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PHYSICAL REVIEW A **91**, 043822 (2015)

Cross-Kerr nonlinearity in optomechanical systems

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We consider the response of a nanomechanical resonator interacting with an electromagnetic cavity via a radiation-pressure coupling and a cross-Kerr coupling. Using a mean-field approach we solve the dynamics of the system and show the different corrections coming from the radiation pressure and the cross-Kerr effect on the usually considered linearized dynamics.

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I. INTRODUCTION

Cavity optomechanics offers a framework to study the coupling between an electromagnetic field and the vibrations of a mechanical resonator. The interaction between these two systems is usually mediated by a radiation-pressure-type coupling proportional, through a coupling constant g, to the number of photons n_c in the cavity and the displacement of the mechanical resonator. The radiation-pressure coupling offers the possibility of altering the resonant frequency of the mechanical resonator and its damping. The latter can be used for cooling [1–3] or amplification [4]. Moreover, the nonlinearity of the interaction may allow for the observation of macroscopic quantum phenomena such as quantum superposition of states [5,6] or quantum squeezed states [7]. The requirements for observing these quantum phenomena are the necessity of being close to the ground state and being in the strong-coupling regime [8,9], where g is larger than the cavity and the mechanical resonator decay rate. However, g is usually weak, and to bypass this constraint a strong drive to the cavity is applied at the cost of losing the nonlinear property of the interaction.

Our recent proposal [10], in which the cavity and the resonator are coupled to a Josephson junction, shows that the interaction between the cavity and the resonator can be enhanced via the nonlinearity of the Josephson effect. There, the nonlinearity of the Josephson effect leads to an additional nonlinear interaction, namely, a cross-Kerr coupling g_{ck} between the cavity and the resonator. Quadratic and higher-order interactions in the displacement have been investigated also in different setups such as the membrane in the middle geometries [11–13] and in atomic arrays placed inside a cavity [14]. In the Josephson-junction setup the relative value of g_{ck} and g depends on the value of the gate charge to a superconducting island, whereas in [12–14], it reflects the position of the resonator within the cavity.

In optics, the Kerr effect refers to the change in the refractive index of a nonlinear medium when an electric field is applied to it, with the change being proportional to the square of the amplitude of the electric field [15]. When two electric fields are applied simultaneously on a nonlinear medium, they undergo an optical effect named the cross-Kerr effect. In this case the refractive index associated with the propagation of one of the electric fields is changed with respect to the square amplitude of the other [15,16]. In quantum information, this effect plays an important role since it allows for creating entanglement between photons [17,18]. In the context of optomechanical systems, the cross-Kerr coupling between the resonator and the cavity induces a change in the refractive index of the cavity that depends on the *number* of phonons in the resonator, whereas the radiation -pressure coupling gives rise to an analogous effect that depends, however, on the *displacement* of the mechanical resonator.

In this paper we solve the dynamics of the cavity and the mechanical resonator in the presence of the cross-Kerr and the radiation-pressure couplings. We determine the effects of the cross-Kerr coupling on the red and blue sidebands within a mean-field approach. In particular, we demonstrate that the sideband peak is shifted due to the cross-Kerr coupling. In addition, the cross-Kerr coupling induces a nonmonotonous response of the effective mechanical damping as a function of the number of photons pumped into the cavity.

II. MEAN-FIELD APPROACH

We consider an electromagnetic cavity with frequency ω_c and linewidth κ coupled to a mechanical resonator with frequency ω_m and linewidth γ. The number of phonons in the cavity n_c is coupled to the vibration amplitude of the mechanical resonator $\hat{x}$ via a radiation-pressure-type coupling g. In addition the number of photons n_c is coupled to the number of phonons n_m in the mechanical resonator through a cross-Kerr coupling g_{ck} (Fig. 1). The Hamiltonian of the system is ($\hbar = 1$)

$$H = \omega_c a^\dagger a + \omega_m b^\dagger b - g a^\dagger a (b^\dagger + b) - g_{ck} a^\dagger a b^\dagger b, \quad (1)$$

