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Connection between electric and magnetic coherence in free electromagnetic fields

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We introduce quantitative measures for the description of the electric and magnetic coherence in a stationary, random electromagnetic field at two points, in a volume, and in the Fourier space. These quantities are applied to free electromagnetic fields, and several theorems regarding the relationship between the two types of coherences in such fields are established. Fields which are statistically homogeneous, and those which, in addition, are statistically isotropic are considered separately. Furthermore, the connection between the electric and magnetic coherence is exemplified for some specific statistically homogeneous fields.

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I. INTRODUCTION

The principles and tools of the optical coherence theory are traditionally formulated within the scalar framework [1]. This significantly limits their usability, since today's physical and engineering problems in optics often necessitate a fully electromagnetic treatment. Although some basic concepts of the electromagnetic coherence theory have been introduced, such as the coherence tensors both in the space-time and space-frequency domains, the wave equations that govern their propagation [1], as well as the coherence-polarization matrices for nonuniform beams [2], many results familiar from the scalar theory have not been verified for the general, nonplanar electromagnetic fields. The situation has started to change only quite recently [3–9], and along with it, also a discussion on the nature of the electromagnetic coherence has arisen [10,11].

A common feature for all of those recent works is that they consider the coherence properties of the electric field component only. In particular, the degree of electric coherence that is a measure for the correlation of the electric field vectors at two points was introduced in Refs. [3,4]. In this paper, we extend that quantity to the magnetic field vector and put forward measures that characterize, in general, the electric and magnetic coherence in a volume and in the Fourier space. Besides purely for fundamental reasons, the magnetic field and its coherence properties are of interest, for instance, in connection with tightly focused electromagnetic waves and magnetic interactions in semiconductor quantum dots. Our specific aim is to employ the two quantities to investigate the connection between the electric and magnetic coherence in free electromagnetic fields [12,13].

The paper is organized as follows. In Sec. II we present the quantities for the characterization of the electric and magnetic spectral (spatial) coherence in general electromagnetic fields. Next, we employ these quantities for free electromagnetic fields (Sec. III), for statistically homogeneous fields (Sec. IV), and for fields that are statistically homogeneous and isotropic (Sec. V). Section VI contains specific examples of statistically homogenous fields, and, finally, Sec. VII summarizes the main results and conclusions of the work. Certain mathematical details are presented in Appendixes A–E.

II. BASIC DEFINITIONS

The spatial coherence (correlation) properties of a statistically stationary, random, electromagnetic field, at a frequency ω, are described by the cross-spectral density tensors [1]. In this work, only the electric and magnetic tensors are of relevance to us. They are defined, specifically, by the relations

$$\overleftrightarrow{W}^{(e)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \mathbf{E}^*(\mathbf{r}_1,\omega)\mathbf{E}(\mathbf{r}_2,\omega)\rangle, \tag{1}$$

$$\overleftrightarrow{W}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle \mathbf{H}^*(\mathbf{r}_1,\omega)\mathbf{H}(\mathbf{r}_2,\omega)\rangle, \tag{2}$$

where $\mathbf{E}(\mathbf{r},\omega)$ and $\mathbf{H}(\mathbf{r},\omega)$ represent the electric and magnetic components of the electromagnetic field realization at frequency ω in a statistical ensemble. Furthermore, the angle brackets and the asterisks denote ensemble averaging and complex conjugation, respectively, and $\mathbf{r}_{1,2}$ refer to two points in space.

It is insightful to investigate the coherence properties of the field in the Fourier space ($\mathbf{k}$-space) [14]. Therefore, we introduce the spatial Fourier transforms of the electric and magnetic field realizations

$$\tilde{\mathbf{E}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^3}\int \mathbf{E}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r, \tag{3}$$

$$\tilde{\mathbf{H}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^3}\int \mathbf{H}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r. \tag{4}$$

These quantities constitute an ensemble of realizations in the $\mathbf{k}$ space, where the coherence tensors are obtained by averaging over the ensemble, i.e.,

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$$\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega) = \langle \tilde{\mathbf{E}}^*(\mathbf{k}_1,\omega)\tilde{\mathbf{E}}(\mathbf{k}_2,\omega)\rangle, \quad (5)$$

$$\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_1,\mathbf{k}_2,\omega) = \langle \tilde{\mathbf{H}}^*(\mathbf{k}_1,\omega)\tilde{\mathbf{H}}(\mathbf{k}_2,\omega)\rangle. \quad (6)$$

Furthermore, the $\mathbf{k}$-space coherence tensors are related to the spatial Fourier transforms of the $\mathbf{r}$-space tensors

$$\tilde{\overleftrightarrow{W}}^{(f)}(\mathbf{k}_1,\mathbf{k}_2,\omega) = \frac{1}{(2\pi)^6}\int\int \overleftrightarrow{W}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \times e^{-i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}d^3r_1d^3r_2, \quad f=(e,h), \quad (7)$$

by the relations

$$\overleftrightarrow{W}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_2,\omega) = \tilde{\overleftrightarrow{W}}^{(f)}(-\mathbf{k}_1,\mathbf{k}_2,\omega), \quad f=(e,h). \quad (8)$$

The minus sign in front of $\mathbf{k}_1$ on the right-hand side of Eq. (8) is simply a consequence of the definition of the coherence tensors with complex conjugation on the first field variable.

A quantitative measure for the degree of coherence of the electric component of a random electromagnetic field was recently introduced in the space-time domain [3], and soon after in the space-frequency domain [4] (see also Refs. [5,6,10,11]). Analogous quantities can readily be written for the magnetic field component as well. Thus, we define the spectral degree of electric coherence, μ_e, and the spectral degree of magnetic coherence, μ_h, by the relations

$$\mu_f^2(\mathbf{r}_1,\mathbf{r}_2,\omega) = \frac{\mathrm{tr}\left[\overleftrightarrow{W}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega)\cdot\overleftrightarrow{W}^{(f)}(\mathbf{r}_2,\mathbf{r}_1,\omega)\right]}{\mathrm{tr}\left[\overleftrightarrow{W}^{(f)}(\mathbf{r}_1,\mathbf{r}_1,\omega)\right]\mathrm{tr}\left[\overleftrightarrow{W}^{(f)}(\mathbf{r}_2,\mathbf{r}_2,\omega)\right]} = \frac{\sum_{i,j}|W_{ij}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2}{\sum_i W_{ii}^{(f)}(\mathbf{r}_1,\mathbf{r}_1,\omega)\sum_i W_{ii}^{(f)}(\mathbf{r}_2,\mathbf{r}_2,\omega)},$$

$$f=(e,h), \quad (9)$$

where $W_{ij}^{(f)}$, with $(i,j)=(x,y,z)$, are the elements of the cross-spectral density tensors. The latter form of the above formula indicates that the electric and magnetic degrees of coherence describe, respectively, the average correlation of the Cartesian electric or magnetic field components at two points [4]. The values of μ_f are bounded between zero and one, with the limits corresponding to complete incoherence (noncorrelation) and complete coherence (correlation), respectively [4,5].

