Department of Mathematics and Systems Analysis

Model-based structural damage identification using vibration measurements

Antti Huhtala

Aalto University

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Abstract

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Abstract

In structural health monitoring (SHM), a structure is continuously monitored with a set of embedded sensors. Damage identification is the part of a SHM system, in which the damage state of the structure is determined from obtained measurements. More specifically, the presence of damage is detected, and its location and severity are estimated. Even if the measured quantities are known to be sensitive to damage, the reconstruction of damage from the measurement is generally not well-posed, since significantly different damage states may still produce similar measurement results. Damage identification is thus an inverse problem.

In this thesis, a model-based approach using vibration measurements is taken. The vibration of the structure is measured using several sensors, which can be for instance strain gauges, gyroscopes or accelerometers. A model of the structure, including a model of how the damage affects the structure and a model of the measurement sensors, is then used to simulate the measurements. Damage identification is achieved through finding a plausible damage state of the model which reproduces the actual measurements as simulated measurements.

Most of the work in this thesis is on damage identification using Bayesian inference, while taking the measurements as mode frequencies and mode shapes of the structure. A multivariate normal distributed noise term is included in the measurement model, which allows taking into account the measurement error and also a large part of the model error. The knowledge of plausible damage states is described using a prior distribution, which is merged with the information obtained through measurement using Bayesian inference.

Other approaches to the damage identification problem are also discussed in the work. The Kalman filter can be used for damage identification by augmenting the state vector of the vibrating structure with parameters of the damage state. The state estimate then gives the damage parameters along with the other state components. While this approach is more sensitive to model errors, it could be used for real-time damage identification for a continuous assessment of the damage state.

The method of sigma algebras on contour maps (SACOM) uses the same noise distribution as the Bayesian approach, and like the Bayesian approach also gives a probability distribution for the damage state. However, in this approach the distribution is obtained by mapping the noise distribution through the set-valued inverse of the structure model.

Finally, a brief discussion is given on the possibility of formulating the damage identification problem as an inverse source problem. As the resulting problem is linear, it gives greater opportunity for a thorough mathematical analysis.

Keywords structural health monitoring, damage identification, inverse problems, Bayesian inference, finite element method

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Tiivistelmä

Rakenteiden kunnonvalvonnassa rakennetta tarkkaillaan jatkuvasti siihen upotetuin anturein. Vauriontunnistus on rakenteiden kunnonvalvonnan osa, jossa rakenteen vauriotila päätellään mittauksista. Tarkemmin sanottuna vaurion läsnäolo tunnistetaan, ja sen paikka ja suuruus arvioidaan. Vaikka mitatun suureen tiedettäisiin olevan herkkä vauriolle, vaurion rekonstruointi mittauksesta ei yleisesti ole hyvin määrätty, koska moni merkittävästi toisistaan poikkeava vauriotila voi tuottaa hyvin samanlaisia mittaustuloksia. Vauriontunnistus on siis käänteisongelma.

Tässä väitöstyössä otetaan mallipohjainen lähestymistapa käyttäen värähtelymittauksia. Rakenteen värähtelyä mitataan useilla antureilla, jotka voivat olla esimerkiksi venymäliuskoja, gyroskooppeja tai kiihtyvyysantureita. Rakenteen mallilla, joka sisältää mallit vaurion vaikutuksista sekä anturien toiminnasta, tuotetaan simuloituja mittauksia. Vauriontunnistus tehdään etsimällä uskottava vauriotila mallille siten, että sen tuottamat simuloidut mittaukset vastaavat todellisia mittauksia.

Suurin osa työstä tässä väitöskirjassa käsittelee vauriontunnistusta käyttäen bayesiläistä päättelyä, kun mittaukset ovat rakenteen mooditaajuuksia ja -muotoja. Mittausmalliin lisätään monimuuttuja-normaalijakautunut kohinatermi, joka mahdollistaa mittavirheen sekä suuren osan mallivirhettä huomioon ottamisen. Tieto uskottavista vauriotiloista kirjoitetaan priorijakaumana, joka yhdistetään mittauksessa saatuun uuteen tietoon käyttäen bayesiläistä päättelyä.