where a and b are the annihilation operators of the cavity and the mechanical resonator, respectively. We treat the interactions with a mean-field (MF) approach. Within it, the radiation-pressure interaction becomes

$$g a^\dagger a (b^\dagger + b) = g[(\langle a^\dagger \rangle a + \langle a \rangle a^\dagger - \langle a^\dagger a \rangle)(b^\dagger + b) + (a^\dagger a - \langle a \rangle a^\dagger - \langle a^\dagger \rangle a)\langle b^\dagger + b \rangle], \quad (2)$$

where $\langle A \rangle$ stands for the average of A over the static nonequilibrium state of the system (mean field). The negative terms in Eq. (2) are included to suppress double counting. The first line

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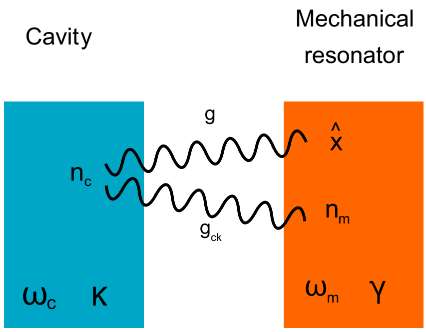

FIG. 1. (Color online) Schematic picture of the system. A cavity and a mechanical resonator coupled via a radiation-type coupling g and a cross-Kerr coupling g_{ck}. The number of photons in the cavity n_c is coupled to the oscillations of the mechanical resonator $\hat{x}$ and the number of phonons in the mechanical resonator n_m.

of Eq. (2) describes exchange processes between the resonator and the cavity, while the second line gives a frequency shift of the cavity, which is proportional to the average displacement of the resonator. This decomposition allows us to find the usual results of the weak radiation-pressure coupling [4,19]. The validity range of such an approach is thus similar to the usual scheme of optomechanics, i.e., $|\langle a^\dagger a\rangle - \langle a^\dagger\rangle\langle a\rangle|/\langle a^\dagger a\rangle \ll 1$ [3,20]. In MF, the cross-Kerr coupling becomes

$$g_{ck}a^\dagger a b^\dagger b = g_{ck}[\langle a^\dagger a\rangle b^\dagger b + \langle b^\dagger b\rangle a^\dagger a + \langle b^\dagger a\rangle b a^\dagger + \langle b a^\dagger\rangle b^\dagger a + \langle b a\rangle b^\dagger a^\dagger + \langle b^\dagger a^\dagger\rangle b a]. \quad (3)$$

The term $\langle a^\dagger a\rangle b^\dagger b$ ($\langle b^\dagger b\rangle a^\dagger a$) describes a Hartree-like interaction between the resonator and the cavity. It corresponds to an effective field induced by the average number of photons (phonons) coupling to the number of mechanical excitations (the number of cavity photons). The other terms describe exchange processes between the resonator and the cavity. Thus we can rewrite the Hamiltonian as

$$\begin{aligned} H = {} & [\omega_c - g_{ck}\langle b^\dagger b\rangle]a^\dagger a + [\omega_m - g_{ck}\langle a^\dagger a\rangle]b^\dagger b \\ & - G[\langle a^\dagger\rangle a b^\dagger + \langle a\rangle a^\dagger b^\dagger] - G^*[\langle a^\dagger\rangle a b + \langle a\rangle a^\dagger b] \\ & + g[(a^\dagger a - \langle a\rangle a^\dagger - \langle a^\dagger\rangle a)\langle b^\dagger + b\rangle - \langle a^\dagger a\rangle(b^\dagger + b)], \end{aligned} \quad (4)$$

where the expectation values of the different operators have to be determined self-consistently within the MF picture and $G = g + g_{ck}\langle b\rangle$. We assume the usual experimental situation where $\omega_c \gg \omega_m$ and where the cavity is driven with a coherent field of strength f_p oscillating at frequency $\omega_p = \omega_c + \Delta$. Using the input-output formalism [19], we get for the equations of motion