It is also useful to introduce corresponding measures for the degrees of coherence of the field components in the Fourier space. Thus, we define the spectral degree of electric coherence in the $\mathbf{k}$ space, $\nu_{\tilde{e}}$, and an analogous quantity for the magnetic field, $\nu_{\tilde{h}}$, via the expressions

$$\nu_{\tilde{f}}^2(\mathbf{k}_1,\mathbf{k}_2,\omega) = \frac{\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\cdot\overleftrightarrow{W}^{(\tilde{f})}(\mathbf{k}_2,\mathbf{k}_1,\omega)\right]}{\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_1,\omega)\right]\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{f})}(\mathbf{k}_2,\mathbf{k}_2,\omega)\right]} = \frac{\sum_{i,j}|W_{ij}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_2,\omega)|^2}{\sum_i W_{ii}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_1,\omega)\sum_i W_{ii}^{(\tilde{f})}(\mathbf{k}_2,\mathbf{k}_2,\omega)},$$

$$f=(e,h). \quad (10)$$

The values of these quantities are bounded to the interval $0 \leqslant \nu_f \leqslant 1$ with the upper and lower limits corresponding to complete coherence and complete incoherence between the Fourier components.

In order to quantify the field correlations within a volume, we define the effective spectral degrees of coherence for the electric and magnetic fields by the formulas [6,15,16]

$$\mu_{f,\mathrm{eff}}^2(\omega) = \frac{\int_D\int_D S_f(\mathbf{r}_1,\omega)S_f(\mathbf{r}_2,\omega)\mu_f^2(\mathbf{r}_1,\mathbf{r}_2,\omega)d^3r_1d^3r_2}{\int_D\int_D S_f(\mathbf{r}_1,\omega)S_f(\mathbf{r}_2,\omega)d^3r_1d^3r_2},$$

$$f=(e,h), \quad (11)$$

where D is the volume in which the field is considered (in our analysis D is the whole space), and

$$S_f(\mathbf{r},\omega) = \mathrm{tr}\left[\overleftrightarrow{W}^{(f)}(\mathbf{r},\mathbf{r},\omega)\right], \quad f=(e,h). \quad (12)$$

The quantities S_e and S_h correspond to the spectral densities of the electric and magnetic fields, respectively.

The above formulas are valid for any electromagnetic field. However, in this work we consider specifically free electromagnetic fields, i.e., fields that consist of a superposition of propagating plane waves only. The realizations of such a field are expressible in the form

$$\mathbf{E}(\mathbf{r},\omega) = \int_\Omega \mathbf{e}(\hat{u},\omega)e^{ik\hat{u}\cdot\mathbf{r}}d\Omega, \quad (13)$$

$$\mathbf{H}(\mathbf{r},\omega) = \int_\Omega \mathbf{h}(\hat{u},\omega)e^{ik\hat{u}\cdot\mathbf{r}}d\Omega, \quad (14)$$

where $\mathbf{e}(\hat{u},\omega)$ and $\mathbf{h}(\hat{u},\omega)$ are, respectively, the electric and magnetic field components of the plane wave propagating in the direction specified by the unit vector $\hat{u}$. Furthermore, $k=\omega/c_0$ is the wave number of the field, with c_0 being the speed of light in vacuum, and the integration is performed over the solid angle Ω. The realizations of the free electromagnetic field obey Maxwell's equations, written in an infinite source-free space as (in SI units)

$$\nabla\cdot\mathbf{E}(\mathbf{r},\omega) = 0, \quad (15)$$

$$\nabla\cdot\mathbf{H}(\mathbf{r},\omega) = 0, \quad (16)$$

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = i\omega\mu_0\mathbf{H}(\mathbf{r},\omega), \quad (17)$$

$$\nabla\times\mathbf{H}(\mathbf{r},\omega) = -i\omega\epsilon_0\mathbf{E}(\mathbf{r},\omega), \quad (18)$$

where μ_0 and ϵ_0 are the vacuum permeability and permittivity, respectively. According to Eq. (17), we can write the following connection between the electric and magnetic plane wave amplitudes in the field realization

$$\mathbf{h}(\hat{u},\omega)=\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\hat{u}\times\mathbf{e}(\hat{u},\omega). \tag{19}$$

By making use of Eqs. (12)–(14), (19), and (A1), one readily finds that

$$\int S_h(\mathbf{r},\omega)d^3r=\frac{\epsilon_0}{\mu_0}\int S_e(\mathbf{r},\omega)d^3r, \tag{20}$$

with the integration performed over the whole space. Finally, we note that in the $\mathbf{k}$ space, Eqs. (15)–(18) take on the forms

$$\mathbf{k}\cdot\tilde{\mathbf{E}}(\mathbf{k},\omega)=0, \tag{21}$$

$$\mathbf{k}\cdot\tilde{\mathbf{H}}(\mathbf{k},\omega)=0, \tag{22}$$

$$\mathbf{k}\times\tilde{\mathbf{E}}(\mathbf{k},\omega)=\omega\mu_0\tilde{\mathbf{H}}(\mathbf{k},\omega), \tag{23}$$

$$\mathbf{k}\times\tilde{\mathbf{H}}(\mathbf{k},\omega)=-\omega\epsilon_0\tilde{\mathbf{E}}(\mathbf{k},\omega), \tag{24}$$

where $\tilde{\mathbf{E}}(\mathbf{k},\omega)$ and $\tilde{\mathbf{H}}(\mathbf{k},\omega)$ are the Fourier transforms defined in Eqs. (3) and (4).

III. ELECTRIC AND MAGNETC COHERENCE IN FREE ELECTROMAGNETIC FIELDS

In this section we first establish a general relationship between the electric and magnetic coherence of free electromagnetic fields in the $\mathbf{k}$ space. Later we use that result to obtain a general connection for the two types of coherence in the $\mathbf{r}$ space.

A. Connection between the electric and magnetic coherence in k space

By making use of Eqs. (6), (23), and (A2), we obtain

$$\begin{aligned}\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\right]&=\frac{1}{\omega^2\mu_0^2}\left\langle\left[\mathbf{k}_1\times\tilde{\mathbf{E}}^*(\mathbf{k}_1,\omega)\right]\cdot\left[\mathbf{k}_2\times\tilde{\mathbf{E}}(\mathbf{k}_2,\omega)\right]\right\rangle\\&=\frac{1}{\omega^2\mu_0^2}\Big\{(\mathbf{k}_1\cdot\mathbf{k}_2)\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\right]\\&\quad-\mathbf{k}_2\cdot\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\cdot\mathbf{k}_1\Big\},\end{aligned} \tag{25}$$

which, when $\mathbf{k}_1=\mathbf{k}_2=\mathbf{k}$, simplifies to

$$\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k},\mathbf{k},\omega)\right]=\frac{\epsilon_0}{\mu_0}\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k},\mathbf{k},\omega)\right]. \tag{26}$$