Työssä tarkastellaan myös muita lähestymistapoja ongelmaan. Kalman-suodinta voi käyttää vauriontunnistukseen lisäämällä värähtelevän rakenteen tilavektoriin vauriotilan muuttujat. Vauriotilan saa tällöin selville tila-arviosta muiden tilakomponenttien ohessa. Vaikka tämä lähestymistapa on herkempi mallivirheille, sitä voisi käyttää reaaliaikaisessa vauriontunnistuksessa jatkuvasti toimivana arviona vauriotilasta.

Sigma-algebrat tasa-arvokuvauksilla -menetelmä (SACOM) käyttää samaa kohinajakaumaa kuin bayesiläinen lähestymistapa, ja kuten bayesiläinen lähestymistapa se myös antaa todennäköisyysjakauman vauriotilalle. Tässä lähestysmistavassa kuitenkin jakauma saadaan kuvaamalla kohinajakauma rakenteen mallin joukkoarvoisen käänteiskuvauksen läpi.

Lopulta esitetään lyhyt tarkastelu mahdollisuudesta esittää vauriontunnistusongelma käänteislähdeongelmana. Koska tällä tavalla muodostuva ongelma on lineaarinen, sen matemaattiseen analyysiin on parempi mahdollisuus.

Avainsanat rakenteiden kunnonvalvonta, vauriontunnistus, käänteisongelmat, bayesiläinen päättely, elementtimenetelmä

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Preface

In the summer of 2008, when I started working on my master's thesis at the department, little did I know what lay ahead of me. I was hired by prof. Rolf Stenberg to work in the Intelligent Structural Health Monitoring System (ISMO) project, which the Department of Mathematics and Systems Analysis was involved with, and which my thesis advisor prof. Sven Bossuyt was already a part of. This project turned out to be an interesting multidisciplinary effort, which involved many of my interests, such as electronics, mechanical engineering and mathematics to name a few. Sven and I were tasked with the problem of damage identification, given acceleration data obtained from a wireless sensor network. This was the first time I had heard of inverse problems, so there were many new things to learn.

Working with the ISMO project was a great time, for which I thank everyone involved, especially Dr. Jyrki Kullaa, who was the leader of the project. After ISMO ended, I still continued working on the damage identification problem and other problems related to it, now as a graduate student. I had a lot of ideas for new ways to approach the problems, though most of them turned out to be infeasible. So, one failed idea after another, I finally got to the point where we are today. Sven helped me a great deal by allowing me to reflect upon my ideas by letting me explain them to him, but also by providing his opinions and ideas at key points. Over the years, we have had many discussions on just about everything, which have definitely shaped the way I think of many things. A big thank you goes also to Rolf for giving me the opportunity to do this project, as well as his guidance along the way.

I thank prof. Ville Kolehmainen for taking the time from his many duties to act as my opponent. I also thank the pre-examiners prof. Dr. Herbert Egger and prof. Tuomo Kauranne for their feedback on my thesis.

At times during this project I was unsure of myself, but support from colleagues, especially Dr. Mika Juntunen and prof. Antti Hannukainen, helped me continue my work. My colleagues at the department also made working there a fun experience. This holds especially true for the members of Mekaniikan pallogrilliseura, with whom the fun also continued outside of work. Other groups of friends, such as the canonical helluposse provided a further break from routines. I thank you all for this.

I thank my parents for all their support both before and during this project. I literally wouldn't be here without them. Finally, I thank my wife Jenni and my son Leo for everything else.

Espoo, February 17, 2015,

Antti Huhtala

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Antti Huhtala and Sven Bossuyt. A Bayesian approach to vibration based structural health monitoring with experimental verification. *Journal of Structural Mechanics*, 44(4):330–344, 2011.

II Antti Huhtala and Sven Bossuyt. Damage localization from vibration data using hierarchical a priori assumptions. *Journal of Physics: Conference Series*, 181:012088, 2009.

III Troy Butler, Antti Huhtala and Mika Juntunen. Quantifying uncertainty in material damage from vibrational data. *Journal of Computational Physics*, 283:414–435, 2015.

IV Antti Huhtala, Sven Bossuyt and Antti Hannukainen. A priori error estimate of the finite element solution to a Poisson inverse source problem. *Inverse Problems*, 30:085007, 2014.

Author's Contribution

Publication I: "A Bayesian approach to vibration based structural health monitoring with experimental verification"

The implementation of the numerical methods, the processing of the measured data and the application of the methods to the data are all due to the author. The author also did the majority of the writing, and was involved in the experiment used to obtain the measurement data.

