Department of Communications and Networking

# Combinatorial Algorithms for Packings, Coverings and Tilings of Hypercubes

Ashik Mathew Kizhakkepallathu

Aalto University

DOCTORAL DISSERTATIONS

Aalto University publication series
**DOCTORAL DISSERTATIONS** 112/2015

# Combinatorial Algorithms for Packings, Coverings and Tilings of Hypercubes

**Ashik Mathew Kizhakkepallathu**

A doctoral dissertation completed for the degree of Doctor of Science (Technology) to be defended, with the permission of the Aalto University School of Electrical Engineering, at a public examination held at the lecture hall S1 of the school on 18 September 2015 at 12.

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Aalto University publication series
**DOCTORAL DISSERTATIONS** 112/2015



ISBN 978-952-60-6326-3 (printed)
ISBN 978-952-60-6327-0 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)
http://urn.fi/URN:ISBN:978-952-60-6327-0

Unigrafia Oy
Helsinki 2015

Finland

NORDIC ECOLABEL
441 697
Printed matter

**Aalto University**

Aalto University, P.O. Box 11000, FI-00076 Aalto www.aalto.fi

# Abstract

**Author**
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**Name of the doctoral dissertation**
Combinatorial Algorithms for Packings, Coverings and Tilings of Hypercubes

**Publisher** School of Electrical Engineering

**Unit** Department of Communications and Networking

**Series** Aalto University publication series DOCTORAL DISSERTATIONS 112/2015

**Field of research** Information Theory

**Manuscript submitted** 10 April 2015 **Date of the defence** 18 September 2015

**Permission to publish granted (date)** 15 June 2015 **Language** English

☐ **Monograph** ☒ **Article dissertation (summary + original articles)**

**Abstract**

Packing, covering and tiling problems not only form a fundamental theme of study in a number of diverse branches of mathematics such as combinatorics and geometry but several applications of them are well established in fields ranging from information theory and computer science to material sciences. Several existence, counting and classification problems related to packings, coverings and tilings of hypercubes in a torus form the main focus of study in this thesis.

The Shannon capacity of a graph $G$ is an important information theoretic parameter of the graph and is defined as $c(G) = \sup_{d\geq 1}(\alpha(G^d))^{\frac{1}{d}}$, where $\alpha(G)$ is the independence number of $G$. Finding the Shannon capacity of odd cycles of length greater than $5$ is a well known open problem and is in close connection with the problem of packing hypercubes in a torus. The packing and covering problems that we study are also respectively equivalent to clique and dominating set problems in certain dense graphs, problems which are known to be computationally hard to solve.

New lower bounds for the Shannon capacity of odd cycles and triangular graphs are obtained using local and exhaustive search algorithms. We also study a problem related to holes in non-extensible cube packings and verify a conjecture related to holes for $5$-dimensional packings. The classification problem of $5$-dimensional cube tilings in the discrete torus of width $4$ is settled as part of our research by representing this problem as an exact cover problem and using computational methods. Some results on the cubicity of interval graphs are also obtained.

**Keywords** Combinatorial algorithms, Cube packing, Shannon capacity, Cube tiling

**ISBN (printed)** 978-952-60-6326-3 **ISBN (pdf)** 978-952-60-6327-0

**ISSN-L** 1799-4934 **ISSN (printed)** 1799-4934 **ISSN (pdf)** 1799-4942

**Location of publisher** Helsinki **Location of printing** Helsinki **Year** 2015

**Pages** 104 **urn** http://urn.fi/URN:ISBN:978-952-60-6327-0

# Preface

Majority of the research that has led to this thesis was conducted at the Department of Communications and Networking, Aalto University. I am deeply indebted to Prof. Patric Östergård for supervising my work during this period (May 2011 to March 2015). His broad knowledge of the research area and deep insights in combinatorics have always enlightened and motivated me. Dr. Alexandru Popa, who has co-authored two of the papers in this thesis has also played the role of a mentor in the initial phases of my research work at Aalto University. I thank Alex for all the fruitful discussions that we had and the energy and enthusiasm that he has always spread around. The work on translations of a hypercube to represent a graph was done at the Indian Institute of Science with Prof. Sunil Chandran. I thank Sunil for introducing me to the world of advanced graph theory.

I thank all my coworkers, the faculty and other staff at Aalto University for their direct and indirect contributions that have helped the speedier completion of my research work and the writing of this thesis. I thank the Academy of Finland for partially supporting my research (grant 132122). I also thank the Nokia Foundation for their scholarship (one-time grant) that I received in 2013 (Project 201410349). I would like to thank the preliminary examiners of the thesis, Prof. A. Vesel and Dr. M. Dutour Sikirić for their constructive comments.

My daughter Nila's birth is probably the most pleasing moment to me during these years and I thank Nila for her beautiful smile. I am out of words to thank my wife Neenu without whose support and sacrifices this thesis would not have come to reality. I also thank my parents, sister and in-laws who were all very supportive during the entire period that I spent at research. My friends in India, Finland and around the world has always supported me in every possible way that they could. In particular,

I would like to remember two of my dearest friends, Rajesh and Knappan who passed away during these years. One lost his life to cancer and the other passed away in a terrible accident. I thank both of them for those wonderful years that we spent together, the memories of which will stay alive forever.

Finally, I would like to emphasize that most of the work that led to this thesis and its writing was done using free and open source software from the Free Software Foundation and several other contributors. I would like to thank every single one of those programmers who has invested their valuable time and skills in building a better future for humanity in general and tools for scientific research using computers in particular.

Helsinki, August 16, 2015,

Ashik Mathew Kizhakkepallathu

# Contents

# List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

**[I]** K. Ashik Mathew, L. Sunil Chandran. An upper bound for cubicity in terms of boxicity. *Discrete Mathematics*, 309, 2571–2574, 2009.

**[II]** K. Ashik Mathew, Patric R. J. Östergård, Alexandru Popa. On the Shannon capacity of triangular graphs. *Electronic Journal of Combinatorics*, 20(2), #P27, 2013.

**[III]** K. Ashik Mathew, Patric R. J. Östergård, Alexandru Popa. Enumerating cube tilings. *Discrete and Computational Geometry*, 50, 1112–1122, 2013.

**[IV]** K. Ashik Mathew, Patric R. J. Östergård. Hypercube packings and their holes. *Congressus Numerantium*, 219, 89–95, 2014.

**[V]** K. Ashik Mathew, Patric R. J. Östergård. On hypercube packings, blocking sets and a covering problem. *Information Processing Letters*, 115(2), 141–145, 2015.

**[VI]** K. Ashik Mathew, Patric R. J. Östergård. New lower bounds for the Shannon capacity of odd cycles. *Submitted to IEEE Transactions on Information Theory*, 2015.

# Author's Contribution

### Publication [I]: "An upper bound for cubicity in terms of boxicity"

The problem was originally suggested by Prof. Sunil Chandran, but the author invented the algorithm for representing any interval graph as intersection of cubes. The author also took part in the writing of the paper.

### Publication [II]: "On the Shannon capacity of triangular graphs"

The author took part in the discussions which led to the theoretical results. He was responsible for the internal details of the tabu search and the exhaustive search algorithms and their implementation. He also took part in writing the paper.

### Publication [III]: "Enumerating cube tilings"

The author co-designed the algorithms with the other authors of the paper and took care of the implementation details himself. He also took part in writing the paper.

### Publication [IV]: "Hypercube packings and their holes"

The problem and the methodology arose from a discussion between the author and Prof. Patric Östergård. The author took care of the implemenation and also took part in writing the paper.