In obtaining the above result, Eq. (21) was employed. Furthermore, by a straightforward computation one finds that

$$\begin{aligned}\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\cdot\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_2,\mathbf{k}_1,\omega)\right]&=\frac{1}{\omega^4\mu_0^4}\left\langle\left[\mathbf{k}_1\times\tilde{\mathbf{E}}^*(\mathbf{k}_1,\omega)\right]_n\left[\mathbf{k}_2\times\tilde{\mathbf{E}}(\mathbf{k}_2,\omega)\right]_m\right\rangle\\&\quad\times\left\langle\left[\mathbf{k}_2\times\tilde{\mathbf{E}}^*(\mathbf{k}_2,\omega)\right]_m\left[\mathbf{k}_1\times\tilde{\mathbf{E}}(\mathbf{k}_1,\omega)\right]_n\right\rangle\\&=\epsilon_{nij}\epsilon_{nuv}\epsilon_{mkl}\epsilon_{mpq}k_{1i}k_{1u}k_{2k}k_{2p}\\&\quad\times W_{jl}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)W_{qv}^{(\tilde{e})}(\mathbf{k}_2,\mathbf{k}_1,\omega),\end{aligned} \tag{27}$$

where ϵ_{nij} is the Levi-Civita symbol, and Einstein's summation notation is invoked. Using the following relation for the Levi-Civita symbol [17]

$$\epsilon_{nij}\epsilon_{nuv}=\delta_{iu}\delta_{jv}-\delta_{iv}\delta_{ju}, \tag{28}$$

together with Eq. (21), we obtain the result

$$\begin{aligned}\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\cdot\overleftrightarrow{W}^{(\tilde{h})}(\mathbf{k}_2,\mathbf{k}_1,\omega)\right]&=\left(\frac{\epsilon_0}{\mu_0}\right)^2W_{jl}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)W_{lj}^{(\tilde{e})}(\mathbf{k}_2,\mathbf{k}_1,\omega)\\&=\left(\frac{\epsilon_0}{\mu_0}\right)^2\mathrm{tr}\left[\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)\cdot\overleftrightarrow{W}^{(\tilde{e})}(\mathbf{k}_2,\mathbf{k}_1,\omega)\right].\end{aligned} \tag{29}$$

Equations (26) and (29), when substituted into Eq. (10), imply the following theorem:

Theorem I. For any free electromagnetic field, the electric and magnetic degrees of coherence in the $\mathbf{k}$ space, defined in Eq. (10), are equal, i.e.,

$$\nu_{\tilde{h}}(\mathbf{k}_1,\mathbf{k}_2,\omega)=\nu_{\tilde{e}}(\mathbf{k}_1,\mathbf{k}_2,\omega). \tag{30}$$

In other words, at all temporal frequencies, two spatial Fourier components of the electric field are as correlated as the corresponding Fourier components of the magnetic field.

Theorem I is, of course, understandable, since Eq. (23) indicates that the Fourier transforms of the magnetic field realizations are obtained from the electric ones by rotating them in the $\mathbf{k}$ space by an angle of $\pi/2$ and by scaling. Put slightly differently, both vectors are proportional to the same random process, but pointing in different (orthogonal) directions.

B. Connection between the electric and magnetic coherence in r space

We note that the $\mathbf{k}$-space relation given in Eq. (29) can be rewritten in the form

$$\sum_{jk}|W_{jk}^{(\tilde{h})}(\mathbf{k}_1,\mathbf{k}_2,\omega)|^2=\left(\frac{\epsilon_0}{\mu_0}\right)^2\sum_{jk}|W_{jk}^{(\tilde{e})}(\mathbf{k}_1,\mathbf{k}_2,\omega)|^2. \tag{31}$$

According to Eqs. (7) and (8), the functions $W_{jk}^{(\tilde{f})}(\mathbf{k}_1,\mathbf{k}_2,\omega)$ and $W_{jk}^{(f)}(-\mathbf{r}_1,\mathbf{r}_2,\omega)$ constitute a Fourier transform pair. Furthermore, we assume that these functions are square integrable. Therefore, Eq. (31) together with Parseval's theorem, Eqs. (A3) and (A4), imply that

$$\sum_{jk}\int\int|W_{jk}^{(h)}(-\mathbf{r}_1,\mathbf{r}_2,\omega)|^2d^3r_1d^3r_2=\left(\frac{\epsilon_0}{\mu_0}\right)^2\sum_{jk}\int\int|W_{jk}^{(e)}(-\mathbf{r}_1,\mathbf{r}_2,\omega)|^2d^3r_1d^3r_2, \tag{32}$$

where the integration is performed over the whole space. On making the change of variable $\mathbf{r}_1\to-\mathbf{r}_1$, and using Eqs. (9) and (12), we obtain the result

$$\int\int \mu_h^2(\mathbf{r}_1,\mathbf{r}_2,\omega)S_h(\mathbf{r}_1,\omega)S_h(\mathbf{r}_2,\omega)d^3r_1d^3r_2 = \left(\frac{\epsilon_0}{\mu_0}\right)^2 \int\int \mu_e^2(\mathbf{r}_1,\mathbf{r}_2,\omega)S_e(\mathbf{r}_1,\omega)S_e(\mathbf{r}_2,\omega)d^3r_1d^3r_2. \tag{33}$$

This formula, together with Eqs. (11) and (20), implies the following theorem:

Theorem II. For any free electromagnetic field, the effective spectral degrees of electric and magnetic coherence, calculated over the whole space, are equal at all frequencies, i.e.,

$$\mu_{e,\mathrm{eff}}(\omega) = \mu_{h,\mathrm{eff}}(\omega), \tag{34}$$

where $\mu_{f,\mathrm{eff}}(\omega)$, with $f=(e,h)$, are defined by Eq. (11).

As seen from Eq. (30), the connection between the electric and magnetic coherence is local in the **k** space, while in the **r** space the relationship is established via the effective degrees of coherence that are averaged quantities and take into account the degrees of coherence between all pairs of points in the space.

IV. ELECTRIC AND MAGNETIC COHERENCE IN STATISTICALLY HOMOGENEOUS ELECTROMAGNETIC FIELDS

For a statistically homogeneous electromagnetic field, the cross-spectral density tensors depend on the positions $\mathbf{r}_1$ and $\mathbf{r}_2$ only through the displacement $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$,

$$\overleftrightarrow{W}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \equiv \overleftrightarrow{W}^{(f)}(\mathbf{R},\omega), \quad f=(e,h). \tag{35}$$