Publication II: "Damage localization from vibration data using hierarchical a priori assumptions"

The author implemented the numerical methods as well as performed the numerical experiments presented in the article. The author was also responsible for most of the writing.

Publication III: "Quantifying uncertainty in material damage from vibrational data"

The author implemented most of the numerical methods, and was responsible for performing all of the computations. This includes generating the simulated measurement data, processing the measured data and applying the presented methods to the data. The author also contributed to a large part of the writing, and was involved in the experiment used to obtain the measurement data.

Publication IV: "A priori error estimate of the finite element solution to a Poisson inverse source problem"

Most of the analysis presented in the article is due to the author, as well as the majority of the writing. The author is also responsible for implementing all of the numerical methods and performing the numerical experiments presented in the article.

1. Introduction

Structural health monitoring has been the subject of active study for the past 35 years [13, 10, 14]. In the past, monitoring has been limited to only special cases, such as research purposes or critical applications where cost has not been an issue. Recently, suitable sensors such as accelerometers and gyroscopes and their accompanying electronics have become extremely cheap. Cost is thus no longer an issue for embedding sensors in structures. However, collecting data from the sensors is not useful by itself, but needs to be accompanied with some way to analyze the collected data as well. During the fairly long history of structural health monitoring, a large number of damage detection and damage identification methods have been developed for different situations. Some methods use measurements of the static response of a structure [21, 19], while others make use of the vibration of the structure. This work is about damage identification using vibration measurements.

In vibration based structural health monitoring, suitable vibration related quantities of interest are extracted from the measured vibration data. These quantities need to be sensitive to damage and insensitive to other effects. The presence of damage is then detected from an observed change in the quantities of interest. However, detecting the location and severity of the damage is more involved. One approach is to assume that the damage only causes a local change in the quantities of interest. Using multiple sensors around the structure, one can then localize the damage near the sensor which measured the largest change. Severity can also be estimated based on the amount of observed change [17, 18]. Another approach to the damage localization problem is to employ a model of the structure, including a model of how the damage affects the structure and a model of the measurement process. By carrying out simulated measurements on the structure model, one could then find a damage state such that the simulated measurements are similar to the actual measurements. If the models are good enough, this approach would directly give an estimate of the damage location as well as severity. The model based approach is what is discussed in this thesis.

The raw measurements are typically time domain signals from measurement sensors such as accelerometers, gyroscopes or strain gauges. While these measurements can be directly used as the quantities of interest, it must be noted that such time domain data are generally also dependent on the load conditions. In this case, both the load conditions and damage parameters need to be estimated in the damage identification problem. Often the measurements are preprocessed to extract the modes of vibration, which depend only on the properties of the structure and not on the load conditions. The mode data are then used as the quantities of interest in the damage identification problem.

This thesis takes the approach of considering the damage identification problem as a statistical inverse problem. In the following, an overview of the methods is given.

2. Forward problem

The forward problem is to solve the values of the quantities of interest when the damage state of the structure is known. It is the inverse of this problem, which is the actual damage identification problem.

2.1 Structure model

For our discussion, we assume that the dynamics of the structure are discretized in space using the finite element method. Ignoring damping, this results in the following differential equation for the degrees of freedom $\boldsymbol{u}(t) \in \mathbb{R}^{n_d}$

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = \boldsymbol{f}(t), \tag{2.1}$$

in which $\boldsymbol{M} \in \mathbb{R}^{n_d \times n_d}$ is the mass matrix, $\boldsymbol{K} \in \mathbb{R}^{n_d \times n_d}$ is the stiffness matrix and $\boldsymbol{f}(t) \in \mathbb{R}^{n_d}$ is the (time-dependent) load. For statically determinate systems, the mass and stiffness matrices are symmetric and positive definite.

2.2 Damage model

Many types of damage do not affect the mass of a structure in a significant way, thus in many models the mass matrix $\boldsymbol{M}$ is assumed to remain constant under damage [26, 23, 2]. What is changed, however, is the stiffness of the structure. That is, the differential equation representing the damaged structure is

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{K}(\boldsymbol{d})\boldsymbol{u}(t) = \boldsymbol{f}(t), \tag{2.2}$$

where $\boldsymbol{K}(\boldsymbol{d})$ is the stiffness matrix under damage state $\boldsymbol{d}$.