### Publication [V]: "On hypercube packings, blocking sets and a covering problem"

The central algorithm was conceptualized by Prof. Patric Östergård, but the author took care of the implementation details. He also took part in writing the paper.

### Publication [VI]: "New lower bounds for the Shannon capacity of odd cycles"

A theoretical overview of the techniques that were used in solving the problem was suggested by Prof. Patric Östergård. The author took care of the algorithmic details and the implementation of the part for computing the cyclic subgroups while Prof. Östergård was responsible for the theory and the implementation of the stochastic search. The author also took part in writing the paper.

# List of Symbols

| | |
|---|---|
| $2^X$ | Power set of set $X$ |
| $\mathrm{Aut}(H)$ | Automorphism group of group $H$ |
| $\mathrm{box}(G)$ | The boxicity of graph $G$ |
| $C_p$ | The cycle graph on $p$ vertices |
| $c(G)$ | The Shannon capacity of graph $G$ |
| $\mathrm{cub}(G)$ | The cubicity of graph $G$ |
| $E(G)$ | The edge set of graph $G$ |
| $f(d)$ | The minimum size of a nonextensible cube packings in a discrete $d$-dimensional torus of width $4$ |
| $G_1 \boxtimes G_2$ | The strong product of graphs $G_1$ and $G_2$ |
| $G_1 \square G_2$ | The Cartesian product of graphs $G_1$ and $G_2$ |
| $H(d, p)$ | The maximum number of hypercubes that can be packed in a discrete $d$-dimensional torus of width $p$ |
| $h(d)$ | The minimum size of a blocking set in a discrete $d$-dimensional torus of width $4$ |
| $\mathbb{R}$ | The set of all real numbers |
| $\mathrm{rank}(A)$ | The rank of matrix $A$ |
| $S \times R$ | The Cartesian product of sets $S$ and $R$ |
| $S^n$ | $S \times S \times S \times \ldots \times S$ ($n$ times) |
| $T_p$ | The triangular graph of order $p$ |
| $V(G)$ | The vertex set of graph $G$ |
| $\mathbb{Z}$ | The set of all integers |
| $\mathbb{Z}_p$ | The set of integers $\{0, 1, 2, \ldots, p-1\}$ |
| $\alpha(G)$ | Independence number of graph $G$ |
| $\chi(G)$ | Chromatic number of graph $G$ |
| $\omega(G)$ | Clique number of graph $G$ |

# 1. Introduction

Packing, covering and tiling problems form a fundamental branch of study at the intersection of combinatorics and geometry [7, 23, 54, 75, 81]. Since information and data can often be represented as geometric objects, with their ambiguity (chance of being confused with one another) represented by the amount of overlap between the corresponding objects, such problems are also found to be analogous to certain problems in information and coding theory [6, 57, 71, 79].

Problems related to packings, coverings and tilings of various geometric objects in Euclidean spaces are well studied [23, 33, 54, 67].

Packing problems not only interest mathematicians and computer scientists but several types of packings (especially in 2- and 3-dimensional spaces) find applications in fields like material science and crystallography [40, 53, 74, 85]. Dense packings of polyhedra and their applications in modelling the structure of Platonic and Archimedean solids are studied in [41]. The aid of geometric and combinatorial tools is sought in assisting the study of the flow dynamics of granular materials in [40].

Because of their very natural and often simple structure, tilings in 2- and 3-dimensional spaces are studied by mathematicians and scientists since time immemorial. For example, people often regard the puzzle of Ostomachion (a tiling puzzle designed by Archimedes) as one of the oldest known recreational problems of mathematics [30].

The systematic study of tilings was already initiated in the nineteenth century, mainly due to their applications in the theories regarding the structure of solid matter, with one of the oldest recorded studies in this regard dating back to 1849 [13]. There are several types of tilings which are well-known and are subjects of extensive study by mathematicians. Though Penrose tilings are famous for their aesthetic value [72], they are also examples of quasicrystals that show up in real crystallographic structures [12, 39, 86]. Anisohedral tilings are famous due to their connection with Hilbert's eighteenth problem [34] while Voronoi tilings [83] are famous due to their applications in fields ranging from the study of the distribution of galaxies in the universe to the study of metallic composites on a microscopic scale [68].

Combinatorial problems are broadly divided into the following three categories: existence, counting and classification [43, Chapter 1]. Existence problems are ones in which the objective is to find a combinatorial object which satisfies certain desired properties. Counting problems deal with finding the number of combinatorial objects (up to isomorphism) with a given property while a classification problem asks to describe (up to isomorphism) all the objects that meets a given property. The ideal tools and algorithmic techniques that shall be employed in solving a combinatorial problem also depend on which one of these categories the problem belongs to. We study several existence, counting and classification problems of packings, coverings and tilings of hypercubes of side $2$ in a discrete $d$-dimensional torus of width $p$ for fixed $p$ and $d$.

Chapter 2 deals with cube packings, coverings and tilings primarily. Their applications, like finding the Shannon capacity of odd cycles are also mentioned.

In Chapter 3, we provide a more formal insight into the underlying mathematical concepts. This mainly comprises of a formal description of the data structures that we use frequently throughout the remaining thesis.

In Chapter 4, we have a more in-depth look at the actual combinatorial problems that we study and describe the algorithmic tools employed in solving them.

In Chapter 5, a model where sets of cubes are used to represent a given graph is presented. Rather than having a fixed volume of space and placing cubes in it (as in the case of packings, coverings and tilings), we have a fixed number of cubes here, one corresponding to each vertex of the graph. The objective here is to minimize the number of dimensions required to represent the graph.

In Chapter 6, we see a summary of the state of research in the area and discuss some interesting open problems.

# 2. Packings, Coverings and Tilings

This chapter begins with a formal introduction to the concepts of packings, coverings and tilings. We briefly describe also the basic concepts in graph theory which we need throughout the remaining of this thesis. Some specific details related to packings, coverings and tilings of hypercubes and their applications are unraveled in the later parts of this chapter.

## 2.1 Preliminaries

For vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, we denote by $\mathbf{x} + \mathbf{y}$ the vector $(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$. For any scalar $k \in \mathbb{R}$, the *scalar product* $k\mathbf{x}$ is defined as the vector $(kx_1, kx_2, \ldots, kx_n)$. For two sets $U, V \subseteq \mathbb{R}^n$, the set $U + V$ is defined as the set $\{\mathbf{u} + \mathbf{v} : \mathbf{u} \in U, \mathbf{v} \in V\}$. The *translation* of a set $U \subseteq \mathbb{R}^n$ by a vector $\mathbf{x}$ is defined as the set $\mathbf{x} + U = \{\mathbf{x} + \mathbf{u} : \mathbf{u} \in U\}$.

Two translations $U_1$ and $U_2$ of a set $U$ are called *disjoint* if $U_1 \cap U_2$ is contained in at most an $(n-1)$-dimensional space. A set of translations $\mathcal{P} = \{S_1, S_2, \ldots, S_m\}$ of a set is called a *packing* if $S_i$ and $S_j$ are disjoint for every $1 \leq i < j \leq m$. The packing $\mathcal{P}$ is called a *tiling* if every point in $\mathbb{R}^n$ is contained in at least one $S_i$ for $1 \leq i \leq m$. A set of translations $\mathcal{C} = \{S_1, S_2, \ldots, S_m\}$ of a set is called a *covering* of $\mathbb{R}^n$ if every point in $\mathbb{R}^n$ is contained in at least one $S_i$ for $1 \leq i \leq m$. Clearly, a set of translations of a set is a tiling if and only if it is a packing and a covering.

