 707 369 495 492 278 980
70	11 732 720 069 619 981 848 276 481 860 820
80	571 656 214 754 754 601 748 236 271 618 957 360
90	28 468 263 497 152 261 665 942 607 175 776 424 807 490
100	1 442 574 085 791 104 356 152 892 805 954 046 070 482 270 300

Table 1: Results using the exact formula for the number of configurations.

Corollary 5.2. *The number of configurations $|C(L)|$ for a level L edges wide can be estimated as*

$$\frac{\sqrt{2}\left(2\sqrt{2}\right)^L}{\pi(L/4+3/2)^2} < |C(L)| < \frac{3}{14}\left(3^L + 11\sqrt{2}^{L-1}\right),$$

where the lower bound can be derived by applying Stirling's formula to the factorials of $|C(L)|$ and the upper bound from the walk interpretation by first considering the trivial upper bound 3^L and lower bounds for walks that reach $X(k) = -1$ for some k.

5.2 Assembling the transfer matrix

The construction of the transfer matrices between two consecutive levels is done elementwise according to the algorithm already seen in proposition 4.6 and can be reduced to the following phases:

- Take configurations $\alpha \in C(L), \beta \in C(L+1)$

- Reindex the configuration of larger basis with the mapping $x \mapsto 2x - 1$ and the configuration of the smaller basis with the mapping $x \mapsto 2x$.

- Take the union of the reindexed configurations, and form pairs such that the edge with the smallest coordinate in the union is paired with the second smallest, the third smallest with the fourth smallest etc. until all edges in the union have been paired. These pairs have a direct interpretation as the set of walk segments that forms the part of the path the self-avoiding walk assumes between the configurations.

- Check the following conditions, and return 0 if any of them occur:

  1. There is an upper configuration arc between the locations of the threads
  2. There is a walk segment from an end point of an arc of the upper configuration to another arc of the upper configuration
  3. There is a walk segment from an end point of an arc of the upper configuration to the thread of the upper configuration
  4. There is a closed system of arcs in the lower configurations that forms a loop, i.e. there is an arc in the lower configuration such the left end of the arc starts a walk segment, the right end of the arc ends a walk segment