A very simple damage model is to assume that damage causes a piecewise constant reduction of stiffness in the structure. As we are using the finite element method, we may parametrize the loss of stiffness element-wise. Suppose the undamaged stiffness matrix is given as

$$\boldsymbol{K}(0) = \sum_{i=1}^{n_e} \boldsymbol{K}_i, \tag{2.3}$$

where $\boldsymbol{K}_i$ are the element-wise stiffness matrix contributions and n_e is the number of elements. The damage model then takes the form

$$\boldsymbol{K}(\boldsymbol{d}) = \sum_{i=1}^{n_e} (1 - d_i)\boldsymbol{K}_i. \tag{2.4}$$

The components d_i of the damage state $\boldsymbol{d}$ thus describe the proportional loss of stiffness in a particular element, with $d_i = 0$ meaning no damage and $d_i = 1$ corresponding to a complete loss of stiffness. This idea can be extended further to obtain more elaborate damage models [26, 25].

2.3 Modes of vibration

Consider the generalized eigenvalue problem

$$\boldsymbol{K}\boldsymbol{X} = \boldsymbol{M}\boldsymbol{X}\boldsymbol{\Lambda}, \tag{2.5}$$

in which $\boldsymbol{X}$ is a matrix such that the columns are the eigenvectors, denoted by $\boldsymbol{x}_i$, and $\boldsymbol{\Lambda}$ is a diagonal matrix of the corresponding eigenvalues, denoted by λ_i. As the stiffness matrix $\boldsymbol{K}$ is symmetric and the mass matrix $\boldsymbol{M}$ is symmetric and positive definite, it follows that the decomposition (2.5) exists and that the eigenvectors can be normalized so that

$$\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{X} = \boldsymbol{I}. \tag{2.6}$$

Then, taking the change of variables $\boldsymbol{u}(t) = \boldsymbol{X}\boldsymbol{\eta}(t)$ in (2.2), and multiplying with $\boldsymbol{X}^T$ from the left gives

$$\boldsymbol{X}^T\boldsymbol{M}\boldsymbol{X}\ddot{\boldsymbol{\eta}}(t) + \boldsymbol{X}^T\boldsymbol{K}\boldsymbol{X}\boldsymbol{\eta}(t) = \boldsymbol{X}^T\boldsymbol{f}(t), \tag{2.7}$$

which is simplified to

$$\ddot{\boldsymbol{\eta}}(t) + \boldsymbol{\Lambda}\boldsymbol{\eta}(t) = \boldsymbol{X}^T\boldsymbol{f}(t). \tag{2.8}$$

Denoting $\omega_i^2 = \lambda_i$, each of the equations has the form

$$\ddot{\eta}_i(t) + \omega_i^2\eta_i(t) = \boldsymbol{x}_i^T\boldsymbol{f}(t), \tag{2.9}$$

where η_i are the components of $\boldsymbol{\eta}$. Each equation thus describes an independent harmonic oscillator with oscillation frequency ω_i driven by the external load. These independent oscillating components are called the modes of the structure. The solution to the original equation is given by

$$\boldsymbol{u}(t) = \boldsymbol{X}\boldsymbol{\eta}(t) = \sum_i \boldsymbol{x}_i \eta_i(t), \tag{2.10}$$

where each term in the sum is the response of a single mode. The shape of the mode is thus defined by the eigenvector $\boldsymbol{x}_i$ while the frequency is related to the eigenvalue λ_i.

2.4 Damping

Free oscillations of a real structure will decay over time. Without damping our model will not have this behavior. In a linear structure model, the damping appears as an additional term

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{C}\dot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}, \tag{2.11}$$

where $\boldsymbol{C} \in \mathbb{R}^{n_d \times n_d}$ is called the damping matrix.

A very useful simplifying assumption is that of modal damping, in which we assume that

$$\boldsymbol{X}^T \boldsymbol{C} \boldsymbol{X} = \boldsymbol{D}, \tag{2.12}$$

where $\boldsymbol{X}$ is the matrix of eigenvectors as defined in the previous section and $\boldsymbol{D}$ is a diagonal matrix with diagonal entries D_i.