The *unit vector* $\mathbf{e}_i$ along the $i$th coordinate axis is defined as the vector $(x_1, x_2, \ldots, x_n)$, where $x_i = 1$ and $x_j = 0$ for all $j \neq i$. A set $S$ is said to be *periodic* with period $p$ if $\mathbf{x} \in S \implies \mathbf{x} + p\mathbf{e}_i \in S$, for all $i$, $1 \leq i \leq n$. From these definitions, one shall observe that a periodic packing of period $p$ of $\mathbb{R}^n$ with hypercubes of side $2$ with their centers having integer coordinates corresponds to a packing of the discrete $n$-dimensional torus of side $p$. For

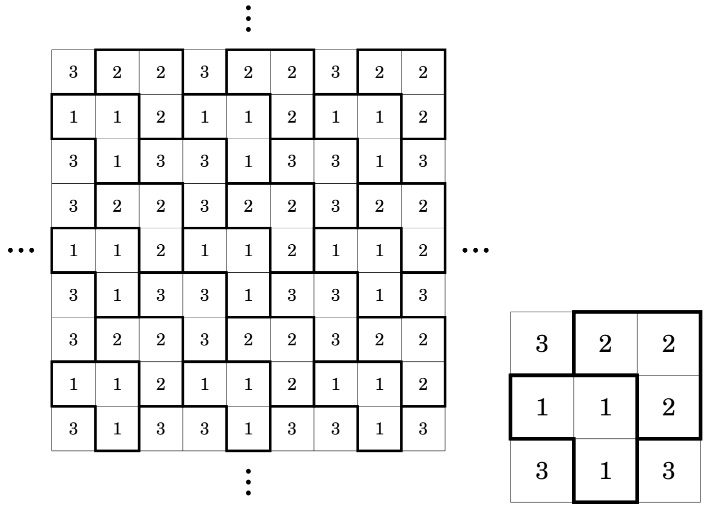


**Figure 2.1.** A periodic packing of $R^2$ and a corresponding packing in a 2-dimensional torus

a periodic tiling of period 3 of $R^2$ with L-shaped bricks and a corresponding tiling of the discrete 2-dimensional torus of width 3, see Figure 2.1.

See [23, 34] for a more detailed overview and examples for packings, coverings and tilings.

## 2.2 Graphs

A *graph* is defined as an ordered pair $G = (V, E)$ where $V$ is called the set of *vertices* of $G$ and $E$, a set of 2-subsets of $V$ is called the set of *edges* of $G$.

For a graph $G$, $V(G)$ denotes the set of vertices of $G$ and $E(G)$ denotes the set of edges of $G$. These notations are independent of the actual names of the vertex and edge sets: the vertex set of a graph $K = (\alpha, \beta)$ shall be denoted by $V(K)$.

If $\{u, v\} \in E(G)$, the vertices $u$ and $v$ are said to be *adjacent* in $G$. An *independent set* in a graph $G$ is a set $I \subseteq V(G)$ such that for any $u, v \in I$, $\{u, v\} \notin E(G)$. A *clique* in a graph $G$ is a set $C \subseteq V(G)$ such that for any $u, v \in C$, $\{u, v\} \in E(G)$. The *independence number* $\alpha(G)$ of a graph $G$ is the size of the biggest independent set in $G$, while the *clique number* $\omega(G)$ is the size of the biggest clique in $G$.

The *chromatic number* $\chi(G)$ of $G$ is the minimum number of colors with which it is possible to color the vertices of $G$ such that adjacent vertices are always given different colors.

The *complement* $\overline{G}$ of a graph $G$ is the graph with vertex set $V(G)$ and edge set $\{\{x, y\} : \{x, y\} \notin E(G)\}$.

The *line graph* $L(G)$ of a graph $G$ is the graph with vertex set $E(G)$ with two vertices in $L(G)$ being adjacent if and only if the corresponding edges have a common vertex in $G$.

For two graphs $G_1$ and $G_2$, their *strong product* $G_1 \boxtimes G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ with an edge between two distinct vertices $(x_1, x_2)$ and $(y_1, y_2)$ if and only if $x_i = y_i$ or $\{x_i, y_i\} \in E(G_i)$ for $i = 1, 2$. Unless otherwise specified, by $G^d$ we denote the graph $G \boxtimes G \boxtimes \cdots \boxtimes G$ ($d$ times).

For two graphs $G_1$ and $G_2$, the Cartesian product $G_1 \square G_2$ is the graph $G$ with $V(G) = V(G_1) \times V(G_2)$ and $E(G) = \{\{(u_1, u_2), (v_1, v_2)\} : (u_1 = v_1, \{u_2, v_2\} \in E(G_2))$ or $(u_2 = v_2, \{u_1, v_1\} \in E(G_1))\}$.

Two graphs $G_1$ and $G_2$ are said to be *isomorphic* with each other if there is a bijection $\pi : V(G_1) \to V(G_2)$ such that $\{u, v\} \in E(G_1)$ if and only if $\{\pi(u), \pi(v)\} \in E(G_2)$ for any $u, v \in V(G_1)$.

A *path* from a vertex $v_1$ to a vertex $v_n$ in a graph $G$ is a non-empty sequence of vertices $v_1, v_2, \ldots, v_n$ such that $\{v_i, v_{i+1}\} \in E(G)$ for every $i \in \{1, 2, \ldots, n-1\}$. A graph $G$ is called *connected* if there is a path from vertex $u$ to vertex $v$ for every pair of $u, v \in V(G)$. A graph on vertices $\{v_1, v_2, \ldots, v_p\}$ is called a *cycle* on $p$ vertices if $E(G) = \{\{v_i, v_{i+1 \bmod p}\} : i \in \{1, 2, \ldots, p\}\}$. A *complete graph* $K_n$ on $n$ vertices is a graph where there is an edge between every pair of vertices. The line graph of $K_n$ is called the *triangular graph* $T_n$.

See [22, Chapter 1] for a detailed overview of the fundamental concepts in graph theory.

## 2.3 Cube Packings

The maximum number of hypercubes of side 2 that can be packed in a discrete $d$-dimensional torus of width $p$ is denoted by $H(d, p)$. As one may quickly infer, $H(d, p) = (\frac{p}{2})^d$, if $p$ is even. Also, $H(1, p) = \lfloor \frac{p}{2} \rfloor$. But finding $H(d, p)$ is highly nontrivial for a general $d$ and odd $p$.

The following theorems from [6] give upper and lower bounds for $H(d, p)$.

**Theorem 2.3.1** (Baumert *et. al.*). $H(d, p) \leq \left\lfloor \frac{p}{2} H(d-1, p) \right\rfloor$.

**Theorem 2.3.2** (Baumert *et. al.*). $H(d, p) \geq H(d_1, p) H(d - d_1, p)$ *for any* $d > 1$ *and* $1 \leq d_1 < d$.

See also [15, 23, 24] for more studies on cube packings.

## 2.4 Shannon Capacity

The zero error capacity of a noisy communication channel is the maximum rate at which it is possible to transmit information (repeatedly) via the channel with zero probability of error [79]. A channel transmitting messages over an alphabet $S$ with $n$ symbols may be modeled as a graph $G$ on $n$ vertices. We let $V(G) = S$ and $E(G) = \{\{v_1, v_2\} : v_1, v_2 \in S$ and $v_1$ and $v_2$ may lead to the same output symbol$\}$. In other words, two symbols are adjacent if and only if they are indistinguishable.

The *Shannon capacity* of $G$ is defined as

$$c(G) = \sup_{d \geq 1} (\alpha(G^d))^{\frac{1}{d}},$$

where $\alpha(G)$ is the independence number of $G$.