Equivalently, the realizations of such a field are of the form given in Eqs. (13) and (14), but the plane waves propagating in different directions are completely uncorrelated [18], i.e., the angular coherence tensors are of the form

$$\langle \mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u}',\omega)\rangle = \langle \mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle\delta(\hat{u}-\hat{u}'), \tag{36}$$

$$\langle \mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u}',\omega)\rangle = \langle \mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle\delta(\hat{u}-\hat{u}'), \tag{37}$$

where $\mathbf{e}(\hat{u},\omega)$ and $\mathbf{h}(\hat{u},\omega)$ are the angular plane-wave amplitudes. Inserting the representations (13) and (14) into Eqs. (1) and (2), and using the angular correlation tensors above, it follows that the electric and magnetic cross-spectral density tensors of a statistically homogeneous field are of the form

$$\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega) = \int_\Omega \langle \mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle e^{-ik\hat{u}\cdot\mathbf{R}}d\Omega, \tag{38}$$

$$\overleftrightarrow{W}^{(h)}(\mathbf{R},\omega) = \int_\Omega \langle \mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle e^{-ik\hat{u}\cdot\mathbf{R}}d\Omega, \tag{39}$$

where, according to Eq. (19),

$$\langle \mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle = -\frac{\epsilon_0}{\mu_0}\hat{u}\times\langle \mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle\times\hat{u}. \tag{40}$$

Formulas (38)–(40) are useful in establishing various connections between the electric and magnetic cross-spectral density tensors as we shall see shortly. For example, with the help of Eq. (A2), we at once find that

$$\mathrm{tr}\left[\overleftrightarrow{W}^{(h)}(\mathbf{R},\omega)\right] = \frac{\epsilon_0}{\mu_0}\mathrm{tr}\left[\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)\right], \tag{41}$$

and, therefore, also that

$$S_h(\omega) = \frac{\epsilon_0}{\mu_0}S_e(\omega), \tag{42}$$

indicating that the electric and magnetic energy densities are constant throughout the space. This is, of course, what one would intuitively expect.

For statistically homogeneous fields it is possible to establish a general, yet compact relation between the electric and magnetic cross-spectral density tensors. A straightforward calculation, presented explicitly in Appendix B, leads to the formula

$$\overleftrightarrow{W}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\left\{\left[\overleftrightarrow{U}+\frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\right]\times\mathrm{tr}\left[\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)\right]-\overleftrightarrow{W}^{(e)T}(\mathbf{R},\omega)\right\}, \tag{43}$$

where T denotes transpose. Hence the functional forms of the electric and magnetic cross-spectral density tensors of a homogeneous electromagnetic field may be completely different. Consequently, the degrees of electric and magnetic coherence may have different values. However, as theorem II states, the effective degrees of electric and magnetic coherence are equal.

V. ELECTRIC AND MAGNETIC COHERENCE IN STATISTICALLY HOMOGENEOUS AND ISOTROPIC ELECTROMAGNETIC FIELDS

Consider next free electromagnetic fields that not only are statistically homogeneous, but also isotropic. The realizations of such a field are expressible as Eqs. (13) and (14), but in addition to the plane waves being angularly uncorrelated (homogeneity), they are also unpolarized and have the same intensity (see theorem IV). A general form for both the electric and the magnetic cross-spectral density tensor of a statistically homogeneous and isotropic field is given by [19,20]

$$\overleftrightarrow{W}^{(f)}(\mathbf{R},\omega) = A_f(R,\omega)\overleftrightarrow{U}+B_f(R,\omega)\hat{R}\hat{R}, \quad f=(e,h), \tag{44}$$

where, as before, $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$, and in addition, $R=|\mathbf{R}|$, and $\hat{R}=\mathbf{R}/R$. The above tensors are symmetric, $\overleftrightarrow{W}^{(f)\mathrm{T}}(\mathbf{R},\omega)=\overleftrightarrow{W}^{(f)}(\mathbf{R},\omega)$, and their form is invariant under rotation of the coordinate system [19]. Furthermore, the coefficients $A_f(R,\omega)$ and $B_f(R,\omega)$ are not independent, but are connected by a divergence equation of the form

$$\boldsymbol{\nabla}\cdot\overleftrightarrow{W}^{(f)}(\mathbf{R},\omega) = \mathbf{0}, \quad f=(e,h), \tag{45}$$

and a Helmholtz equation given in Eq. (B4). Under these conditions and using Eq. (43) we find (as shown in Appendix C) that

$$A_h(R,\omega) = \frac{\epsilon_0}{\mu_0} A_e(R,\omega), \quad (46)$$

$$B_h(R,\omega) = \frac{\epsilon_0}{\mu_0} B_e(R,\omega). \quad (47)$$

This result implies the following theorem:

Theorem III. For any statistically homogeneous and isotropic electromagnetic field, the electric and magnetic cross-spectral density tensors are of the form of Eq. (44), and differ only by a constant factor ϵ_0/μ_0,

$$\overleftrightarrow{W}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0} \overleftrightarrow{W}^{(e)}(\mathbf{R},\omega). \quad (48)$$

Consequently, at any pair of points $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$ and at any frequency ω,

$$\mu_h(\mathbf{R},\omega) = \mu_e(\mathbf{R},\omega), \quad (49)$$

i.e., the degrees of electric and magnetic coherence defined by Eq. (9) are equal.

Note that according to Eq. (42) the factor ϵ_0/μ_0 $(=1/\eta_0^2$, with η_0 being the free-space impedance) is equal to the ratio of the magnetic and electric energy densities of the field.

VI. CONNECTION BETWEEN ELECTRIC AND MAGNETIC COHERENCE IN SPECIFIC STATISTICALLY HOMOGENEOUS ELECTROMAGNETIC FIELDS

Next, we employ the plane-wave representation of free electromagnetic fields, Eqs. (13) and (14), to investigate the connection between the electric and magnetic coherence in some specific statistically homogeneous fields. For this purpose, it is useful to express the electric amplitude vector of each plane wave in the field realization in terms of the s and p polarized components, which are mutually orthogonal and perpendicular to the propagation direction $\hat{u}$. Hence, for the amplitude of the electric field, we write

$$\mathbf{e}(\hat{u},\omega) = e_s(\hat{u},\omega)\hat{s} + e_p(\hat{u},\omega)\hat{p}, \quad (50)$$

where $e_s(\hat{u},\omega)$ and $e_p(\hat{u},\omega)$ denote the amplitudes of the s and p polarized components, respectively. Furthermore, the corresponding unit vectors are obtained as $\hat{s}=\hat{u}\times\hat{u}_z/|\hat{u}\times\hat{u}_z|$, and $\hat{p}=\hat{s}\times\hat{u}$, with $\hat{u}_z$ being the unit vector along the z axis of the Cartesian coordinate system.