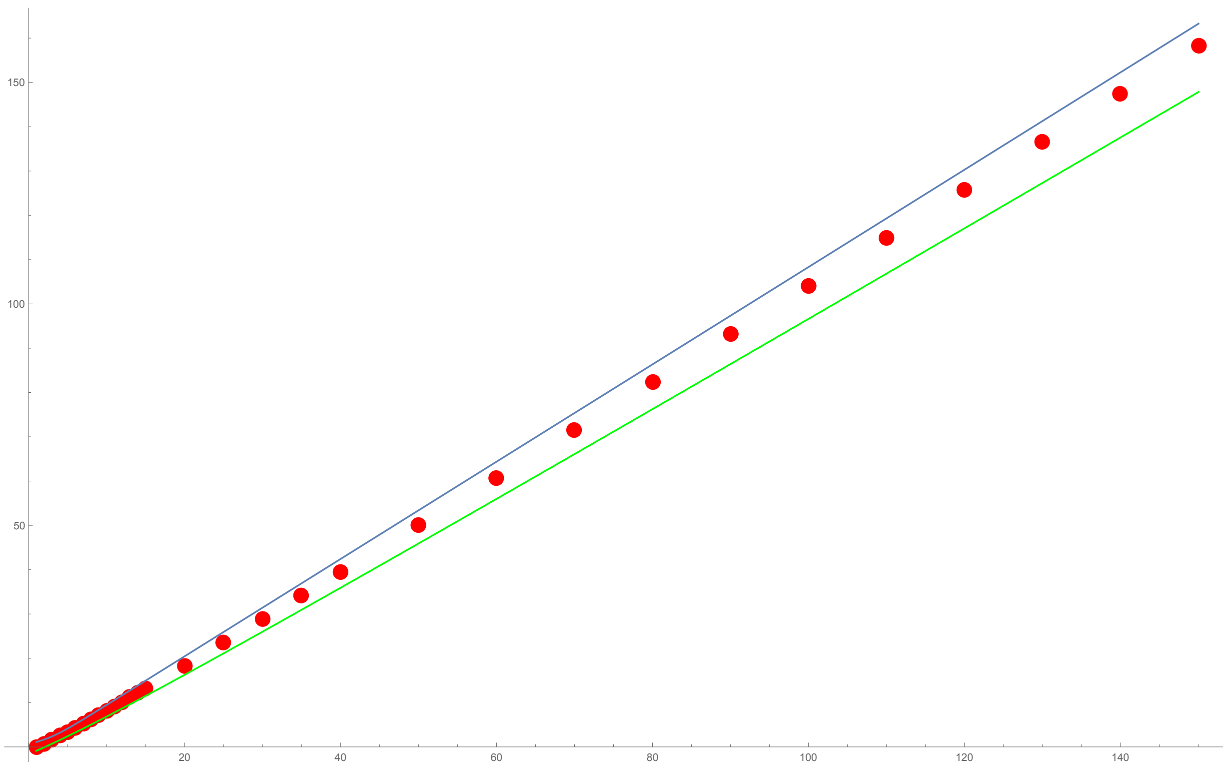

Figure 17: $\log|C(L)|$ plotted against L. The blue curve is the upper bound $\frac{3}{14}\left(3^L + 11\sqrt{2}^{L-1}\right)$ and the green curve is the lower bound $\frac{\sqrt{2}\left(2\sqrt{2}\right)^L}{\pi(L/4+3/2)^2}$.

and there is no arc in the upper configuration between the end points of the arc.

- If none of the above conditions are met, calculate the total length of the walk segments connecting the two configurations, i.e. the number of diagonal edges in the walk segments plus the total number of edges in the configurations times 1/2 one half corresponding to the half-edge steps in the walk segments.

- Returning the total length $l(\alpha \to \beta)$ and saving it in the matrix element $T^{\star}_{(\beta,\alpha)}$.

- Loop through all of $C(L) \times C(L+1)$ to construct the complete matrix $T^{\star}$. Similarly loop through all of $C(L+1) \times C(L)$ to construct the matrix T°.

The number of configurations $C(L) \times C(L+1)$ grows exponentially fast by corollary 5.2, and saving the matrices became too memory-intensive for my hardware when L reached the value 11. Checking the conditions 2, 3 and 4 above is slow, so it is more efficient to first drop out elements of the matrix that satisfy the condition 1.

```
width = 10;
longb = 2*basis[width + 1, 1] - 1;
shortb = 2*basis[width, 1];
```

First fix the width L and calculate the two bases $C(L)$ and $C(L+1)$, map the narrower basis $C(L)$ to a set of even indices and the wider basis $C(L+1)$ to a set of odd indices using the existing main loop basis of the previous section.

```
indexNW = Block[{indices},
    indices = {};
    For[j = 1, j <=  width + 1, j++,
     jpos = Flatten[Position[longb, {2*j - 1}][[All, 1]]];
     For[i = 1, i <=  width, i++,
      ijfree = {};
      Do[
       If[
        IntersectingQ[Flatten[longb[[jpos[[k]], 2]]],
          Range[Min[2*i + 1, 2*j + 1], Max[2*i - 1, 2*j - 3], 2]] ==
         False, ijfree = Union[ijfree, {jpos[[k]]}]],{k, 1, Length[jpos]}];
      indices =
       Union[indices,
        cartesianProd[ijfree[[Range[1, Length[ijfree]]]],
         Flatten[Position[shortb, {2*i}][[All, 1]]]]];

      ];
     ];
    indices
    ]; // Timing
```

Run through all thread positions `jpos` in the wider basis of configurations $C(L+1)$ and for all configurations β_j associated so $\texttt{jpos} = j$ check all thread locations i of the narrower basis $C(L)$. Collect to list `ijfree` all configurations in the upper configuration with nothing between i and j and take the cartesian product of `ijfree` and the list of configurations α_i in $C(L)$ with the thread location i. This yields the list of pairs of configurations $(\beta, \alpha) \in C(L+1) \times C(L)$ for which the transition $\alpha \to \beta$ is plausible according to the first condition. The naming `indexNW` comes from the transition Narrow to Wide.

```
zcrit = N[1/Sqrt[2 + Sqrt[2]], 10];
pairDiff[a_List] := Range[a[[1]], a[[2]]];
```