Taking the change of variables $\boldsymbol{u} = \boldsymbol{X}\boldsymbol{\eta}$ now in (2.11), and multiplying with $\boldsymbol{X}^T$ from the left gives

$$\ddot{\boldsymbol{\eta}} + \boldsymbol{D}\dot{\boldsymbol{\eta}} + \boldsymbol{\Lambda}\boldsymbol{\eta} = \boldsymbol{X}^T \boldsymbol{f}, \tag{2.13}$$

which now describes a set of independent damped harmonic oscillators driven by the external load. Hence, with the modal damping assumption the mode shapes of the structure are unaffected by the damping, and are still defined by the eigenvectors $\boldsymbol{x}_i$. Although this is not true in the general case, i.e. when $\boldsymbol{D}$ is not diagonal, it is still a reasonably good approximation if the damping is small [4].

The damping will cause a slight change in the oscillation frequency of the modes. Denoting $D_i = 2\zeta_i \omega_i$, each independent harmonic oscillator of equation (2.13) can be written in the form

$$\ddot{\eta}_i + 2\zeta_i \omega_i \dot{\eta}_i + \omega_i^2 \eta_i = \boldsymbol{x}_i^T \boldsymbol{f}. \tag{2.14}$$

Assuming sub-critical damping, i.e. $\zeta_i < 1$, the oscillation frequency of the damped system is

$$\omega_{i,d} = \omega_i \sqrt{1 - \zeta_i^2}. \tag{2.15}$$

Since the damping ratio ζ for metallic structures is in the order of 0.02 to 0.05 [4], the resulting frequency shift is thus in the order of 0.1%, which is quite insignificant compared to measurement and model errors. In many cases an undamped model of the structure is thus sufficient for describing the modes of vibration.

2.5 Measurement sensor model

Many types of measurement sensors can be used to monitor the structure. Some sensors of interest are for instance accelerometers, strain gauges and gyroscopes. To accommodate these types of sensors, we assume that each measurement sensor can be modeled as a linear functional of either $\boldsymbol{u}(t)$ or one of its time derivatives. However, any time derivative of $\boldsymbol{u}(t)$ still describes essentially the same dynamics as $\boldsymbol{u}(t)$ itself. This follows simply from differentiating (2.11) in time, and noticing that the coefficient matrices are independent of time. It is thus enough to consider the measurement as

$$\boldsymbol{m}(t) = \boldsymbol{H}\boldsymbol{u}(t), \tag{2.16}$$

where $\boldsymbol{H} \in \mathbb{R}^{n_m \times n_d}$ and n_m is the number of measurement channels.

Taking a change of variables to the modal basis with $\boldsymbol{u} = \boldsymbol{X}\boldsymbol{\eta}$ gives us

$$\boldsymbol{m}(t) = \boldsymbol{H}\boldsymbol{X}\boldsymbol{\eta}(t) = \sum_i \boldsymbol{H}\boldsymbol{x}_i \eta_i(t), \tag{2.17}$$

where each term in the sum is the response of a single mode. The shapes of the modes as seen by the measurement, called the observable mode shapes, are thus defined by the vectors $\boldsymbol{H}\boldsymbol{x}_i$.

2.6 Modal measurement

The vibration modes of a structure are not dependent on the applied initial conditions or loading conditions. The mode frequencies as well as the mode shapes also exhibit a change due to damage. This makes them suitable for use as the quantities of interest in the damage identification problem. However, deducing modal information from time domain measurements is somewhat problematic, especially since the time domain measurements are dependent on the loading conditions which often are unknown.

Assuming that the measured data follows the measurement model of

(2.17), and that the external load is a stochastic process of a suitable type, then there exist methods for extracting modal information from the measurement data [1, 6, 20, 5, 24]. With these methods, modal information is generally obtained for a few modes of the highest amplitude. The obtained modal information contains the mode frequencies, damping coefficients and observable mode shapes.

Suppose we have obtained modal information for p modes, for which let $\{\nu_i\}_{i=1}^p$ and $\{\boldsymbol{y}_i\}_{i=1}^p$ denote the identified mode frequencies and the identified observable mode shapes, respectively. The data is then collected as the modal measurement vector

$$\boldsymbol{m}_\omega = \begin{pmatrix} \nu_1 \\ \boldsymbol{y}_1 \\ \vdots \\ \nu_p \\ \boldsymbol{y}_p \end{pmatrix}. \tag{2.18}$$