Using Eqs. (40) and (50), the angular coherence tensors take on the forms

$$\langle\mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle = \mathcal{A}_{ss}(\hat{u},\omega)\hat{s}\hat{s} + \mathcal{A}_{sp}(\hat{u},\omega)\hat{s}\hat{p} + \mathcal{A}_{ps}(\hat{u},\omega)\hat{p}\hat{s} + \mathcal{A}_{pp}(\hat{u},\omega)\hat{p}\hat{p}, \quad (51)$$

$$\langle\mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle = \frac{\epsilon_0}{\mu_0}[\mathcal{A}_{ss}(\hat{u},\omega)\hat{p}\hat{p} - \mathcal{A}_{sp}(\hat{u},\omega)\hat{p}\hat{s} - \mathcal{A}_{ps}(\hat{u},\omega)\hat{s}\hat{p} + \mathcal{A}_{pp}(\hat{u},\omega)\hat{s}\hat{s}], \quad (52)$$

where

$$\mathcal{A}_{ij}(\hat{u},\omega) = \langle e_i^*(\hat{u},\omega)e_j(\hat{u},\omega)\rangle, \quad (i,j)=(s,p). \quad (53)$$

The diagonal elements $\mathcal{A}_{ii}(\hat{u},\omega)$ correspond to the intensities (spectral densities) of the s and p polarized electric field components, and the off-diagonal elements $\mathcal{A}_{ij}(\hat{u},\omega)$, with $i\neq j$, characterize their correlations. By varying the value of these elements as a function of $\hat{u}$, we can change the angular distribution and the state of partial polarization of the plane waves. We recall that when the s and p polarized components of a plane wave are completely uncorrelated and have the same intensity, the wave is fully unpolarized. On the other hand, when the two components are fully correlated, the plane wave is fully polarized, irrespective of the intensities. In any other case, the wave is partially polarized. For a more thorough discussion on the partial polarization of plane waves (two-dimensional fields) and their degree of polarization, we refer to Ref. [1]. Below we consider some specific plane-wave distributions, in which, due to the assumption of statistical homogeneity, the waves are angularly uncorrelated.

A. Statistically homogeneous and isotropic field and its plane-wave representation

First we study what is required of the plane waves, or more specifically of the angular coherence tensors in Eqs. (51) and (52), in order for the electromagnetic field to be statistically homogeneous and isotropic. We note that Eq. (48) of theorem III demands that $\mathcal{A}_{ss}(\hat{u},\omega)=\mathcal{A}_{pp}(\hat{u},\omega)$, and $\mathcal{A}_{sp}(\hat{u},\omega)=-\mathcal{A}_{ps}(\hat{u},\omega)$ in this case. Furthermore, the symmetry of the tensors implies that $\mathcal{A}_{sp}(\hat{u},\omega)=\mathcal{A}_{ps}(\hat{u},\omega)$, which, when combined with the former condition, indicates that $\mathcal{A}_{sp}(\hat{u},\omega)=\mathcal{A}_{ps}(\hat{u},\omega)=0$. Hence the angular coherence tensors must be of the form

$$\langle\mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle = \mathcal{A}(\hat{u},\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}), \quad (54)$$

$$\langle\mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle = \frac{\epsilon_0}{\mu_0}\mathcal{A}(\hat{u},\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}), \quad (55)$$

where $\mathcal{A}(\hat{u},\omega)$ is the intensity of the s and p polarized components. Making use of the relations $\hat{s}\hat{s}+\hat{p}\hat{p}+\hat{u}\hat{u}=\overleftrightarrow{U}$ and $\hat{u}\exp(-ik\hat{u}\cdot\mathbf{R})=(i/k)\boldsymbol{\nabla}\exp(-ik\hat{u}\cdot\mathbf{R})$, the electric and magnetic cross-spectral density tensors can be written as

$$\overleftrightarrow{W}^{(e)}_{iso}(\mathbf{R},\omega) = \left(\overleftrightarrow{U} + \frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\right)I(\mathbf{R},\omega), \quad (56)$$

$$\overleftrightarrow{W}^{(h)}_{iso}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\left(\overleftrightarrow{U} + \frac{1}{k^2}\boldsymbol{\nabla}\boldsymbol{\nabla}\right)I(\mathbf{R},\omega), \quad (57)$$

where the subscript *iso* refers to statistical isotropy, and

$$I(\mathbf{R},\omega) = \int_{4\pi}\mathcal{A}(\hat{u},\omega)e^{-ik\hat{u}\cdot\mathbf{R}}d\Omega. \quad (58)$$

Next, we note that the cross-spectral density tensors in Eqs. (56) and (57) are of the form of Eq. (44) if, and only if, $I(\mathbf{R},\omega)=I(R,\omega)$, i.e., if $\mathcal{A}(\hat{u},\omega)=\mathcal{A}(\omega)$, requiring that the distribution of the plane waves be uniform within the full 4π solid angle (proof is found in Appendix D). In such a case, employing Eqs. (A5)–(A8), the cross-spectral density tensors are found to be given by

$$\overleftrightarrow{W}_{iso}^{(e)}(\mathbf{R},\omega) = 4\pi\mathcal{A}(\omega)\left\{\left[j_0(kR) - \frac{j_1(kR)}{kR}\right]\overleftrightarrow{U} + j_2(kR)\hat{R}\hat{R}\right\}, \tag{59}$$

$$\overleftrightarrow{W}_{iso}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\overleftrightarrow{W}_{iso}^{(e)}(\mathbf{R},\omega), \tag{60}$$

where $j_i(kR)$, with $i=(0,1,2)$, are spherical Bessel functions of order i. The above analysis then results in the following theorem:

Theorem IV. An electromagnetic field expressed as a superposition of plane waves is statistically homogeneous and isotropic if, and only if, the plane waves are fully unpolarized and uniformly distributed within the full solid angle. Furthermore, the electric and magnetic cross-spectral density tensors are necessarily of the form given in Eqs. (59) and (60), respectively.

Note that the cross-spectral density tensors in Eqs. (59) and (60) are proportional to the imaginary part of the free-space Green tensor [21,22]. Furthermore, it should be noted that this example covers the important case of blackbody radiation for which the coefficient $4\mathcal{A}(\omega)$ is determined by Planck's law [21,22]. The theorem is, however, more generally valid; the spectrum of the radiation may differ from Planck's spectrum since thermal equilibrium is not assumed.

B. Nonuniform distribution of unpolarized, angularly uncorrelated plane waves

For a statistically homogeneous electromagnetic field consisting of a nonuniform distribution of unpolarized plane waves, the elements of the tensors in Eqs. (51) and (52) are given by

$$\mathcal{A}_{ss}(\hat{u},\omega) = \mathcal{A}_{pp}(\hat{u},\omega) = \mathcal{A}(\hat{u},\omega), \tag{61}$$

$$\mathcal{A}_{sp}(\hat{u},\omega) = \mathcal{A}_{ps}^{*}(\hat{u},\omega) = 0, \tag{62}$$

and, therefore, the angular coherence tensors are those given by Eqs. (54) and (55). Although the functional forms of the electric and magnetic cross-spectral density tensors depend on the quantity $\mathcal{A}(\hat{u},\omega)$, Eqs. (38), (39), (54), and (55) indicate that the tensors always are connected by the relation

$$\overleftrightarrow{W}_{B}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\overleftrightarrow{W}_{B}^{(e)}(\mathbf{R},\omega), \tag{63}$$

where the subscript B refers to "example B." Thus, irrespective of the angular nonuniformity, the coherence properties of the electric and magnetic fields are the same.