Use a ten-decimal numerical approximation for the critical constant $z_c = \frac{1}{\mu}$. Define the function `pairDiff` as the difference of the two members of the pair.

```
cNW = Function[{wind, nind},
   (*The transfer matrix element from the shorter basis to the longer one:
   a\[Rule]b corresponds to element (b,a)*)
   Block[{longvect = Flatten[longb[[wind]]],
     shortvect = Flatten[shortb[[nind]]], element},
    upp = Sort[longvect];
    path = Sort[Join[longvect, shortvect]];
    (*The start and end of the path*)
    pathden =
     Transpose[{path[[Range[1, Length[path], 2]]],
       path[[Range[2, Length[path], 2]]]}];
```

The building of a transfer matrix NW from the narrow basis $C(L)$ to the wide basis $C(L+1)$ begins with the initiation. `wind` is the configuration $\beta \in C(L+1)$ and `nind` is the configuration $\alpha \in C(L)$. The variables `longvect` associated to `wind` and `shortvect` associated to `nind` keep track of occupied edges but nothing else. The union of `shortvect` and `longvect` is named `path` and the variable `path` has the information of the canonical way to connect the two walks with walk segments. The variable `pathden` is a list of the end points of these walk segments.

```
(*Checks that the upper conf. thread is not
    connected to an upper conf. arc*)
    (*upper conf. arcs*)
    upc = longvect[[Range[2, Length[longvect]]]];
    longden =
     Transpose[{upc[[Range[1, Length[upc], 2]]],
       upc[[Range[2, Length[upc], 2]]]}];
    overden = Complement[posProd[upp, upp], longden];
    (*Checks that the upper conf. arcs are not connected to each other*)
If[IntersectingQ[pathden, overden], element = 0.  Break];
```

The variable `upc` is a list of edges occupied by the upper configuration arcs. `longden` is a list of the upper configuration arcs. `overden` is a list of all ways to form walk segments between the edges of `upc` that are not arcs in the upper configuration. If some of the walk segments in `overden` is in the set `pathden`, the configurations cannot be joined and the program returns 0. This amounts to checking the conditions 2 and 3.

```
(*The lower conf. arcs*)
lowc = shortvect[[Range[2, Length[shortvect]]]];
shoden =
 Transpose[{lowc[[Range[1, Length[lowc], 2]]],
   lowc[[Range[2, Length[lowc], 2]]]}];

rottenarcs =
 Intersection[
  posProd[path[[Range[1, Length[path], 2]]],
   path[[Range[2, Length[path], 2]]]], shoden];
```

The variable `lowc` gives the edges occupied by the lower configuration arcs, while `shoden` gives the arcs themselves. The variable `rottenarcs` gives the intersection of the lower configuration arcs with pairs formed by combining left endpoints of the walk segments with the right endpoints of the walk segments, i.e. it gives the lower configuration arcs whose left end is the left end of a walk segment and right end the right end of a walk segment. Every such arc must have an upper configuration arc in between its end points.

```
If[  rottenarcs == {} ||
  FreeQ[Map[IntersectingQ[pairDiff[#], upc] &, rottenarcs], False],
 element =
  zcrit^(Length[path]/2 +
     Total[path[[Range[2, Length[path], 2]]] -
       path[[Range[1, Length[path], 2]]]]), element = 0.
  ]
 (*return z^length of the walk segments if the walk segments are OK, otherwise 0*)

]];
```