2.7 Forward operator

In order to use a measurement as defined in (2.18) we need a forward operator, which maps the damage state onto this measurement space. The modes identified in the measurement need to be paired up with modes obtained from the structure model. An automated way to do this is to for example compare the modes using the modal assurance criterion [15]. Let k_i be the index of structure model's mode which has been paired up with the ith identified mode. We may then define the forward operator $\boldsymbol{f}(\boldsymbol{d})$ as

$$\boldsymbol{f}(\boldsymbol{d}) = \begin{pmatrix} \omega_{k_1} \\ \boldsymbol{H}\boldsymbol{x}_{k_1} \\ \vdots \\ \omega_{k_p} \\ \boldsymbol{H}\boldsymbol{x}_{k_p} \end{pmatrix}. \tag{2.19}$$

The forward operator is a mapping from the damage parameters to the modal frequencies and the observable mode shapes, which correspond to the components of the modal measurement in (2.18).

3. Damage identification using Bayesian inference

3.1 Bayesian inference

To identify the damage, we seek a damage state of the model such that the measurements obtained from the model match the measurements obtained from the actual structure. In other words we seek the inverse of the forward operator $\boldsymbol{f}(\boldsymbol{d})$. In general this is an ill-posed problem, as the solution does not necessarily exist, is not necessarily unique and is not continuously dependent on the measurement data. The problem is an example of an inverse problem.

One way to obtain a meaningful solution to such a problem is to take the Bayesian approach [22, 16], in which we regard the measurement and the damage state as random variables $\boldsymbol{\mathcal{M}}$ and $\boldsymbol{\mathcal{D}}$ respectively.

We assume that the measurement is in the range of the forward operator corrupted by additive noise $\boldsymbol{\mathcal{E}}$, so that

$$\boldsymbol{\mathcal{M}} = \boldsymbol{f}(\boldsymbol{\mathcal{D}}) + \boldsymbol{\mathcal{E}}. \tag{3.1}$$

In addition, we assume that $\boldsymbol{\mathcal{E}}$ is independent of $\boldsymbol{\mathcal{D}}$ and that we know the probability distribution of $\boldsymbol{\mathcal{E}}$. It then follows, that

$$p(\boldsymbol{\mathcal{M}} = \boldsymbol{m} | \boldsymbol{\mathcal{D}} = \boldsymbol{d}) = p(\boldsymbol{\mathcal{E}} = \boldsymbol{m} - \boldsymbol{f}(\boldsymbol{d})). \tag{3.2}$$

That is, under the condition $\boldsymbol{\mathcal{D}} = \boldsymbol{d}$ the distribution of $\boldsymbol{\mathcal{M}}$ is the same as the distribution of $\boldsymbol{\mathcal{E}}$, but shifted with the constant $\boldsymbol{f}(\boldsymbol{d})$. This distribution is called the likelihood of observing $\boldsymbol{\mathcal{M}} = \boldsymbol{m}$ when $\boldsymbol{\mathcal{D}} = \boldsymbol{d}$.

A simple assumption for the distribution of $\boldsymbol{\mathcal{E}}$ is the multivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix Σ. The likelihood will then be

$$p(\boldsymbol{\mathcal{E}} = \boldsymbol{m} - \boldsymbol{f}(\boldsymbol{d})) = \exp\left(-\frac{1}{2}\|\boldsymbol{S}(\boldsymbol{m} - \boldsymbol{f}(\boldsymbol{d}) - \boldsymbol{\mu})\|^2\right), \tag{3.3}$$

where $\boldsymbol{S}$ is a matrix such that $\boldsymbol{S}^T\boldsymbol{S} = \boldsymbol{\Sigma}^{-1}$.

However, we wish to know the *probability* of having $\boldsymbol{\mathcal{D}} = \boldsymbol{d}$ after having observed $\boldsymbol{\mathcal{M}} = \boldsymbol{m}$. Using Bayes' theorem, we can write this conditional probability as

$$p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}|\boldsymbol{\mathcal{M}} = \boldsymbol{m}) = \frac{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m}|\boldsymbol{\mathcal{D}} = \boldsymbol{d})p(\boldsymbol{\mathcal{D}} = \boldsymbol{d})}{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m})}. \tag{3.4}$$

It states that the posterior distribution, as the result is called, can be written using the likelihood function and the prior distribution $p(\boldsymbol{\mathcal{D}} = \boldsymbol{d})$. The factor $p(\boldsymbol{\mathcal{M}} = \boldsymbol{m})$ is a constant which acts to normalize the posterior distribution. The prior distribution, however, describes all the knowledge of $\boldsymbol{\mathcal{D}}$ that we have prior to the measurement outcome of $\boldsymbol{m}$. An interpretation is that the additional information obtained in the measurement is used to update the prior knowledge. This update gives the posterior distribution, which is a combination of information from both sources.