C. Uniform distribution of partially polarized, angularly uncorrelated plane waves

Next we consider a statistically homogeneous field composed of a uniform distribution of partially polarized plane waves, for which the tensor elements in Eqs. (51) and (52) are of the form

$$\mathcal{A}_{ss}(\hat{u},\omega) = \mathcal{A}_{pp}(\hat{u},\omega) = \mathcal{A}(\omega), \tag{64}$$

$$\mathcal{A}_{sp}(\hat{u},\omega) = \mathcal{A}_{ps}^{*}(\hat{u},\omega) = \mathcal{B}(\omega), \tag{65}$$

where $\mathcal{B}(\omega)$ is a complex quantity. The waves have the same state of partial polarization and the intensities associated with the s and p polarized components are equal. The angular coherence tensors are now given by

$$\langle \mathbf{e}^{*}(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle = \mathcal{A}(\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}) + [\mathcal{B}(\omega)\hat{s}\hat{p} + \mathcal{B}^{*}(\omega)\hat{p}\hat{s}], \tag{66}$$

$$\langle \mathbf{h}^{*}(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle = \frac{\epsilon_0}{\mu_0}\mathcal{A}(\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}) - \frac{\epsilon_0}{\mu_0}[\mathcal{B}^{*}(\omega)\hat{s}\hat{p} + \mathcal{B}(\omega)\hat{p}\hat{s}]. \tag{67}$$

We see that the first terms correspond to the angular coherence tensors, which in example A above were shown to lead to electric and magnetic cross-spectral density tensors that are proportional to the imaginary part of the free-space Green tensor. Furthermore, the terms in the brackets are transposes of each other. Thus we can write

$$\overleftrightarrow{W}_{C}^{(e)}(\mathbf{R},\omega) = \overleftrightarrow{W}_{iso}^{(e)}(\mathbf{R},\omega) + \overleftrightarrow{F}(\mathbf{R},\omega), \tag{68}$$

$$\overleftrightarrow{W}_{C}^{(h)}(\mathbf{R},\omega) = \overleftrightarrow{W}_{iso}^{(h)}(\mathbf{R},\omega) - \frac{\epsilon_0}{\mu_0}\overleftrightarrow{F}^{\mathrm{T}}(\mathbf{R},\omega), \tag{69}$$

where the subscript C refers to "example C," and

$$\overleftrightarrow{F}(\mathbf{R},\omega) = \int_{4\pi}[\mathcal{B}(\omega)\hat{s}\hat{p} + \mathcal{B}^{*}(\omega)\hat{p}\hat{s}]e^{-ik\hat{u}\cdot\mathbf{R}}d\Omega. \tag{70}$$

Eliminating the tensor $\overleftrightarrow{F}(\mathbf{R},\omega)$ from Eqs. (68) and (69), and making use of Eq. (60) together with the fact that $\overleftrightarrow{W}_{iso}^{(e)}(\mathbf{R},\omega)$ is symmetric [see Eq. (59)], one obtains

$$\overleftrightarrow{W}_{C}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\Big[2\overleftrightarrow{W}_{iso}^{(e)}(\mathbf{R},\omega) - \overleftrightarrow{W}_{C}^{(e)\mathrm{T}}(\mathbf{R},\omega)\Big]. \tag{71}$$

Hence, in general, the electric and magnetic cross-spectral density tensors in this case have different functional forms. However, the degrees of electric and magnetic coherence are equal; $\mu_e(\mathbf{R},\omega)=\mu_h(\mathbf{R},\omega)$, for all $\mathbf{R}$. This can be verified by inserting Eq. (71) into the expression of $\mu_h(\mathbf{R},\omega)$ given in Eq. (9), then using Eqs. (41) and (68), and the fact that $\overleftrightarrow{F}(-\mathbf{R},\omega)=-\overleftrightarrow{F}(\mathbf{R},\omega)$, which is proven in Appendix E.

D. Nonuniform distribution of partially polarized, angularly uncorrelated plane waves

The final example concerns a statistically homogeneous field consisting of a nonuniform distribution of partially polarized plane waves, for which

$$\mathcal{A}_{ss}(\hat{u},\omega) = \mathcal{A}_{pp}(\hat{u},\omega) = \mathcal{A}(\hat{u},\omega), \tag{72}$$

$$\mathcal{A}_{sp}(\hat{u},\omega) = \mathcal{A}^*_{ps}(\hat{u},\omega) = \mathcal{B}(\omega). \tag{73}$$

Thus the intensities of the s and p polarized components associated with each plane wave are equal, but differ for waves propagating in different directions, as do their states of partial polarization. The angular coherence tensors can readily be written as

$$\langle \mathbf{e}^*(\hat{u},\omega)\mathbf{e}(\hat{u},\omega)\rangle = \mathcal{A}(\hat{u},\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}) + [\mathcal{B}(\omega)\hat{s}\hat{p} + \mathcal{B}^*(\omega)\hat{p}\hat{s}], \tag{74}$$

$$\langle \mathbf{h}^*(\hat{u},\omega)\mathbf{h}(\hat{u},\omega)\rangle = \frac{\epsilon_0}{\mu_0}\mathcal{A}(\hat{u},\omega)(\hat{s}\hat{s}+\hat{p}\hat{p}) - \frac{\epsilon_0}{\mu_0}[\mathcal{B}^*(\omega)\hat{s}\hat{p} + \mathcal{B}(\omega)\hat{p}\hat{s}]. \tag{75}$$

Consequently, by making use of the results of the previous examples B and C, we at once obtain that

$$\overleftrightarrow{W}^{(e)}_D(\mathbf{R},\omega) = \overleftrightarrow{W}^{(e)}_B(\mathbf{R},\omega) + \overleftrightarrow{F}(\mathbf{R},\omega), \tag{76}$$

$$\overleftrightarrow{W}^{(h)}_D(\mathbf{R},\omega) = \overleftrightarrow{W}^{(h)}_B(\mathbf{R},\omega) - \frac{\epsilon_0}{\mu_0}\overleftrightarrow{F}^{\mathrm{T}}(\mathbf{R},\omega), \tag{77}$$

where the subscript D refers to "example D," the tensors $\overleftrightarrow{W}^{(f)}_B(\mathbf{R},\omega)$ are those encountered in example B, and $\overleftrightarrow{F}(\mathbf{R},\omega)$ is given by Eq. (70). Eliminating the tensor $\overleftrightarrow{F}(\mathbf{R},\omega)$ from the pair of equations above, employing Eq. (63), and noting that $\overleftrightarrow{W}^{(e)}_B(\mathbf{R},\omega)$ is symmetric (since the corresponding angular coherence tensor is so), we get

$$\overleftrightarrow{W}^{(h)}_D(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\Big[2\overleftrightarrow{W}^{(e)}_B(\mathbf{R},\omega) - \overleftrightarrow{W}^{(e)\mathrm{T}}_D(\mathbf{R},\omega)\Big]. \tag{78}$$

We therefore find that, in general, the electric and magnetic cross-spectral density tensors have different functional forms. In addition, the degrees of the electric and magnetic coherence are generally different, i.e., $\mu_e(\mathbf{R},\omega) \neq \mu_h(\mathbf{R},\omega)$. One can trace the origin of this difference in the present case to the angular dependence of the parameter $\mathcal{A}(\hat{u},\omega)$. Unless $\mathcal{A}(\hat{u},\omega)=\mathcal{A}(-\hat{u},\omega)$, the two degrees of coherence acquire different values.