$$\begin{aligned} \dot{a} = {} & -i[-\Delta - g_{ck}\langle b^\dagger b\rangle]a - \frac{\kappa}{2}a + \sqrt{\kappa} f_p \\ & + iG^*\langle a\rangle b + iG\langle a\rangle b^\dagger - ig\langle b^\dagger + b\rangle[a - \langle a\rangle], \end{aligned} \quad (5)$$

$$\begin{aligned} \dot{b} = {} & -i[\omega_m - g_{ck}\langle a^\dagger a\rangle]b - \frac{\gamma}{2}b + \sqrt{\gamma} b_{in} \\ & + iG\langle a^\dagger\rangle a + iG\langle a\rangle a^\dagger - ig\langle a^\dagger a\rangle. \end{aligned} \quad (6)$$

Here we have written the cavity operator a in a frame rotating with frequency ω_p, neglecting the fast oscillating terms. We define b_{in} as the thermal input of the resonator satisfying $\langle b_{in}(t)\rangle = 0$ and $\langle b_{in}^\dagger(t) b_{in}(t')\rangle = n^{th}\delta(t-t')$, where n^{th} is the number of phonons in the thermal bath damping the resonator. We split the cavity and the mechanical operators into a sum of coherent and fluctuation parts, i.e., $a \equiv \delta a + \alpha$ and $b \equiv \delta b + \beta$, with $\alpha = \langle a\rangle$, $\beta = \langle b\rangle$, and $\langle \delta a\rangle = \langle \delta b\rangle = 0$. As usual, we assume that α and β oscillate at the same frequency as the coherent drive so that $\dot{\alpha} = \dot{\beta} = 0$. In addition $|\alpha|^2 \gg \langle \delta a^\dagger \delta a\rangle$ to ensure the validity of the mean-field approach. With these approximations, the solutions of Eqs. (5) and (6) are

$$\alpha = \frac{\sqrt{\kappa} f_p}{\frac{\kappa}{2} - i[\Delta - g_{ck}\langle b^\dagger b\rangle - (G^*\beta + G\beta^*)]}, \quad (7)$$

$$\beta = \frac{i(2G - g)|\alpha|^2 - ig\langle \delta a^\dagger \delta a\rangle}{\gamma/2 + i(\omega_m - g_{ck}\langle a^\dagger a\rangle)}. \quad (8)$$

In the derivation of Eqs. (7) and (8), we have assumed, in agreement with what is usually done in the optomechanical literature (see, e.g., [21]), $\Delta + g\langle b^\dagger + b\rangle \approx \Delta$. The equations of motion for the fluctuations in the Fourier space are given by

$$\left[\frac{\kappa}{2} - i(\omega + \tilde{\Delta})\right]\delta a = iG\alpha\delta b^\dagger + iG^*\alpha\delta b, \quad (9)$$

$$\left[\frac{\gamma}{2} - i(\omega - \tilde{\omega}_m)\right]\delta b = iG\alpha^*\delta a + iG\alpha\delta a^\dagger + \sqrt{\gamma} b_{in}, \quad (10)$$

where $\tilde{\Delta} = \Delta + g_{ck}\langle b^\dagger b\rangle$ and $\tilde{\omega}_m = \omega_m - g_{ck}\langle a^\dagger a\rangle$. The effect of the thermal drive b_{in} on the response of the cavity is mediated by the coupling G. Through this coupling the oscillations of the mechanical resonator produce sideband peaks at $\omega_d \pm \tilde{\omega}_m$ in the cavity response. They allow for the exchange of energy between the cavity and the resonator when $\Delta \approx \pm\omega_m$ [2,22]. These processes are depicted in Fig. 2. For $\Delta \approx -\omega_m$ the system is in the red sideband regime, and one