The notion of a perfect graph was introduced by Berge in [8]. A graph $G$ is called a *perfect graph* if $\omega(G) = \chi(G)$. It is known that the Shannon capacity of a perfect graph is same as its independence number. It was conjectured in [8] that a graph is perfect if and only if it does not have $C_p$ or $\overline{C_p}$ as an induced subgraph (for any odd $p \geq 5$). This conjecture, known as the *perfect graph conjecture* was proved by Chudnovsky *et. al.* [17]. This result gives evidence that odd cycles and their complements are the simplest graphs for which the estimation of Shannon capacity is nontrivial.

It was proved by Lovász [55] in 1979 that $c(C_5) = \sqrt{5}$, but finding $c(C_p)$ for general odd $p$ remains one of the most celebrated open problems in extremal combinatorics [10].

Various authors have provided upper and lower bounds for the Shannon capacity of odd cycles using theoretical [6, 11] and computational [82] methods.

The problem of Shannon capacity of odd cycles is closely related to the problem of packing cubes in a torus, because $\alpha(C_p^d)$ is same as the number of cubes that can be packed in the discrete $d$-dimensional torus of width $p$ [6]. In a similar fashion, the problem of packing unconnected cubes (Cartesian products of unions of two disjoint intervals of unit length) is related to the problem of finding the Shannon capacity of triangular graphs [15].

The *union* of two graphs $G_1$ and $G_2$, $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If the graphs $G_1$ and $G_2$ are graph representations of two channels, $G_1 \cup G_2$ is the graph which represents the channel which is the union of the channels represented by

$G_1$ and $G_2$. Here, the union of two channels is defined as a new channel which corresponds to a situation where either one of the channels may be used for each transmitted alphabet. Shannon showed that $c(G_1 \cup G_2) \geq c(G_1) + c(G_2)$ [79]. He conjectured that the equality holds whenever the vertex set of one of the graphs can be covered by a set of cliques whose cardinality is the independence number of the graph. Alon [3] disproved this conjecture. The Shannon capacity of digraphs was also investigated by Alon in [4].

Algebraic tools for investigating the Shannon capacity of graphs were developed by Haemers in the 1970s [35, 36]. For a graph $G$ on $n$ vertices and an $n \times n$ matrix $A$ where two vertices $i$ and $j$ are adjacent if and only if $A_{ij} \neq 0$, it is shown that $c(G) \leq \text{rank}(A)$ [35].

The Lovász's $\vartheta$-function

$$\vartheta(p) = \frac{p \cos \frac{\pi}{p}}{1 + \cos \frac{\pi}{p}}$$

gives upper bounds for the Shannon capacity of the odd cycle $C_p$ [55]. For more studies on the Shannon capacity of graphs, see also [9, 48, 50, 58, 64].

## 2.5 Blocking Sets

Suppose $T(n, p, q)$ is the minimum number of copies of the grid graph $P_q^n$ (where $P_q$ is a path on $q$ vertices) that can cover all the vertices of the $n$-dimensional torus $C_p^n$, where the exponents of graphs are assumed to be with respect to the Cartesian product.

A *blocking set* $\mathcal{S}$ is a set of hypercubes of side 2 in a discrete torus with the property that every possible hypercube of side 2 has a nonempty intersection with some hypercube in $\mathcal{S}$. The minimum cardinality of a blocking set in a discrete $d$-dimensional torus of width $4$ is denoted by $h(d)$. It is shown in [V] that $h(d)$ is same as $T(d, 4, 3)$.

The related problem of finding non-extensible packings (which are blocking sets with the additional constraint that overlaps between participating cubes are not allowed) is also studied in [V]. The minimum cardinality of a non-extensible packing is denoted by $f(d)$. Clearly, $h(d) \leq f(d)$.

An example for a blocking set in a $2$-dimensional torus is the set of cubes with centers (0,0), (1,1) and (2,2). Also observe that there is no non-extensible cube packing in the $2$-dimensional torus which is not a tiling.

The study of $T(n,p,q)$ for other values of $p$ and $q$, especially for $p=3, q=2$, was popularized already a few decades ago in the context of covering codes and football pools [14, 18, 38, 69, 71].

## 2.6 Holes in Cube Packings

For a packing $\mathcal{P}$ of hypercubes of side $2$ in $\mathbb{R}^d$, the complement of the space occupied by the cubes in $\mathcal{P}$

$$\mathbb{R}^d - \bigcup_{P \in \mathcal{P}} P$$

is called the *hole* of $P$.

Given a $d$-dimensional non-extensible cube packing, its lifting is a $(d+1)$-dimensional non-extensible cube packing obtained by adding a layer of cube tiling.

A conjecture regarding certain structural properties of holes of non-extensible cube packings was proposed by Dutour Sikirić, Itoh and Poyarkov [25]. The conjecture states that the hole of any non-extensible cube packing in a $d$-dimensional torus which is near-complete (of cardinalities greater than $2^d - 8$) is same as the hole of some non-extensible packing that can be obtained by a sequence of liftings of a non-extensible cube packing in a $3$- or $4$-dimensional torus. For the complete statement and a discussion about this conjecture, we refer the reader to the original article [25]. The authors of [25] support this conjecture by extensive numeric computations. This conjecture is verified partly for the $5$-dimensional case in [IV].

## 2.7 Cube Tilings

Tilings of the space $\mathbb{R}^d$ by hypercubes of side $2$ are called $d$-dimensional *general cube tilings* [23]. A *special cube tiling* is a general cube tiling which is periodic with period $4$ and where the centers of all cubes are restrained to have only integer coordinates. Because of their periodicity, we may also talk about special cube tilings as tilings of the discrete $d$-dimensional torus of width $4$ by cubes of side $2$. In the sequel, *cube tiling* stands for a special cube tiling. See Figure 2.2 for a $3$-dimensional cube tiling with $8$ cubes. The cube labeled 8 is not visible from the reader's angle of view.

Several formulations of the problem of finding cube tilings to other well

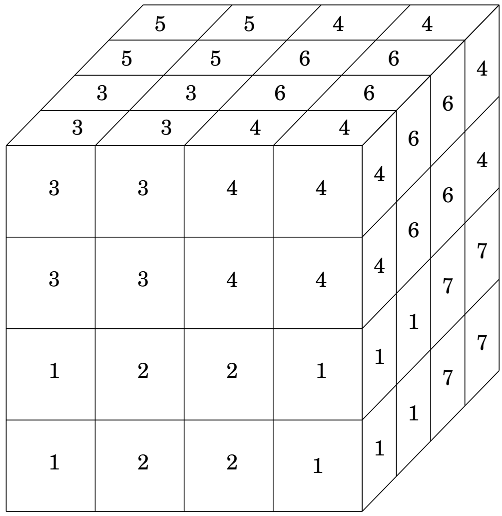


**Figure 2.2.** A 3-dimensional cube tiling

known problems in algebra and geometry are known; Hajos [37] formulated the problem of cube tilings as a problem of factorizations of abelian groups.

## 2.8 Keller's conjecture

A significant number of studies on cube tilings are related to a conjecture proposed by Keller in 1930 [46]. Keller's conjecture states that in any $n$-dimensional cube tiling, there is at least one pair of cubes that share an $(n-1)$-dimensional face. For example, in the 3-dimensional cube tiling in Figure 2.2, the cubes labeled 1 and 2 share a 2-dimensional face.

Keller's cube tiling conjecture was proved for $n \leq 6$ by Perron in 1940 [73], but the conjecture remained open for higher dimensions for more than half a century.