VII. SUMMARY AND CONCLUSIONS

In this work, we introduced quantities that are fundamental for characterizing the electric and magnetic spectral (spatial) coherence in general, three-dimensional, random electromagnetic fields. We employed them for free electromagnetic fields, i.e., fields that consist only of propagating plane waves, and proved that in such fields two spatial Fourier components of the electric field are as coherent as the corresponding magnetic components. Furthermore, in any free field, the effective degrees of electric and magnetic coherence, evaluated over the whole space, are equal to each other at all frequencies. We also derived a compact, general relation that connects the electric and magnetic cross-spectral density tensors in the case of a statistically homogeneous field. When the electromagnetic field is statistically isotropic in addition to being homogeneous, the two cross-spectral density tensors differ only by a constant factor of ϵ_0/μ_0, indicating that the behaviors of the electric and magnetic coherence in such fields are identical. We also proved that a superposition of plane waves leads to a statistically homogeneous and isotropic electromagnetic field if, and only if, the waves are fully unpolarized and uniformly distributed within the full 4π solid angle. The results and examples establish a useful basis for a more complete treatment of the electromagnetic theory of optical coherence in general, three-dimensional fields, for which the scalar theory is not applicable.

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APPENDIX A: USEFUL FORMULAS

An integral involving exponential functions (Sec. 15.3 in Ref. [23])

$$\frac{1}{(2\pi)^3}\int e^{i(\mathbf{k}_1-\mathbf{k}_2)\cdot\mathbf{r}}d^3r = \delta(\mathbf{k}_1-\mathbf{k}_2), \tag{A1}$$

where the integration is performed over the whole space.

A vector identity [17]

$$[\mathbf{a}\times\mathbf{b}]\cdot[\mathbf{c}\times\mathbf{d}] = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}). \tag{A2}$$

Parseval's theorem for functions of several variables (p. 67 in Ref. [24]): Assume that the functions $g(\mathbf{r}_1,\mathbf{r}_2,\omega)$ and $G(\mathbf{k}_1,\mathbf{k}_2,\omega)$ are square integrable and constitute a Fourier transform pair, i.e.,

$$G(\mathbf{k}_1,\mathbf{k}_2,\omega) = \frac{1}{(2\pi)^6}\int\int g(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}d^3r_1d^3r_2. \tag{A3}$$

Then

$$\frac{1}{(2\pi)^6}\int\int |g(\mathbf{r}_1,\mathbf{r}_2,\omega)|^2d^3r_1d^3r_2 = \int\int |G(\mathbf{k}_1,\mathbf{k}_2,\omega)|^2d^3k_1d^3k_2. \tag{A4}$$

A relation involving derivatives of a spherically symmetric, twice differentiable function $g(R)$

$$\nabla\nabla g(R) = \frac{1}{R}g'(R)\overleftrightarrow{U} + \left[g''(R) - \frac{1}{R}g'(R)\right]\hat{R}\hat{R}, \tag{A5}$$

where the primes denote derivatives with respect to the spatial variable R.

An integral over the full solid angle [Eq. (12) in Ref. [22]]

$$\int_{4\pi}\exp(-ik\hat{u}\cdot\mathbf{R})d\Omega = 4\pi\frac{\sin kR}{kR} = 4\pi j_0(kR). \tag{A6}$$

Relations for spherical Bessel functions (Eqs. 11.161–11.163 in Ref. [23])

$$j_{n-1}(x) + j_{n+1}(x) = \frac{2n+1}{x} j_n(x), \tag{A7}$$

$$n j_{n-1}(x) - (n+1) j_{n+1}(x) = (2n+1) j_n'(x). \tag{A8}$$

APPENDIX B: PROOF OF EQ. (43)

An element of the magnetic cross-spectral density tensor can be developed, with the help of Eq. (17), as follows:

$$W_{ij}^{(h)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = \langle H_i^*(\mathbf{r}_1,\omega) H_j(\mathbf{r}_2,\omega)\rangle = \frac{1}{\omega^2\mu_0^2}\epsilon_{imn}\epsilon_{jpq}\partial_m^1\partial_p^2\langle E_n^*(\mathbf{r}_1,\omega)E_q(\mathbf{r}_2,\omega)\rangle, \tag{B1}$$

where ϵ_{imn} is the Levi-Civita tensor, summation convention is used, and ∂_m^1 and ∂_p^2 act on the coordinates of $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. In the case of a statistically homogeneous field, the cross-spectral density tensors are functions of $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$ only, and consequently, we can write

$$\begin{aligned} W_{ij}^{(h)}(\mathbf{R},\omega) &= -\frac{1}{\omega^2\mu_0^2}\epsilon_{imn}\epsilon_{jpq}\partial_m\partial_p W_{nq}^{(e)}(\mathbf{R},\omega) \\ &= -\frac{1}{\omega^2\mu_0^2}[\delta_{nq}(\delta_{ij}\delta_{mp}-\delta_{ip}\delta_{mj}) \\ &\quad + \delta_{np}(\delta_{iq}\delta_{mj}-\delta_{ij}\delta_{mq}) \\ &\quad + \delta_{nj}(\delta_{ip}\delta_{mq}-\delta_{iq}\delta_{mp})]\partial_m\partial_p W_{nq}^{(e)}(\mathbf{R},\omega) \\ &= -\frac{1}{\omega^2\mu_0^2}\Big\{(\nabla^2\delta_{ij}-\partial_i\partial_j)\mathrm{tr}\Big[\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)\Big] \\ &\quad - \nabla^2 W_{ji}^{(e)}(\mathbf{R},\omega)\Big\}. \end{aligned} \tag{B2}$$

The cross-spectral density tensors satisfy the following Helmholtz equations [1]

$$\nabla_p^2 W_{ij}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega) + k^2 W_{ij}^{(f)}(\mathbf{r}_1,\mathbf{r}_2,\omega) = 0, \quad f=(e,h), \tag{B3}$$

where ∇_p, with $p=(1,2)$, operates on the coordinates $\mathbf{r}_p$. For a statistically homogeneous field, the above formulas can be written as

$$\nabla^2 W_{ij}^{(f)}(\mathbf{R},\omega) + k^2 W_{ij}^{(f)}(\mathbf{R},\omega) = 0, \quad f=(e,h). \tag{B4}$$

Using this equation in Eq. (B2), one obtains

$$W_{ij}^{(h)}(\mathbf{R},\omega) = \frac{\epsilon_0}{\mu_0}\left\{\left[\delta_{ij}+\frac{1}{k^2}\partial_i\partial_j\right]\mathrm{tr}\Big[\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)\Big] - W_{ji}^{(e)}(\mathbf{R},\omega)\right\}, \tag{B5}$$

which, when written in the dyadic notation, is the same as Eq. (43).