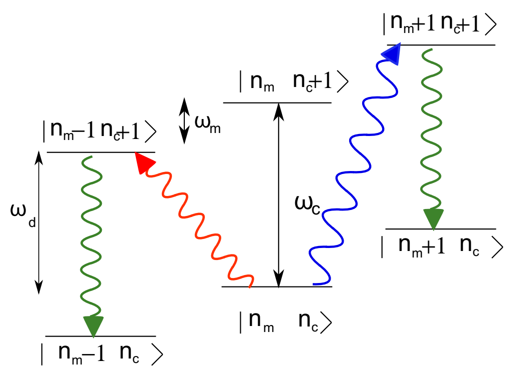

FIG. 2. (Color online) Cooling (heating) process. The cavity is driven with a frequency $\omega_d = \omega_c - \omega_m$ ($\omega_d = \omega_c + \omega_m$). The drive does not allow a transition from $|n_m,n_c\rangle$ to $|n_m,n_c+1\rangle$ but allows the transition from $|n_m,n_c\rangle$ to $|n_m - 1,n_c+1\rangle$ ($|n_m,n_c\rangle$ to $|n_m+1,n_c+1\rangle$). The cavity relaxes then to the state $|n_m - 1,n_c\rangle$ ($|n_m+1,n_c\rangle$), resulting in cooling (heating) of the mechanical resonator.

can transfer energy from the resonator to the cavity; thus the mechanical resonator is damped and cooled. For $\Delta \approx \omega_m$, the system is in the blue sideband regime, and one can transfer energy from the cavity to the resonator; thus the mechanical resonator is excited and heated. In order to find the correction to the damping, we solve the response function of δa for the thermal input δb_{in}. We find that it is a Lorentzian function peaked at $\tilde{\omega}_m + \omega_{\text{shift}}$, with

$$\omega_{\text{shift}} = -\frac{|G|^2|\alpha|^2\left(\tilde{\Delta}^2 - \tilde{\omega}_m^2 + \frac{\kappa^2}{4}\right)}{\tilde{\omega}_m} \times \left(\frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m + \tilde{\Delta})^2} - \frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m - \tilde{\Delta})^2}\right), \quad (11)$$

whose linewidth is $\gamma + \Gamma_{\text{opt}}$, with

$$\Gamma_{\text{opt}} = |G|^2|\alpha|^2\kappa\left(\frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m + \tilde{\Delta})^2} - \frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m - \tilde{\Delta})^2}\right). \quad (12)$$

Integrating the Lorentzian function obtained above, we obtain the number of phonons and photons coming from the thermal vibrations of the resonator. We get [3]

$$\langle \delta b^\dagger \delta b \rangle = \frac{\gamma n^{th} + \Gamma_{\text{opt}} n_{m_0}}{\gamma + \Gamma_{\text{opt}}}, \quad (13)$$

$$\langle \delta a^\dagger \delta a \rangle = G^2|\alpha|^2 \langle \delta b^\dagger \delta b \rangle \times \left(\frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m + \tilde{\Delta})^2} + \frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m - \tilde{\Delta})^2}\right), \quad (14)$$

with

$$n_{m_0} = -\frac{(\tilde{\omega}_m + \tilde{\Delta})^2 + \frac{\kappa^2}{4}}{4\tilde{\Delta}\tilde{\omega}_m}. \quad (15)$$

The validity of the mean field in Eq. (3) is given by the condition $\langle \delta a^\dagger \delta a \rangle \ll |\alpha|^2$, which translates to the following condition:

$$\langle \delta b^\dagger \delta b \rangle \ll G^{-2}\left(\frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m + \tilde{\Delta})^2} + \frac{1}{\frac{\kappa^2}{4} + (\tilde{\omega}_m - \tilde{\Delta})^2}\right)^{-1} \approx \left(\frac{\kappa}{2G}\right)^2, \quad (16)$$

where the latter approximation is valid in the resolved sideband regime $\kappa \ll \omega_m \approx |\Delta|$. Equations (7), (8), (13), and (14) form a set of self-consistency equations. This constitutes a nonlinear system that hence has multiple solutions. It should thus show some bistability as the numbers of photons and phonons are increased [4]. However, when solving the system, we limit ourselves close to the stable solution of the radiation pressure in the weak-coupling regime without cross-Kerr coupling, as discussed in [4]. We now focus on the different sidebands.