In 1992, it was shown that Keller's conjecture is false for dimensions greater than 9 [51]. MacKey [56] proved that Keller's conjecture is false even for dimensions 8 and 9. This was done using a graph theoretic reformulation of the problem using certain class of graphs called *Keller graphs*. Keller's conjecture was proved for special cube tilings for dimension 7 by Debroni et al. in 2011 [21]. In the original conjecture, cubes are allowed to have coordinates having any real number value, leaving the original conjecture still open for dimension 7.

## 2.9 Switching Graph and Its Connectedness

Switching is a local transformation that when applied to a given combinatorial object gives a new object with the same set of parameters as the original object. Switching is a well studied topic [43] and the history of this concept can be traced to back to works by Norton [66] and Fisher [28] in the 1930s. For a set of combinatorial objects $\mathcal{X}$, their switching graph is a graph defined on vertex set $\mathcal{X}$ with edges between two vertices (objects) present if and only if one object can be obtained from the other by a switching operation. If this graph is connected, it implies that one can obtain the whole classification (the entire set $\mathcal{X}$) from any single object $X \in \mathcal{X}$ and by applying a sequence of switches.

For a $d$-dimensional cube tiling that has two cubes sharing a $(d-1)$-dimensional face, it is possible to define such a switching operation as one that gives a new tiling by moving the pair of cubes by one unit across the axis perpendicular to the shared face. Now, one may ask the following question: Is the switching graph of $d$-dimensional cube tilings connected? Obviously, for a dimension $d$ greater than 7, the answer is in the negative because there are tilings with no cubes which share a $(d-1)$-dimensional face (since Keller's conjecture is known to be false).

Apart from finding a complete classification of $5$-dimensional cube tilings, the connectedness of their switching graph is also studied in [III]. It is found that the switching graph is connected in this case.

# 3. Mathematical Background and Representations

This chapter describes some of the data structures that are used for representing the packings, coverings and tilings of hypercubes that are described in Chapter 2.

## 3.1 Code Representation

Though coding theory originated as a tool for solving engineering applications, it later evolved to also a topic of pure mathematical interest and many fundamental problems in coding theory are closely related to construction of combinatorial objects [43, Chapter 2].

The set of integers $\{0, 1, 2, \ldots, p-1\}$ is referred to as $\mathbb{Z}_p$. A non-empty set $C \subseteq \mathbb{Z}_p^d$ is called a $p$-ary (block) *code*. Elements of $C$ are called *codewords*.

For the purpose of formalism and facilitating algebraic discussion, we often represent the sets of cubes as codes. This is done by choosing the sequence of coordinates of the center of each cube as the codeword corresponding to the cube.

For example, the packing of a $2$-dimensional torus of width $5$ by hypercubes of side $2$ in Figure 3.1 consists of $5$ cubes with their centers at $\{(1,1),(3,2),(0,3),(2,4),(4,0)\}$ and can hence be represented by the code $\{11, 32, 03, 24, 40\}$.

## 3.2 Isomorphism and Graph Representation

Two embeddings of cubes in a discrete torus of width $p$ are said to be isomorphic with each other if the code corresponding to one may be mapped to the code corresponding to the other by a permutation of coordinates followed by permutations of the coordinate values (on each coordinate) of

| | | | | |
|---|---|---|---|---|
| | 4 | 4 | 5 | 5 |
| 3 | 4 | 4 | | 3 |
| 3 | | 2 | 2 | 3 |
| 1 | 1 | 2 | 2 | |
| 1 | 1 | | 5 | 5 |

**Figure 3.1.** A packing of $5$ squares in the discrete $2$-dimensional torus of width $5$

the form

$$i \to ai + b \mod p, a \in \{-1, 1\}, b \in \mathbb{Z}_p. \tag{3.1}$$

The technique of checking if two combinatorial objects are isomorphic by checking if their graph representations are isomorphic is a very effective method and is widely employed in classification algorithms [44, 60]. Several software libraries with routines for graph isomorphism checking are known, such as *nauty* [62] and *bliss* [42].

For a packing of $k$ hypercubes $\{C_1, C_2, \ldots, C_k\}$ in a discrete $d$-dimensional torus of width $p$ where each cube $C_i$ can be represented by the codeword $c_1^i c_2^i \ldots c_d^i$, a graph representation $G$ can be constructed as follows: $V(G) = V_0 \cup V_1 \cup V_2 \cup \cdots V_d$, where $V_0 = \{z_1, z_2, \ldots, z_k\}$ and $V_j = \{v_0^j, v_1^j, \ldots, v_{p-1}^j\}$ for $1 \leq j \leq d$. For any $i$, $1 \leq i \leq k$ and $j$, $1 \leq j \leq d$, we add edges $\{z_i, v_{c_j^i}^j\}$. We also add edges from $v_t^j$ to $v_{t+1 \mod p}^j$ for every $1 \leq j \leq d$ and $0 \leq t \leq p-1$.

Figure 3.2 shows a graph representation constructed using the aforementioned rule from the packing in Figure 3.1.

Isomorphism is an equivalence relation on combinatorial objects that plays a big role in classification algorithms [43, Chapter 4]. When asked to present a classification of a combinatorial object with a set of desired properties, we are expected to present the objects only up to isomorphism.

During the course of our research, we have extensively used *nauty* to solve our subproblems of checking for isomorphisms of combinatorial objects, after representing them as graphs. One may use *nauty* to generate a *canonical labeling* of the vertices of a given graph as well as for comput-

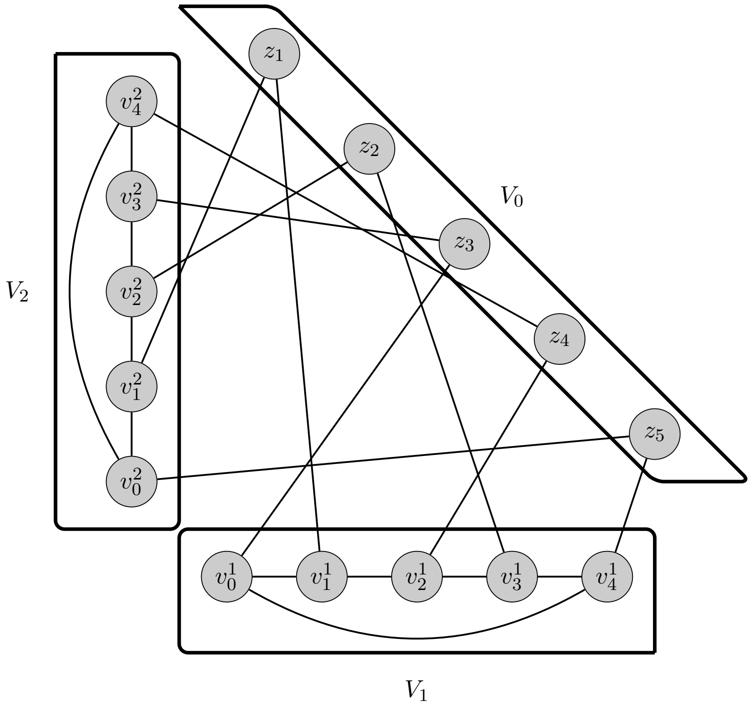


**Figure 3.2.** Graph representation of the cube packing in Figure 3.1

ing the automorphism group of the graph. Once the canonical labelings of two graphs are available, we may check (bitwise) if they are same, to see if the graphs are isomorphic. For the underlying back-track search algorithm and the detailed internal working of the *nauty* program, we refer the reader to [62].

## 3.3 Packings as Cliques

A packing problem can be formulated as a clique search problem by considering the graph $G$ where $V(G)$ is the set of objects that can be present in a packing with edges between two vertices existing only when there is no overlap between the corresponding objects .