The final part of the code tests condition 4 and returns the combined length of the legitimate walk segments if none of the conditions are met. By running the same process with every index in `indexNW` we have the transfer matrix from the narrow basis to the wide basis. The program assembling the transfer matrix from the wide basis to the narrow has the same phases, with only minor changes in defining variables.

```
Parallelize[
   nw1 = cNW[#[[1]], #[[2]]] & /@
     indexNW[[1 ;; Floor[Length[indexNW]/4]]];
   nw2 = cNW[#[[1]], #[[2]]] & /@
     indexNW[[
      Floor[Length[indexNW]/4] + 1 ;; Floor[Length[indexNW]/2]]];
   nw3 = cNW[#[[1]], #[[2]]] & /@
     indexNW[[
```

```
      Floor[Length[indexNW]/2] + 1 ;; Floor[3*Length[indexNW]/4]]];
   nw4 = cNW[#[[1]], #[[2]]] & /@
     indexNW[[Floor[3*Length[indexNW]/4] + 1 ;; Length[indexNW]]];
   , Method -> "CoarsestGrained"
   ]; // AbsoluteTiming

nwval = Catenate[{nw1, nw2, nw3, nw4}]; // Timing


nw = SparseArray[indexNW -> nwval];


Export["nw10.mat", nw]; // Timing
```

To reduce computation time, the matrix elements of viable indices are calculated in four parts and then catenated together. The resulting matrix is not too sparse, which makes saving the transfer matrices for large widths problematic.

With the program code we are able to calculate the transfer matrices between consecutive layers of the hexagonal lattice for SAWs of different values of z.

5.3 Computational results

The theory developed in previous sections for a self-avoiding walk in vertical strips does not depend on the scale or location of the strip. Thus we can fix the left and right boundaries of the strips S_L to be 0 and π, while keeping the lattice such that the real line is at the center of a layer. We get finer and finer approximations of a vertical strip in the complex plane by increasing the number of edges L in a layer. In particular in the approximation S_L the mesh size δ equals $\frac{\pi}{L-1}$. As we increase L we get closer to the possible scaling limit of the SAW. The following conjecture is our main tool in comparing the SAW with its conjectured scaling limit.

Conjecture 5.3. *[JJK15] Consider a simply connected domain $\Omega \subsetneq \mathbb{C}$ and let a, b be two points on the boundary $\partial\Omega$. Furthermore assume that there are sections of the boundary perpendicular to either coordinate axis with points $c_1, c_2, \ldots, c_n$ on these sections. Approximate the set Ω and the points a, b according to definition 2.7 with the triplet $(\Omega_\delta, a_\delta, b_\delta)$ and let $\mathbb{P}_{z_c,\Omega_\delta,a,b}$ be the law of self-avoiding walks on the set Ω_δ. Denote by $c_k^\delta, k = 1, 2 \ldots, n$ the edge on Ω_δ closest to the boundary point c_k. See also fig. 18. If the scaling limit of the critical self-avoiding walk exists and is conformally invariant, by theorem 2.11 it must be the chordal Schramm-Löwner evolution $SLE_{8/3}$. Associating the self-avoiding walk with chordal SLE_κ yields the following estimate:*

$$\frac{1}{\delta^{nh}}\mathbb{P}_{z_c,\Omega_\delta,a,b}\left[\gamma \textit{ visits } c_1^\delta, c_2^\delta, \ldots, c_n^\delta\right] \xrightarrow[\delta\to 0]{} \zeta_\Omega(a, b, c_1, \ldots, c_n), \tag{5.1}$$