III. OPTIMAL COOLING AND HEATING

In order to minimize or maximize the optical damping Γ_{opt}, we set $\tilde{\Delta} = \mp\tilde{\omega}_m$. The upper sign refers to the red sideband ($\Gamma_{\text{opt}} > 0$), and the lower sign refers to the blue sideband ($\Gamma_{\text{opt}} < 0$). In the resolved sideband limit, $\omega_m \gg \kappa \gg \gamma$, the frequency shift and the optical damping become

$$\omega_{\text{shift}} = \mp\frac{|G|^2|\alpha|^2}{\tilde{\omega}_m} = \mp\frac{|G|^2|\alpha|^2}{\omega_m - g_{ck}\langle a^\dagger a\rangle}, \quad (17)$$

$$\Gamma_{\text{opt}} = \pm\frac{4|G|^2|\alpha|^2}{\kappa}. \quad (18)$$

The result for the optical damping equation (18) is identical to the one usually obtained in optomechanics in the absence of the cross-Kerr coupling. The effect of g_{ck} shows up only in the frequency shift, which now depends on the number of coherent and thermal photons in the cavity Eq. (17). In Fig. 3 we show a schematic picture of what happens to the sideband in the presence of the cross-Kerr coupling.

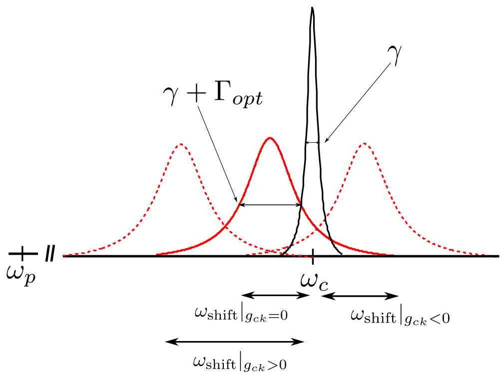

FIG. 3. (Color online) Schematic picture of the red sideband with and without cross-Kerr coupling for $\Delta < 0$. For $g_{ck} > 0$ the sideband peak is shifted to lower values, while for $g_{ck} < 0$ the sideband peak is shifted to higher values.

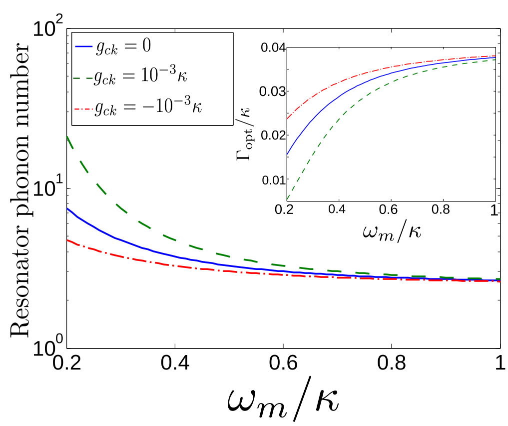

FIG. 4. (Color online) Steady-state phonon number in the resonator and Γ_{opt} (inset) as a function of the ratio ω_m/κ at the red sideband for the optimal case $\tilde{\Delta} = -\tilde{\omega}_m$ with $\gamma = 10^{-3}\kappa, g = 10^{-2}\kappa$. The number of photons pumped into the cavity is fixed to $|\alpha|^2 = 100$, and the bath temperature corresponds to $n^{th} = 100$.