The aforementioned graph $G$ that corresponds to a cube packing $C = \{c_1, c_2, \ldots c_k\}$ in the discrete $d$-dimensional torus of width $p$ has the following structure: The vertex set of $G$ is

$$V(G) = \{0, 1, \ldots, p-1\}^d$$

and the edge set is

$$E(G) = \{\{v_1, v_2\} : \min(|v_{1i} - v_{2i}|, p - |v_{1i} - v_{2i}|) \geq 2 \text{ for some } i(1 \leq i \leq d)\}.$$

Here $v_{ki}$ stands for the value at position $i$ in codeword $v_k$.

Since finding cliques in a graph $G$ is equivalent to finding independent sets in $\overline{G}$, we may also talk about this problem using the notion of independent sets.

It is possible to represent the problem of cube tilings in a torus also as a clique problem in graphs. A tiling in the discrete $d$-dimensional torus of width $4$ with hypercubes of side $2$ corresponds to a maximum clique in a graph $G$ on vertex set $V(G) = \{0,1,2,3\}^d$ with the edge set $E(G) = \{\{x, y\} : x, y \in V(G)$ and the codewords $x$ and $y$ differ exactly by $2$ in at least one coordinate$\}$.

Software programs such as *Cliquer* [65] can be used to find cliques in a graph. Since clique search in graphs is known to be **NP**-hard, no polynomial time exact algorithms are expected to exist (unless **P = NP**). The branch and bound algorithm used by *Cliquer* is found to be much faster than a brute-force search for several real-world situations [70].

## 3.4 Tilings and Exact Cover

A *set cover*, or simply *cover* of a finite set $P$ is a set of non-empty subsets of $P$ whose union is $P$. A cover consisting of pairwise disjoint subsets is an *exact cover*. Given a set $P$ and a set of subsets $\mathcal{S}$ of $P$, the problem of finding all exact covers of $P$ consisting of sets from $\mathcal{S}$ is referred to as the *exact cover problem*.

Now, the problem of finding all $d$-dimensional cube tilings can be stated as an instance of the exact cover problem where $P$ is the set $\{0,1,2,3\}^d$ and elements of $\mathcal{S}$ are subsets of $P$ of the form $\{k_1, k_1+1\} \times \{k_2, k_2+1\} \times \cdots \times \{k_d, k_d+1\}$ for $0 \leq k_1, k_2, \ldots, k_d \leq 3$ (all numbers are taken $modulo$ 4).

For a back-track search algorithm for the exact cover problem, see [43, Algorithm 5.2]. Knuth [49] proposed an idea to speed up back-track search algorithms by using a data structure which he called *dancing links*. The libexact software library [45], which uses the dancing links data structure, helps in solving exact cover problems computationally. The classification problem of 5-dimensional cube tilings was solved by representing the problem as an instance of the exact cover problem and using libexact [III].

# 4. Algorithms and Tools

A general description of the various combinatorial algorithms and computational methods that are used in solving some of the problems related to packings, coverings and tilings of hypercubes is presented in this chapter.

## 4.1 Existence Problems, Exhaustive Search and Local Search

The most straight-forward approach to solve an existence problem is to consider every possible solution sequentially. This method, which can be applied only when the search space is discrete and finite, is called *exhaustive search*. Exhaustive search methods can be used to find a classification of combinatorial objects that satisfy a given property or to prove their non-existence. The proof for the non-existence of projective planes of order 10 is a classical and well known example of the use of exhaustive search [52] using computers. Exhaustive search is also used in the early study of cube packings in the 3-dimensional torus of width 7 in [6] and later for proving several results in [25]. An exhaustive search algorithm was employed in proving the non-existence of packings of size 28 of the 5-dimensional torus of width 5 using unconnected cubes in [II].

Local search methods are primarily applicable to solving existence problems by explicit construction though they do not guarantee to find a solution even if one exists [43, Chapter 1]. A local search algorithm moves from one point in the search space to another point at each step after making certain considerations. Local search methods often outperform exhaustive search methods when the goal is to find an acceptable good solution in a limited amount of time. It can also sometimes happen that the solution which is found by local search can be proven to be the optimum solution.

Simulated annealing is a local search method invented by Černy [16] and Kirckpatrick [47] independently at the same time in the early 1980s. For some early studies involving the use of simulated annealing, see [26, 84]. A lower bound for the Shannon capacity of $C_7$ (which was later improved in [VI]) was obtained using simulated annealing [82].

Tabu search is another meta-heuristic local search method invented by Glover and McMillan [31]. Tabu search avoids the problem of the search getting stuck at a local optimum by keeping track of the recently visited points in the search space and prohibiting the search algorithm from revisiting them (hence the name "tabu" search). See [32] for a detailed overview of this technique. Tabu search was used in [II] for obtaining packings of discrete tori with unconnected cubes.

## 4.2 Counting and Classification

A simple approach for ensuring isomorph-free generation while solving a counting or classification problem is to keep a global record $\mathcal{R}$ of all the objects that are accepted so far, while doing an exhaustive search. When a new object $X$ is obtained, we may check against all objects in $\mathcal{R}$ for isomorphism and accept $X$ and add it to $\mathcal{R}$ if and only if $X$ is non-isomorphic to every $Y \in \mathcal{R}$. Though this technique is easy to implement, the time complexity of this approach is at least quadratic in the size of the final classification, making it inefficient when the expected size of the classification is huge. Orderly generation is a more efficient method which saves the same purpose of isomorph-free exhaustive generation. We refer the reader to [43, Section 4.2.2] for a detailed description about orderly generation.

A method called *canonical augmentation* was invented by B. D. McKay for isomorph-free exhaustive generation [61]. The following section briefly explains the main working principles behind canonical augmentation.

## 4.3 Building the Required Structures from Lower Dimensional Structures

A complete classification of sets of hypercubes in an $(n+1)$-dimensional torus that satisfy a given set of rules can be built (in most cases that we study) by *extending* from a complete classification of sets of hypercubes in an $n$-dimensional torus that satisfy the same set of rules. Here, embedding usually means a function $f : \mathcal{E}_n \to 2^{\mathcal{E}_{n+1}}$, where $\mathcal{E}_d$ denotes the set of all sets of $d$-dimensional hypercubes that satisfy the given set of rules and $E \in \mathcal{E}_n$ is the projection of some $e \in f(E)$ in any $n$-dimensional plane.

For example, one may say that the $2$-dimensional packing in Figure 3.1 is in the extension of the $1$-dimensional packing defined by the cubes $\{0, 1, 2, 3, 4\}$ (The values here are centers of the 1-dimensional cubes which are line segments of length $2$).

We use canonical augmentation to ensure that only one object from its isomorphism class is accepted in the classification. This is achieved by introducing the following checks before accepting an element $X \in f(E)$:

**Condition 1:** Set $X$ is the lexicographic minimum of all the sets under the action of the automorphism group of $E$, Aut(E), on $X$.

**Condition 2:** Set $E$ is the canonical parent of $X$.

These two tests can be performed in any order.

See [43, Section 4.2.3] for a detailed theory behind canonical augmentation and some examples of using it.

## 4.4 Computational Results

New lower bounds for the Shannon capacity of triangular graphs $T_7$, $T_{13}$, $T_{15}$ and $T_{21}$ are obtained in [II] using tabu search while the best currently known lower bounds for $C_7$ and $C_{15}$ are obtained in [VI] by prescribing symmetries for independent sets in powers of these cycles and using computational tools (a stochastic search which internally employs elements of the same branch and bound algorithm that *Cliquer* uses).

The currently best known upper and lower bounds for $\alpha(C_p^d)$ and $\alpha(T_p^d)$ are listed in Table 4.1 and Table 4.2 together with keys. Only one key is provided in cases where the value can be obtained using more than one method. The numbers in **bold** font are new bounds obtained as part of the author's research.