APPENDIX C: PROOF OF EQS. (46) and (47)

Substituting $\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)$ from Eq. (44) into Eq. (43), and making use of Eq. (A5), one can connect the electric coefficients $A_e(R,\omega)$ and $B_e(R,\omega)$ with the magnetic ones, $A_h(R,\omega)$ and $B_h(R,\omega)$. The calculation results in

$$A_h(R,\omega) = \frac{\epsilon_0}{\mu_0}\left\{2A_e(R,\omega) + B_e(R,\omega) + \frac{1}{k^2R}[3A_e'(R,\omega) + B_e'(R,\omega)]\right\}, \tag{C1}$$

$$B_h(R,\omega) = \frac{\epsilon_0}{\mu_0}\left\{-B_e(R,\omega) - \frac{1}{k^2R}[3A_e'(R,\omega) + B_e'(R,\omega)] + \frac{1}{k^2}[3A_e''(R,\omega) + B_e''(R,\omega)]\right\}, \tag{C2}$$

where the primes stand for derivatives with respect to the spatial variable R. Inserting the tensor $\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)$ into the divergence equation in Eq. (45) yields the relation

$$\left[A_e'(R,\omega) + B_e'(R,\omega) + \frac{2B_e(R,\omega)}{R}\right]\hat{R} = 0. \tag{C3}$$

Furthermore, substituting $\overleftrightarrow{W}^{(e)}(\mathbf{R},\omega)$ into the Helmholtz equation, Eq. (B4), gives

$$\left[A_e''(R,\omega) + \frac{2A_e'(R,\omega)}{R} + \frac{2B_e(R,\omega)}{R^2} + k^2A_e(R,\omega)\right]\overleftrightarrow{U} + \left[B_e''(R,\omega) + \frac{2B_e'(R,\omega)}{R} - \frac{6B_e(R,\omega)}{R^2} + k^2B_e(R,\omega)\right]\hat{R}\hat{R} = 0. \tag{C4}$$

In order to satisfy the latter two equations for all values of R, the coefficient of the vector $\hat{R}$ in Eq. (C3) and the coefficients of the tensors $\overleftrightarrow{U}$ and $\hat{R}\hat{R}$ in Eq. (C4), have to be identically zero. Thus, we get three equations to connect the parameters $A_e(R,\omega)$ and $B_e(R,\omega)$ and their derivatives. A fourth equation is obtained by taking the derivative of the coefficient in Eq. (C3). These four equations imply

$$A_e''(R,\omega) = \left(k^2 - \frac{4}{R^2}\right)B_e(R,\omega), \tag{C5}$$

$$A_e'(R,\omega) = \frac{1}{R}B_e(R,\omega) - \frac{k^2R}{2}[A_e(R,\omega) + B_e(R,\omega)], \tag{C6}$$

$$B_e''(R,\omega) = \frac{12}{R^2}B_e(R,\omega) - k^2[A_e(R,\omega) + 2B_e(R,\omega)], \tag{C7}$$

$$B_e'(R,\omega) = -\frac{3}{R}B_e(R,\omega) + \frac{k^2R}{2}[A_e(R,\omega) + B_e(R,\omega)]. \tag{C8}$$

Use of these formulas in Eqs. (C1) and (C2) verifies Eqs. (46) and (47).

APPENDIX D: PROOF THAT IN EQ. (58) $I(\mathbf{R},\omega)=I(R,\omega)$ IF AND ONLY IF $\mathcal{A}(\hat{u},\omega)=\mathcal{A}(\omega)$

We make use of the fact that $I(\mathbf{R},\omega)=I(R,\omega)$ if, and only if, the value of $I(\mathbf{R},\omega)$ is invariant under rotations. Thus, we introduce a general rotation, described by a real, orthogonal matrix $\overleftrightarrow{T}$, under which the vector $\mathbf{R}$ changes to $\mathbf{R}'=\overleftrightarrow{T}\cdot\mathbf{R}$. Substitution of $\mathbf{R}'$ into Eq. (58) results in

$$I(\mathbf{R}',\omega)=\int_{4\pi}\mathcal{A}(\hat{u},\omega)e^{-ik\hat{u}\cdot\overleftrightarrow{T}\cdot\mathbf{R}}d\Omega = \int_{4\pi}\mathcal{A}(\hat{u}'\cdot\overleftrightarrow{T}^{\mathrm{T}},\omega)e^{-ik\hat{u}'\cdot\mathbf{R}}d\Omega', \qquad \text{(D1)}$$

where $\hat{u}'=\hat{u}\cdot\overleftrightarrow{T}$. Dropping the prime in the last form of this equation and comparing it to Eq. (58) implies that $I(\mathbf{R},\omega)=I(\mathbf{R}',\omega)$ for any rotation if, and only if, $\mathcal{A}(\hat{u}\cdot\overleftrightarrow{T}^{\mathrm{T}},\omega)=\mathcal{A}(\hat{u},\omega)$ for all $\overleftrightarrow{T}$. This is satisfied only if $\mathcal{A}(\hat{u},\omega)=\mathcal{A}(\omega)$.

APPENDIX E: PROOF THAT TENSOR $\overleftrightarrow{F}$ OF EQ. (70) HAS THE PROPERTY $\overleftrightarrow{F}(-\mathbf{R},\omega)=-\overleftrightarrow{F}(\mathbf{R},\omega)$

Since the differential element of the solid angle is $d\Omega=\sin\theta\, d\varphi\, d\theta$, where φ and θ are the azimuthal and polar angles in spherical polar coordinates, we can write

$$\overleftrightarrow{F}(-\mathbf{R},\omega)=\int_0^{\pi}\int_0^{2\pi}[\mathcal{B}(\omega)\hat{s}\hat{p}+\mathcal{B}^*(\omega)\hat{p}\hat{s}]e^{ik\hat{u}\cdot\mathbf{R}}\sin\theta\, d\varphi\, d\theta. \qquad \text{(E1)}$$

Next we transform the integration variable $\hat{u}$ into $-\hat{u}'$, which implies the following changes: $\hat{s}\to-\hat{s}'$, $\hat{p}\to\hat{p}'$, $\varphi\to\varphi'+\pi$, $\theta\to\pi-\theta'$, $d\theta\to-d\theta'$, and $d\varphi\to d\varphi'$. This yields

$$\overleftrightarrow{F}(-\mathbf{R},\omega)=\int_{\pi}^{0}\int_{\pi}^{3\pi}[\mathcal{B}(\omega)(-\hat{s}')\hat{p}'+\mathcal{B}^*(\omega)\hat{p}'(-\hat{s}')] \times e^{-ik\hat{u}'\cdot\mathbf{R}}\sin(\pi-\theta')d\varphi'(-d\theta'). \qquad \text{(E2)}$$

Noting that $\sin(\pi-\theta')=\sin\theta'$ and that the integrand is 2π periodic with respect to φ', we obtain $\overleftrightarrow{F}(-\mathbf{R},\omega)=-\overleftrightarrow{F}(\mathbf{R},\omega)$.

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