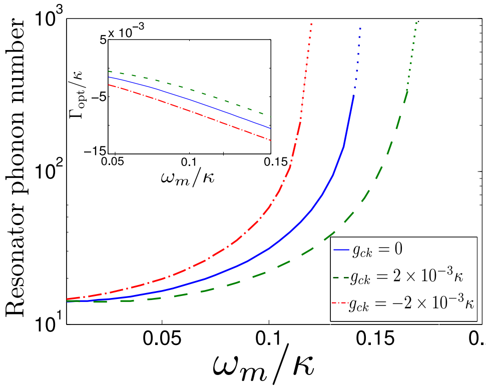

FIG. 5. (Color online) Steady-state phonon number in the resonator and $\Gamma_{\rm opt}$ (inset) as a function of the ratio ω_m/κ at the blue sideband for the optimal case $\tilde{\Delta}=\tilde{\omega}_m$ with $\gamma=10^{-2}\kappa$, $g=10^{-2}\kappa$. The number of photons pumped into the cavity is fixed to $|\alpha|^2=100$, and the bath temperature corresponds to $n^{th}=10$. The dashed lines indicate the onset of the parametric instability for $\Gamma_{\rm opt}=-\gamma$.

In the Doppler limit ($\omega_m\leqslant\kappa$) the frequency shift and optical damping are given by

$$\omega_{\rm shift}=\mp4|G|^2|\alpha|^2\frac{\omega_m-g_{ck}\langle a^\dagger a\rangle}{\frac{\kappa^2}{4}+4(\omega_m-g_{ck}\langle a^\dagger a\rangle)^2}, \tag{19}$$

$$\Gamma_{\rm opt}=\pm\frac{4|G|^2|\alpha|^2}{\kappa}\frac{4(\omega_m-g_{ck}\langle a^\dagger a\rangle)^2}{\frac{\kappa^2}{4}+4(\omega_m-g_{ck}\langle a^\dagger a\rangle)^2}. \tag{20}$$

Now both the frequency shift and the optical damping depend on the cross-Kerr coupling. In Figs. 4 and 5 we plot the number of phonons and the optical damping as a function of ω_m/κ for the red sideband in the Doppler limit. Since the cross-Kerr coupling shifts the mechanical frequency, the value of $\Gamma_{\rm opt}$ is shifted as well. The sign of the shift is given by the sign of g_{ck}. Otherwise, we recover the cooling of the resonator for the red sideband (Fig. 4) and the parametric instability when $\Gamma_{\rm opt}=-\gamma$ for the blue sideband (Fig. 5).

IV. CASE WITH $\Delta=\omega_m$.

In experiments the parameter one can tune directly is the detuning Δ and not $\tilde{\Delta}$ as it can be difficult to set $\tilde{\Delta}=\mp\tilde{\omega}_m$ for each value of $|\alpha|$ as the pump strength is varied. Therefore

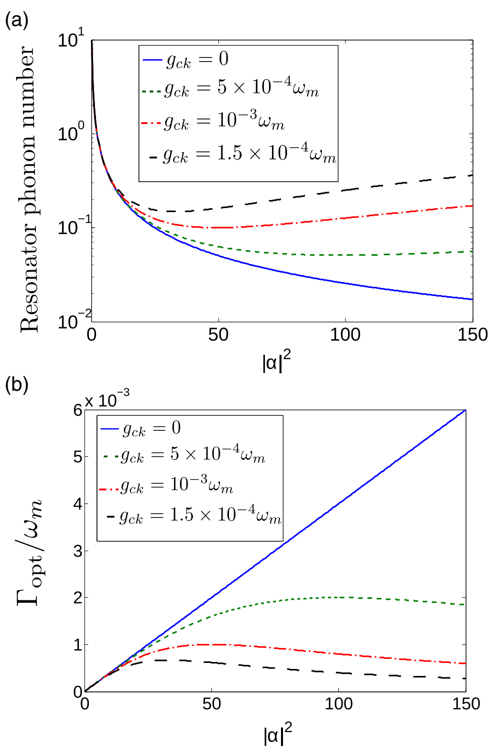

FIG. 6. (Color online) (a) Steady-state phonon number in the resonator and (b) the optical damping $\Gamma_{\rm opt}$ as a function of the number of photons pumped into the cavity in the case where $\Delta=-\omega_m$. The values for the parameters are $\gamma=10^{-4}\omega_m$, $\kappa=10^{-1}\omega_m$, $g=10^{-3}\omega_m$, and the bath temperature corresponds to $n^{th}=10$.