3.2 Prior distribution of the damage state

The form of the prior distribution $p(\boldsymbol{\mathcal{D}} = \boldsymbol{d})$ can be chosen in many ways. However, since damage is a rare event, the distribution should give high probability when no damage is present. Also, a reasonable assumption is that low levels of damage are more probable than high levels.

A simple suitable distribution can be obtained as a truncated normal distribution, i.e.

$$p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}) = \begin{cases} 0 & \exists i \text{ such that } d_i < 0 \text{ or } d_i > 1 \\ C\exp\left(-\frac{1}{2\lambda^2}\|\boldsymbol{d}\|^2\right) & \text{otherwise} \end{cases} \tag{3.5}$$

That is, the distribution gives zero probability if any damage parameter is outside the interval $[0, 1]$, otherwise it follows the normal distribution centered around zero with standard deviation λ.

Complex prior information can be represented as hierarchical prior distributions. The idea in hierarchical priors is that the prior distribution is made dependent on a set of parameters, which then have a prior distribution themselves, called a hyperprior distribution. That is, in addition to the damage parameters described by the random variable $\boldsymbol{\mathcal{D}}$ we have additional parameters described by the random variable $\boldsymbol{\mathcal{L}}$, which have a joint distribution

$$p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}, \boldsymbol{\mathcal{L}} = \boldsymbol{\lambda}) = p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}|\boldsymbol{\mathcal{L}} = \boldsymbol{\lambda})p(\boldsymbol{\mathcal{L}} = \boldsymbol{\lambda}). \tag{3.6}$$

Similar to the simple case, the posterior distribution, which is now a joint distribution for both $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{L}}$, is obtained using Bayes' theorem as

$$p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}, \boldsymbol{\mathcal{L}} = \boldsymbol{\lambda} | \boldsymbol{\mathcal{M}} = \boldsymbol{m}) = \frac{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m} | \boldsymbol{\mathcal{D}} = \boldsymbol{d}, \boldsymbol{\mathcal{L}} = \boldsymbol{\lambda}) p(\boldsymbol{\mathcal{D}} = \boldsymbol{d} | \boldsymbol{\mathcal{L}} = \boldsymbol{\lambda}) p(\boldsymbol{\mathcal{L}} = \boldsymbol{\lambda})}{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m})}. \quad (3.7)$$

This technique allows for instance formulating damage priors, which not only prefer small damage values but also prefer piecewise smoothness of the damage. An example of such a prior is studied in Publication II.

3.3 Maximum a posteriori estimate

Although we have obtained the posterior distribution through Bayes' theorem, and while it contains all information of the damage, it alone is not very satisfactory. The reconstructed damage is more intuitively understood through representative samples from the distribution, i.e. point estimates. An obvious candidate to characterize the distribution is to use its mean value, but unfortunately obtaining the mean is computationally quite demanding as it requires integrating over the possibly very high dimensional damage parameter space. A more easily computable point estimate is the mode of the distribution, which is typically referred to as the maximum a posteriori (MAP) estimate in this context. The estimate is given by

$$\begin{aligned} \boldsymbol{d}_{\mathrm{MAP}} &= \arg\max_{\boldsymbol{d}} p(\boldsymbol{\mathcal{D}} = \boldsymbol{d} | \boldsymbol{\mathcal{M}} = \boldsymbol{m}) \\ &= \arg\max_{\boldsymbol{d}} \frac{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m} | \boldsymbol{\mathcal{D}} = \boldsymbol{d}) p(\boldsymbol{\mathcal{D}} = \boldsymbol{d})}{p(\boldsymbol{\mathcal{M}} = \boldsymbol{m})}. \end{aligned} \quad (3.8)$$

Any reasonable prior will require all the components of $\boldsymbol{d}$ to remain in the interval $[0, 1)$. This can thus be written as a constraint in the optimization. Also, as the factor $p(\boldsymbol{\mathcal{M}} = \boldsymbol{m})$ is a constant, it has no effect on the argument maximum, and hence we can write the MAP estimate as

$$\boldsymbol{d}_{\mathrm{MAP}} = \arg\max_{0 \le d_i \le 1} p(\boldsymbol{\mathcal{M}} = \boldsymbol{m} | \boldsymbol{\mathcal{D}} = \boldsymbol{d}) p(\boldsymbol{\mathcal{D}} = \boldsymbol{d}). \quad (3.9)$$