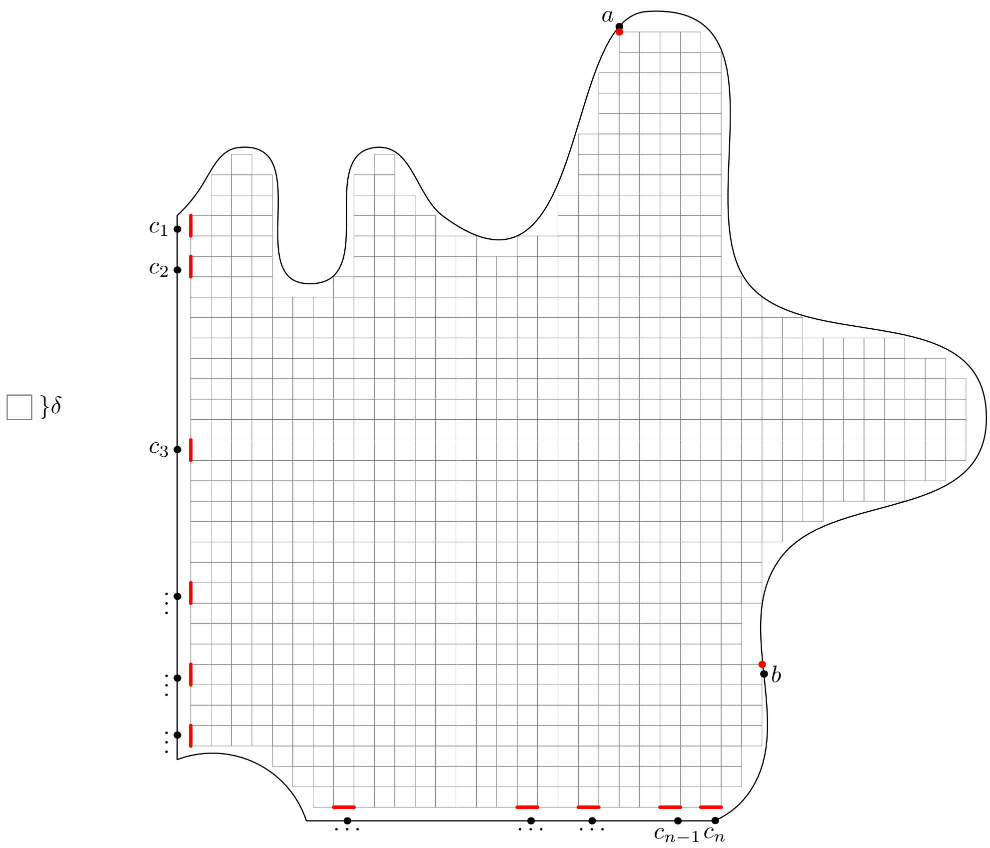

Figure 18: An example of a simply connected domain Ω, its lattice approximation Ω_δ and the set of edges $c_1^\delta, \dots, c_n^\delta$ (red line segments) approximating boundary points $c_1, \dots, c_n$ on parts of the boundary $\partial\Omega$ perpendicular to the coordinate axes.

where

$$\zeta_\Omega(a, b, c_1, \dots, c_n) = C \prod_{j=1}^{n} |g'(c_j)|^h \zeta^{(n)}(g(c_1), g(c_2), \dots g(c_n)),$$

$g : \Omega \to H$ is a conformal mapping onto the uppper halfplane H with $g(a) = 0$, $g(b) = \infty$, C is a constant dependent on the choice of the lattice and the approximation and $\zeta^{(n)}$ is an explicit function for SLE boundary visit amplitudes, universal for all lattice approximations of Ω by the theory of renormalization groups. When $n = 2$ and $0 < g(c_1) < g(c_2)$, we have

$$\zeta^{(2)}(g(c_1), g(c_2)) = |g(c_1)|^{1-8/\kappa} |g(c_2)|^{1-8/\kappa} {}_2F_1(4/\kappa, \frac{\kappa - 8}{\kappa}; \frac{8}{\kappa}; \frac{g(c_2) - g(c_1)}{g(c_2)}).$$

The constant κ and the exponent $h = \frac{8}{8-\kappa}$ are determined by the chordal SLE_κ that the self-avoiding walk is conjectured to converge to, namely $SLE_{8/3}$, to be $\kappa = 8/3$, $h = 2$.

The idea for producing computational support for the conjecture 2.10 using our previous results and conjecture 5.3 is as follows:

- Choose the open strip $S = \{z \in \mathbb{C} \mid 0 < Re(z) < \pi\}$ as the simply connected domain Ω.

- Translate and scale the infinite strip S_L of section 4.1 of the hexagonal lattice so that the real part of the leftmost edge on the real line is 0 and the real part of rightmost edge on the real line is π. This makes the width of the hexagons equal $\frac{\pi}{L-1}$, which implies that the height difference between two consecutive levels is $\frac{\sqrt{3}\pi}{L-1}$. In other words for $\delta = \frac{\pi}{L-1}$ the hexagonal lattice strip S_L equals Ω_δ in the conjecture 5.3.