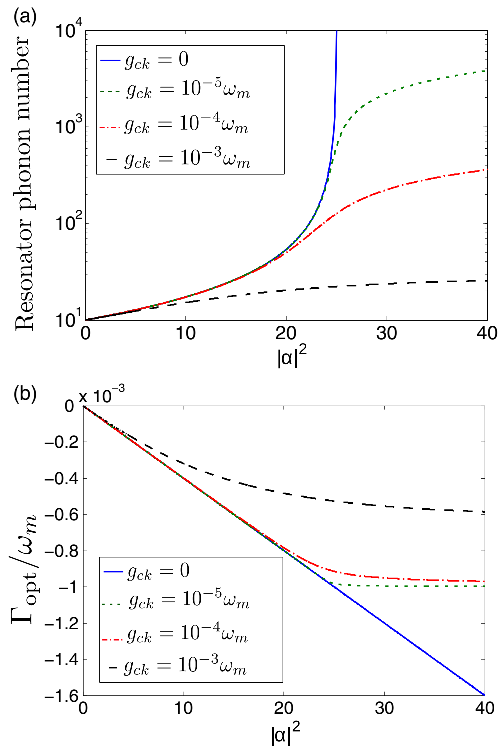

FIG. 7. (Color online) (a) Steady-state phonon number in the resonator and (b) the optical damping $\Gamma_{\rm opt}$ as a function of the number of photons pumped into the cavity in the case where $\Delta=\omega_m$. The values for the parameters are $\gamma=10^{-3}\omega_m$, $\kappa=10^{-1}\omega_m$, $g=10^{-3}\omega_m$, and the bath temperature corresponds to $n^{th}=10$.

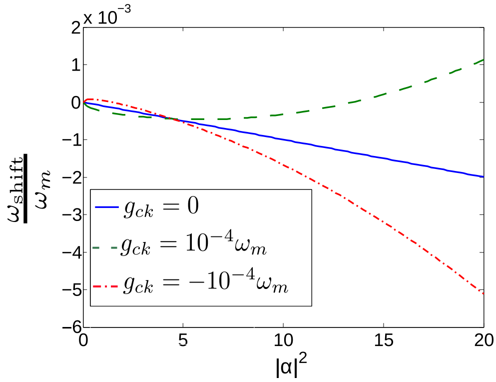

FIG. 8. (Color online) Frequency shift as a function of the number of photons pumped into the cavity when $\Delta = -\omega_m$ with $\gamma = 10^{-3}\omega_m$, $\kappa = 10^{-1}\omega_m$, $g = 10^{-2}\omega_m$ and the bath temperature corresponds to $n^{th} = 100$.

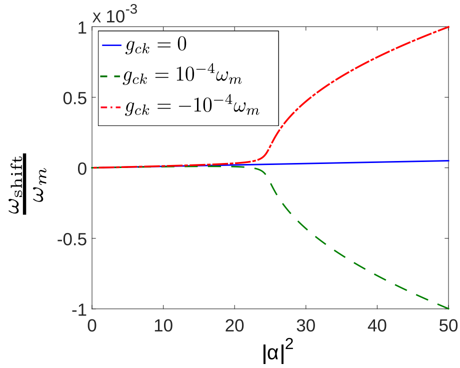

FIG. 9. (Color online) Frequency shift as a function of the number of photons pumped into the cavity when $\Delta = \omega_m$ with $\gamma = 10^{-3}\omega_m$, $\kappa = 10^{-1}\omega_m$, $g = 10^{-2}\omega_m$ and the bath temperature corresponds to $n^{th} = 100$.