Key to Table 4.1 and Table 4.2.

**Table 4.1.** Bounds on $\alpha(C_p^d)$ for $1 \le d \le 5$ and $5 \le p \le 15$

| $p \backslash d$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 5 | $^{a}2^{a}$ | $^{a}5^{a}$ | $^{c}10^{f}$ | $^{c}25^{d}$ | $^{c}50$–$55^{j}$ |
| 7 | $^{a}3^{a}$ | $^{a}10^{a}$ | $^{f}33^{f}$ | $^{h}108$–$115^{d}$ | $^{k}$**350**–$401^{j}$ |
| 9 | $^{a}4^{a}$ | $^{a}18^{a}$ | $^{e}81^{d}$ | $^{c}324$–$361^{j}$ | $^{c}1458$–$1575^{j}$ |
| 11 | $^{a}5^{a}$ | $^{a}27^{a}$ | $^{e}148^{d}$ | $^{k}$**748**–$814^{d}$ | $^{c}3996$–$4477^{d}$ |
| 13 | $^{a}6^{a}$ | $^{a}39^{a}$ | $^{g}247^{i}$ | $^{k}$**1534**–$1605^{d}$ | $^{c}9633$–$10432^{d}$ |
| 15 | $^{a}7^{a}$ | $^{a}52^{a}$ | $^{k}$**381**–$390^{d}$ | $^{b}2720$–$2925^{d}$ | $^{c}$**19812**–$21937^{d}$ |

**Table 4.2.** Bounds on $\alpha(T_p^d)$ for $1 \le d \le 4$ and $5 \le p \le 21$

| $p \backslash d$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 5 | $^{p}2^{p}$ | $^{p}5^{p}$ | $^{a}12^{a}$ | $^{t}$**27**$^{t}$ |
| 7 | $^{p}3^{p}$ | $^{p}10^{p}$ | $^{s}$**35**$^{q}$ | $^{r}$**114**–$122^{q}$ |
| 9 | $^{p}4^{p}$ | $^{p}18^{p}$ | $^{e}81^{q}$ | $^{r}327$–$364^{q}$ |
| 11 | $^{p}5^{p}$ | $^{p}27^{p}$ | $^{e}148^{q}$ | $^{r}776$–$814^{q}$ |
| 13 | $^{p}6^{p}$ | $^{p}39^{p}$ | $^{s}$**248**–$253^{q}$ | $^{r}$**1551**–$1644^{q}$ |
| 15 | $^{p}7^{p}$ | $^{p}52^{p}$ | $^{r}$**384**–$390^{q}$ | $^{r}$**2802**–$2925^{q}$ |
| 17 | $^{p}8^{p}$ | $^{p}68^{p}$ | $^{e}578^{q}$ | $^{e}4913^{q}$ |
| 19 | $^{p}9^{p}$ | $^{p}85^{p}$ | $^{e}807^{q}$ | $^{e}7666^{q}$ |
| 21 | $^{p}10^{p}$ | $^{p}105^{p}$ | $^{d}1092$–$1102$ | $^{r}$**11441**–$11571^{q}$ |

[a] $\alpha(C_p) = \lfloor \frac{p}{2} \rfloor, \alpha(C_p^2) = \lfloor \frac{p^2-p}{4} \rfloor$ [6, Theorem 2]

[b] $\alpha(C_p^d) \ge 1 + \alpha(C_{p-2}^d)\frac{p^d-2^d}{(p-2)^d}$ [6, Corollary 2]

[c] $\alpha(C_p^d) \ge \alpha(C_p^{d_1})\alpha(C_p^{d-d_1})$ [6, Corollary 3]

[d] $\alpha(C_p^d) \le \lfloor \frac{p}{2}\alpha(C_p^{d-1}) \rfloor$ [6, Lemma 2]

[e] Baumert *et al.* [6, Theorem 3]

[f] Baumert *et al.* [6, Theorem 4]

[g] Baumert *et al.* [6, Theorem 6]

[h] Vesel and Žerovnik [82]

[i] Bohman, Holzman, and Natarajan [11]

[j] $\alpha(C_p^d) \le \left[ \frac{p \cos \frac{\pi}{p}}{1+\cos \frac{\pi}{p}} \right]^d$ [55]

[k] Stochastic search [ VI]

[p] $\alpha(T_p) = \lfloor \frac{p}{2} \rfloor, \alpha(T_p^2) = \lfloor \frac{p^2-p}{4} \rfloor$[15, p. 42]

[q] $\alpha(T_p^{d+1}) \le \left\lfloor \frac{p}{2}\alpha(T_p^d) \right\rfloor$ [II, Theorem 4]

[r] $\alpha(T_{p+2}^d) \ge \sum_{i=0}^{d} \binom{d}{i} H(i,p)$ [II, Theorem 5]

[s] Tabu search [II]

[t] Exhaustive search [II]

Algorithms for classifying blocking sets and non-extensible packings in a discrete torus are presented in [V], using which it is shown that $f(5) = 12$, $f(6) = 16$, $h(6) = 15$ and $h(7) \leq 23$. Here, canonical augmentation is used for extending the rows in a covering matrix, until a blocking set in the required dimension is obtained.

Canonical augmentation was used for isomorph-free generation while extending 4-dimensional tilings to 5-dimensional tilings in [III] using an exact cover representation and computations using libexact. It was shown that there are 899,710,227 5-dimensional cube tilings up to isomorphism, and the total number of labeled tilings is 638,560,878,292,512. While computing a complete classification of a certain class of combinatorial objects, it is often possible to validate the results using double-counting. One way to find the total number of labeled objects is to count them while they are generated. Another way to find it is by using the orbit-stabilizer theorem from the objects that are accepted after the canonical augmentation tests (the representatives of each isomorphism class). The total number of labeled cube tilings was validated by double-counting using the orbit-stabilizer theorem in [III]. Validation by double counting was used also in classification results in [II], [IV] and [V].

# 5. Representing Graphs by Translations of a Hypercube

In the previous chapters, we saw problems where we were given a finite amount of space (in a given dimension) and were asked to place cubes into the space, without breaking certain structural constraints (no overlaps in the case of packings and the additional constraint of covering the entire space in the case of tilings etc.). In this chapter we introduce another problem related to translations of cubes. We are given a graph $G$ and we are asked to find a set $\mathcal{C}$ of $|V(G)|$ translations of the hypercube $[-1, 1]^n$ and a bijection $f : V \rightarrow \mathcal{C}$ such that for any $v_1, v_2 \in V$, $f(v_1)$ and $f(v_2)$ are disjoint if and only if $\{v_1, v_2\} \notin E(G)$. One may quickly notice that this is always possible if the dimension of the space is arbitrarily high. For example, if we consider an $n$-dimensional space, where $n = |V|$, this problem is trivial. So, we ask for the minimum dimension in which this can be done.

## 5.1 Formal Definition and Examples

Consider a set system $\mathcal{S} = \{S_1, S_2, \ldots, S_p\}$. An *intersection graph* on $\mathcal{S}$ is a graph $G$ defined together with a bijection $f : V(G) \rightarrow \mathcal{S}$ such that two vertices $v_1, v_2 \in V(G)$ are adjacent in $G$ iff the sets $f(v_1)$ and $f(v_2)$ have a non-empty intersection. Intersection graphs of various geometric objects is a well studied topic [5, 27, 63, 78].