- Choose the boundary points to be $a = -i\infty$ and $b = +i\infty$. The conditions $g(a) = 0, g(b) = \infty$ and the choice $g(\pi) = 1$ then determine the conformal mapping g from domain S to the half-plane $H = \{z \in \mathbb{C} \mid Im(z) \geq 0\}$ to be

$$g(z) = e^{i(\pi - z)},\ g'(x+iy) = ie^{y}e^{-ix}.$$

- Consider the points $c_{L:k} := \pi + i\frac{\sqrt{3}\pi k}{L-1}$ with imaginary coordinate on the $(2k)^{th}$ level of the set S_L and real coordinate on the right boundary of the strip S.

- By section 4.4 the probability measure $\mathbb{P}_L = \mathbb{P}_{z_c, S_L, -i\infty, +i\infty}$ for infinitely long SAWs exists and we can explicitly calculate the probabilities of visiting two points:

$$\mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}] = \frac{w^T P(\pi) T^k P(\pi) v}{\lambda^k w^T v}.$$

- When $n = 2$, $c_1 = c_{L;0}$ and $c_2 = c_{L;k}$ the right-hand side of equation (5.1) is reduced to

$$\begin{aligned}
&\zeta_S(-i\infty, +i\infty, c_{L;0} = \pi, c_{L;k}) \\
&= C|g'(\pi)|^2|g'(c_{L;k})|^2|g(\pi)|^{-2}|g(c_{L;k}) - g(\pi)|^{-2}{}_2F_1(3/2, -2; 3; \frac{g(c_{L;k}) - g(\pi)}{g(c_{L;k})}) \\
&= C|g'(c_{L;k})|^2|g(c_{L;k}) - 1|^{-2}{}_2F_1(3/2, -2; 3; \frac{g(c_{L;k}) - 1}{g(c_{L;k})})
\end{aligned}$$

- Finally we have that if conjecture 5.3 holds

$$\frac{(L-1)^4\mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}]}{|g'(c_{L;k})|^2} \xrightarrow[L\to\infty]{} C|g(c_{L;k}) - 1|^{-2}{}_2F_1(3/2, -2; 3; \frac{g(c_{L;k}) - 1}{g(c_{L;k})}).$$

  We test this conjecture by plotting the points

$$\left(g(c_{L;k}), \frac{(L-1)^4\mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}]}{|g'(c_{L;k})|^2}\right) \tag{5.2}$$

  against the graph of the function $C|x-1|^{-2}{}_2F_1(3/2, -2; 3; \frac{x-1}{x})$ for various k, L with C chosen to minimize the least squares distance between the graph and the points.

Remark 5.4. In the above reasoning we assumed that $S_L = S_{n,L}$, i.e. that the even levels are one hexagon wider than the odd ones. If we approximate the set S_L with the set $S_{w,L}$ where the even levels are one hexagon narrower than the odd ones, we need to make the change $\delta = \frac{\pi}{L}$ which makes the height difference between two levels equal $\frac{\sqrt{3}\pi}{L}$ and thus the points $c_{L;k}$ take the form $\pi + i\frac{\sqrt{3}\pi k}{L}$. Thus we numerically compare the points

$$\left(g(c_{L;k}), \frac{L^4 \mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}]}{|g'(c_{L;k})|^2} \right)$$

to the graph of the function $C|x-1|^{-2}{}_2F_1(3/2, -2; 3; \frac{x-1}{x})$.

$\mathbb{P}_L[\gamma \ni c^\delta_{L;0}, c^\delta_{L;k}]$		k						
$S_L = S_{w,L}$		1	2	3	4	5	6	7
L	2	0.282	0.254	0.251	0.250	0.250	0.250	0.250
	3	0.128	0.0980	0.0911	0.0894	0.0890	0.0889	0.0889
	4	0.0728	0.0484	0.0484	0.0397	0.0390	0.0388	0.0387
	5	0.0470	0.0281	0.0226	0.0206	0.0199	0.0196	0.0195
	6	0.0331	0.0183	0.0138	0.0121	0.0114	0.0111	0.0110
	7	0.0248	0.0130	0.00926	0.00783	0.00720	0.00691	0.00677
	8	0.0196	0.00981	0.00672	0.00549	0.00494	0.00468	0.00454
	9	0.0162	0.00786	0.00520	0.00414	0.00365	0.00341	0.00329
	10	0.0140	0.00664	0.00428	0.00333	0.00289	0.00267	0.00256

Table 2: The values of the probabilities $\mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}]$ listed in a table when the even levels are one edge narrower than the odd ones, i.e. the case $S_{w,L}$. See also fig. 11 of section 4.1.