another regime we consider is the case where $\Delta = \mp\omega_m$, i.e., setting $\tilde{\Delta} = \mp\omega_m + g_{ck}\langle b^\dagger b\rangle$. In this case the frequency shift and optical damping in the red (upper sign) and blue (lower sign) sidebands become

$$\omega_{\rm shift} = \mp\frac{|G|^2|\alpha|^2}{\tilde{\omega}_m}\frac{[\kappa^2/4 - 2g_{ck}\omega_m(\langle b^\dagger b\rangle - \langle a^\dagger a\rangle)]}{\left[g_{ck}^2(\langle b^\dagger b\rangle - \langle a^\dagger a\rangle)^2 + \kappa^2/4\right]}, \quad (21)$$

$$\Gamma_{\rm opt} = \pm\frac{|G|^2|\alpha|^2\kappa}{g_{ck}^2(\langle b^\dagger b\rangle - \langle a^\dagger a\rangle)^2 + \kappa^2/4}. \quad (22)$$

In Figs. 6 and 7 the steady-state phonon number and the optical damping are plotted as a function of the number of photons pumped into the cavity for the red and blue sidebands, respectively. For the red sideband (Fig. 6) the optical damping increases with increasing $|\alpha|$ until it reaches a maximum value at $|\alpha| = \alpha_c$. Assuming that the minimum is reached for $|\alpha_c|^2 \gg \langle b^\dagger b\rangle$, the maximum corresponds to $\alpha_c^2 = \kappa/(2|g_{ck}|)$, implying cooling to $\langle\delta b^\dagger \delta b\rangle \approx \gamma n^{\rm th}/(\gamma + G^2/g_{ck})$ phonons. This estimate is valid when $n^{th} \ll \kappa/(2g_{ck})[1 + g^2/(\gamma g_{ck})]$. When $|\alpha| \gg \alpha_c$, the optical damping becomes inversely proportional to the number of photons pumped into the cavity; consequently, the cooling deteriorates when more photons are pumped into the cavity.

In the blue sideband (Fig. 7) the main effect of a small cross-Kerr coupling is to limit the instability to a finite number of phonons, $\langle b^\dagger b\rangle \approx \sqrt{\kappa/\gamma}|G||\alpha|/|g_{ck}| + |\alpha|^2$. For $g_{ck} \gg \kappa/(4n^{th})$ the cross-Kerr coupling prevents the instability altogether. This effect thus competes with the usual limitation coming from the intrinsic (Duffing) nonlinearity of the resonator.

In Figs. 8 and 9 we plot the frequency shift as a function of the number of photons pumped into the cavity for the red and blue sidebands. For the red sideband (Fig. 8), when $g_{ck} > 0$ ($g_{ck} < 0$), the frequency shift first increases (decreases) as α increases until $\alpha \approx \langle b^\dagger b\rangle$, after which it decreases (increases). For the blue sideband (Fig. 9), when $g_{ck} > 0$, the frequency shift decreases, while for $g_{ck} < 0$ it increases. The difference at small α between the red and blue sidebands arises from the fact that in the red sideband, when more photons are pumped into the cavity, the cooling improves; thus the number of phonons in the mechanical resonator decreases, making it possible to have a number of photons in the cavity of the same order as and larger than the number of phonons in the resonator.

V. CONCLUSION

In conclusion, we have solved the dynamics of a mechanical resonator coupled to an electromagnetic cavity via a radiation-pressure coupling and a cross-Kerr coupling using a mean-field approach. We have shown that the cross-Kerr coupling shifts the frequencies of the mechanical resonator and of the optical cavity, with the shift depending on the number of photons in the cavity and phonons in the resonator. In addition, we have shown that when the detuning of the pump is equal to the frequency of the mechanical resonator, the variation of the optical damping is nonmonotonous instead of being linearly dependent on the number of phonons pumped into the cavity. We also find that the cross-Kerr coupling can suppress the parametric instability to self-oscillations for the blue sideband pumping.

ACKNOWLEDGMENTS

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