The concept of cubicity of a graph $G$, cub$(G)$ shall be defined now as follows:

**Definition 5.1.1.** *The cubicity of a graph $G$ is the smallest positive integer $k$ such that $G$ can be represented as an intersection graph of $k$-dimensional hypercubes of same size.*

A variant of the aforementioned problem is to ask for the minimum dimension in which a given graph can be represented as the intersection

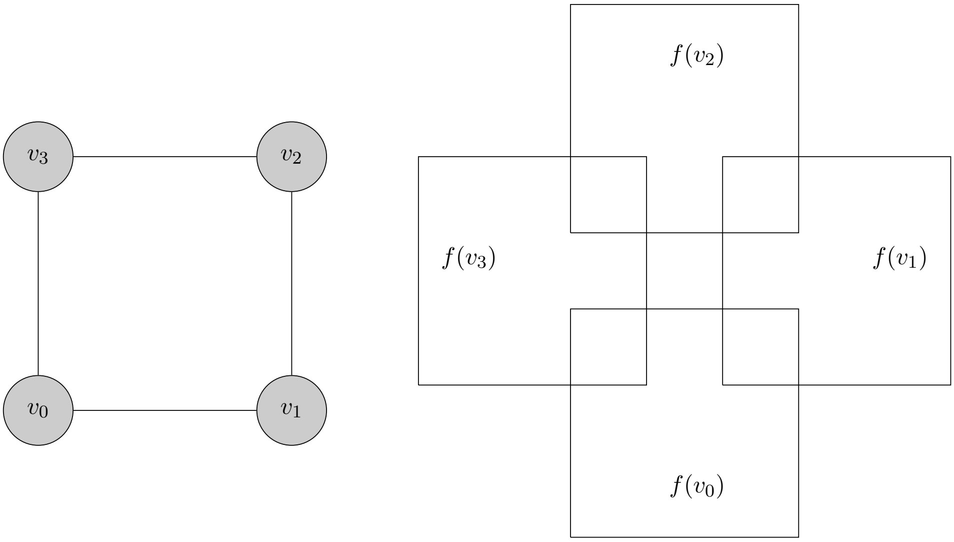


**Figure 5.1.** A 4-cycle $C_4$ and its 2-dimensional cube representation. The function $f$ is a bijection from $V(C_4)$ to a set of 2-dimensional cubes

graph of axis-parallel boxes (Cartesian product of intervals of possibly different lengths). The minimum dimension in which a graph $G$ can be represented as the intersection graph of axis parallel boxes is called the *boxicity* of $G$ and is denoted by box$(G)$. The concepts of cubicity and boxicity were originally proposed by F. S. Roberts in 1969 [76]. The original motivation behind the study of these concepts was their applications in niche overlap in ecology [19, 77] and fleet maintenance problems in operations research [20].

See Figure 5.1 for the representation of the cycle $C_4$ as an intersection graph of 2-dimensional cubes. It is known that cub$(C_4) = 2$.

For the star graph in Figure 5.2 with $2^d+1$ vertices ($p = 2^d$), it is possible to create a $d$-dimensional cube representation by mapping the vertex $v_0$ to a cube of side $2$ centered at $(0, 0, \ldots, 0)$ and the other vertices to cubes of side $2$ with their centers at $(\pm 2, \pm 2, \ldots, \pm 2)$. It is also known that the cubicity of this graph is $d$. See [76] for a proof.

## 5.2 Representing an Interval Graph as a Set of Cubes

An interval graph is a graph $G$ for which there exists a bijection $f$ from the set of vertices of the graph to set of intervals on the real line such that two vertices $v_1$ and $v_2$ are adjacent if and only if the corresponding intervals $f(v_1)$ and $f(v_2)$ overlap. By the virtue of its definition, interval

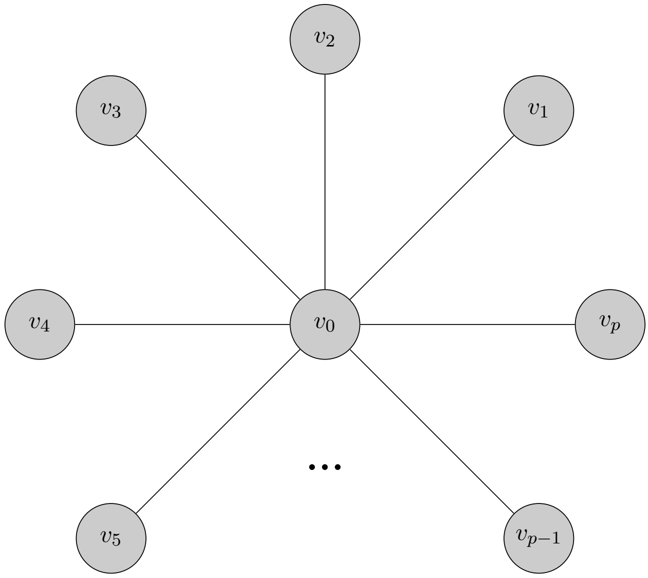


**Figure 5.2.** A star graph on $p + 1$ vertices

graphs are the same class of graphs that have boxicity 1. Also, observe that the star graph mentioned in the last section is an interval graph.

For any interval graph $G$, cub$(G) \leq \lceil \log_2 |V(G)| \rceil$. This was proved in [I]. The proof from [I] also results in a polynomial time algorithm for representing any interval graph as an intersection graph of cubes in $\lceil \log_2 |V(G)| \rceil$ dimensions.

# 6. Conclusions

Computational techniques for finding lower bounds for the independence numbers of powers of odd cycles and triangular graphs have been successful in improving several best known theoretical bounds for the Shannon capacity of these graphs. Prescribing symmetries for independent sets of the powers of odd cycles and using stochastic search techniques has yielded new results [VI].

Though computational techniques have been successful in improving many existing bounds, the exact values for the Shannon capacity of these graphs are still unknown and remains one of the most challenging open problems in information theory and extremal combinatorics.

The study of the Shannon capacity of the complements of odd cycles using computational methods is a problem which remains to be examined in the new light of the computational studies in [II] and [VI].

A generalized version of the cube packing problem is to ask for the maximum number of boxes (for which all sides do not have the same length) that can be packed in a non-uniform discrete $d$-dimensional torus. See [7, 80] for a background study of such box packings. It is a natural question to ask if local and exhaustive search algorithms may be put to use in solving problems related to such generalized box packings.

Packing and tiling problems of hypercubes in a discrete $d$-dimensional torus of width $4$ are studied in [III], [IV] and [V]. The classification problem for cube tilings in the $5$-dimensional torus of width $4$ was settled in [III]. It may be worthwhile investigating the complete classification of cube tilings in the next dimension in which this problem is open now, which is $6$. This problem is extremely challenging using the known set of computational tools (including the ones that we used for settling the $5$-dimensional case). It is estimated that the algorithm that we used in [III] will take billions of years of CPU time, to solve the $6$-dimensional case.

Another interesting problem is related to verifying Keller's conjecture for the 7-dimensional non-discrete case.

The problem of finding sets of cubes that represent a given graph as an intersection graph of cubes (cubicity) has been well studied and many results have been obtained in this area after the publication of [I]. Several upper bounds for the cubicity of graphs are available in terms of functions of more than one graph invariants. Perhaps the most important of all of them is a result from [2] that cub$(G) \leq$ box$(G)\lceil \log_2 \alpha(G) \rceil$. It would be nice to see if cubicity can be bounded from above as a unary function of some important graph invariant, at least for some classes of graphs like planar graphs. For more recent studies and open problems in this field, see also [1, 29, 59].

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Aalto-DD 112/2015

ISBN 978-952-60-6326-3 (printed)
ISBN 978-952-60-6327-0 (pdf)
ISSN-L 1799-4934
ISSN 1799-4934 (printed)
ISSN 1799-4942 (pdf)

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