$\mathbb{P}_L[\gamma \ni c^\delta_{L;0}, c^\delta_{L;k}]$		k						
$S_L = S_{n,L}$		1	2	3	4	5	6	7
L	3	0.0695	0.0562	0.0545	0.0543	0.0542	0.0542	0.0542
	4	0.0309	0.0206	0.0186	0.0181	0.0180	0.0179	0.0179
	5	0.0174	0.0100	0.00828	0.00778	0.00761	0.00756	0.00754
	6	0.0112	0.00579	0.00444	0.00400	0.00383	0.00377	0.00374
	7	0.00790	0.00378	0.00272	0.00235	0.00220	0.00213	0.00210
	8	0.00596	0.00269	0.00184	0.00153	0.00140	0.00134	0.00131
	9	0.00474	0.00206	0.00135	0.00109	9.74e-04	9.21e-04	8.94e-04
	10	0.00395	0.00167	0.00106	8.35e-04	7.35e-04	6.87e-04	6.62e-04
	11	0.00345	0.00143	8.91e-04	6.88e-04	5.98e-04	5.54e-04	5.32e-04

Table 3: The same probabilities $\mathbb{P}_L[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}]$ when the even levels are one edge wider than the odd ones, i.e. the case $S_L = S_{n,L}$.

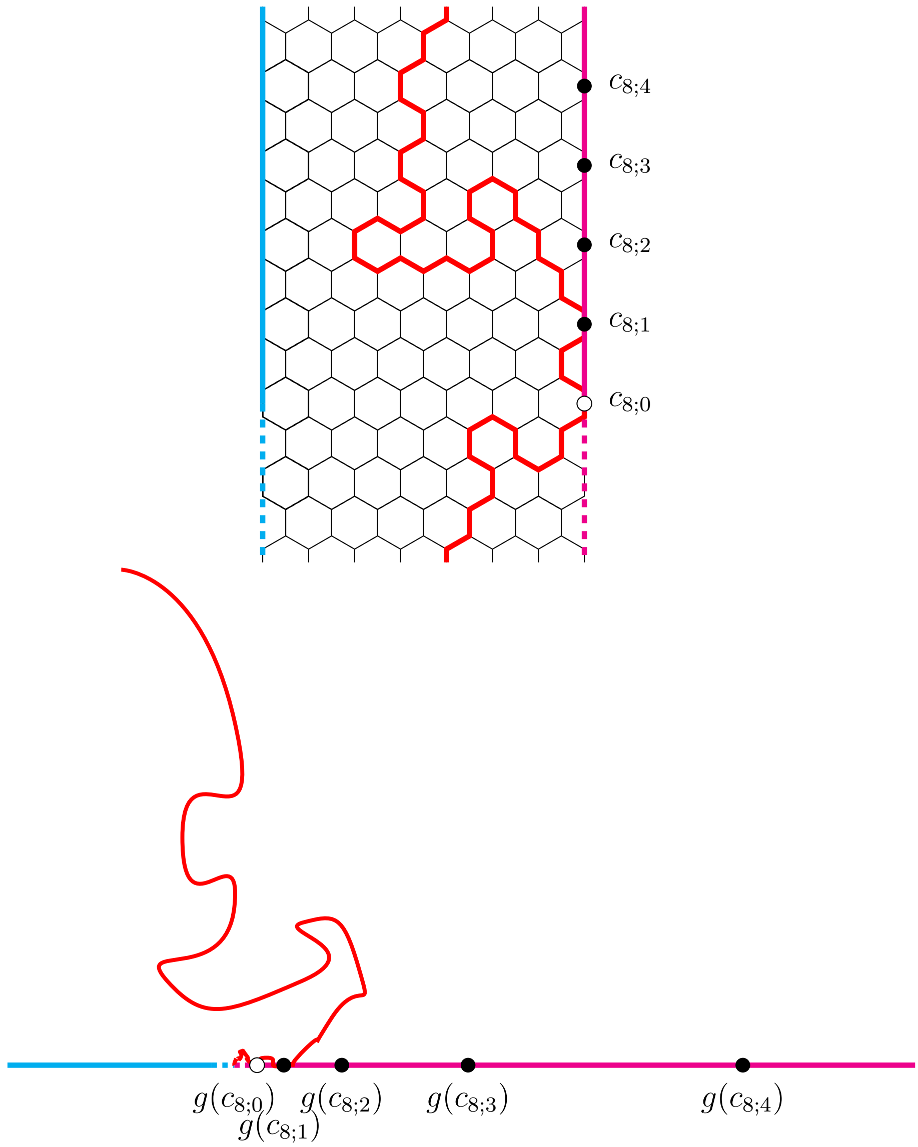

Figure 19: An infinitely long SAW γ in the vertical strip S approximated with $S_{n,8}$ and the points $c_{8;k}, k = 0, 1, 2, 3, 4$, mapped conformally onto the half-plane H

In the case of $S_{n,L}$ the data points (5.2) for $k > 1, L > 8$ are close to the curve $12.009|x-1|^{-2}{}_2F_1(3/2, -2; 3; \frac{x-1}{x})$, while for $S_{w,L}$ the data points for $k > 1, L > 7$ are close to $60.2103|x-1|^{-2}{}_2F_1(3/2, -2; 3; \frac{x-1}{x})$. Figure 20 presents all of the the data points with the hypergeometric functions that they approximate. The upper curve and points correspond to the strip $S_{w,L}$ where even levels are one edge narrower than odd ones, while the lower curve and points correspond to the strip $S_{n,L}$ where the even levels are one edge wider than the odd ones. It should be noted that the first point $k = 1$ from the left for each value of L is not expected to follow the rule and thus the curves have been fitted to the case $k \geq 2$. The points corresponding to $k \geq 2$ fit onto the curves notably well.

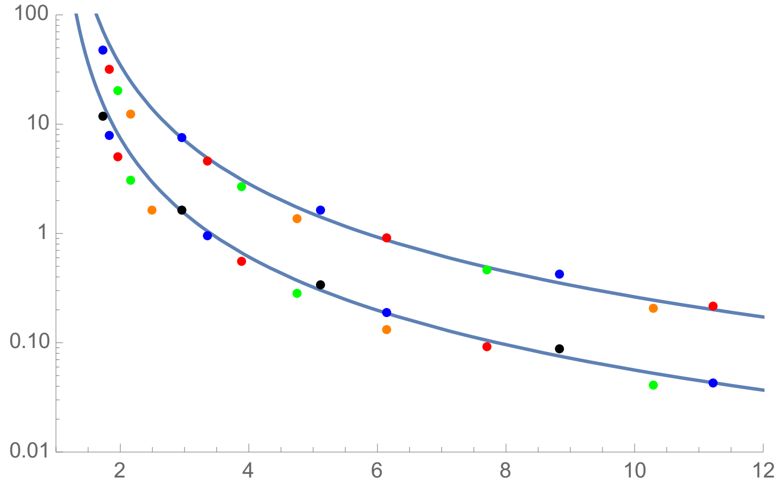

Figure 20: Hypergeometric curves $|x-1|^{-2}{}_2F_1(3/2,-2;3;\frac{x-1}{x})$ fitted against the data points

$$\left(g(c_{L;k}), \frac{(L-1)^4\mathbb{P}_L\left[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}\right]}{|g'(c_{L;k})|^2}\right) \text{ for } S_{n,L},$$

$$\left(g(c_{L;k}), \frac{L^4\mathbb{P}_L\left[\gamma \text{ visits } c^\delta_{L;0}, c^\delta_{L;k}\right]}{|g'(c_{L;k})|^2}\right) \text{ for } S_{w,L}.$$

Color coding: $L = 11$ as black, $L = 10$ as blue, $L = 9$ as red, $L = 8$ as green, $L = 7$ as orange.

The data provides numerical support for conjecture 5.3 that has been derived from the assumption that the scaling limit of the self-avoiding walk is the Schramm-Löwner Evolution $SLE_{8/3}$. Hence we also provide further evidence for the conformal invariance conjecture 2.10.

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