In particular, using (3.5) as the prior distribution, the problem becomes

$$\boldsymbol{d}_{\mathrm{MAP}} = \arg\min_{0 \le d_i \le 1} \left\{ \frac{1}{2} \| \boldsymbol{S}(\boldsymbol{m} - \boldsymbol{f}(\boldsymbol{d}) - \boldsymbol{\mu}) \|^2 + \frac{1}{2\lambda^2} \| \boldsymbol{d} \|^2 \right\}, \quad (3.10)$$

which is equivalent to Tikhonov regularization, but with an interpretation of what the terms and factors represent.

3.4 Statistical parameters of the noise term

The additive noise term in (3.1) contains both measurement noise as well as model noise. Measurement noise is due to the non-ideal behavior of the measurement sensors, as well as noise that comes from the pre-processing of the time domain data to obtain the mode frequencies and observable mode shapes. Model noise, on the other hand, is due to errors in the structure model. This includes for instance discretization errors rising from the use of the finite element method, errors due to simplifications of the model, such as using the beam equation instead of full elasticity, as well as unmodeled environmental effects, such as effects caused by the changing of the ambient temperature.

Assuming the additive noise term indeed follows a multivariate normal distribution, one only needs to estimate the mean $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. This, together with the assumption that the noise is independent of the damage state, makes it possible to estimate the statistical parameters directly from the measurement data.

Writing (3.1) in a different way, we see that

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{M}} - \boldsymbol{f}(\boldsymbol{\mathcal{D}}). \tag{3.11}$$

Considering the undamaged case, the mean of the noise distribution is then given as

$$\boldsymbol{\mu} = E[\boldsymbol{\mathcal{E}}] = E[\boldsymbol{\mathcal{M}} - \boldsymbol{f}(\boldsymbol{\mathcal{D}})|\boldsymbol{\mathcal{D}} = 0]. \tag{3.12}$$

Given a set of measurements taken from the structure in its undamaged state, the mean of the noise term can be estimated simply as the sample mean of the obtained measurements in the set. The same procedure could of course be used regardless of what the damage state is, as long as it has a known value and is constant over the whole measurement set. In practice, however, the actual damage state is only known in the undamaged case.

The noise covariance matrix can be estimated similarly, i.e. by approximating

$$\begin{aligned}\boldsymbol{\Sigma} &= E[(\boldsymbol{\mathcal{E}} - E[\boldsymbol{\mathcal{E}}])(\boldsymbol{\mathcal{E}} - E[\boldsymbol{\mathcal{E}}])^T] \\ &= E[(\boldsymbol{\mathcal{M}} - E[\boldsymbol{\mathcal{M}}|\boldsymbol{\mathcal{D}} = \boldsymbol{d}])(\boldsymbol{\mathcal{M}} - E[\boldsymbol{\mathcal{M}}|\boldsymbol{\mathcal{D}} = \boldsymbol{d}])^T|\boldsymbol{\mathcal{D}} = \boldsymbol{d}].\end{aligned} \tag{3.13}$$

as the sample covariance over a set of measurements in which the damage state is known to have remained constant.

To reliably estimate the statistical parameters, a large amount of measurements are needed, and they should also contain all environmental variation that the structure is expected to experience. This is especially important for the accurate estimation of the covariance matrix.

The mean of the additive noise term can be thought of as compensation against an offset error in the forward model. If this offset were not accounted for, i.e. assumed to be zero, then damage could only be detected after the change in measurement became significantly greater than the initial model error. Even in the most extreme damage cases in our experiments, the observed changes in the measurement were only in the order of one percent. Thus without the offset compensation the model would need to agree with the measurement much better than one percent. However, assuming a nonzero mean value allows the forward model to be much less accurate in this respect. With the compensation the model can still be used as long as it has a reasonably similar sensitivity to damage as the true structure.

In addition to information about the random noise in the measurement, the covariance also contains information about correlated parts of the unmodeled dynamics. Since environmental conditions have a global effect on the structure, they cause changes in the measurement which may have correlation between the components. As an example, a change in temperature will cause changes in all mode frequencies. Since the change is mostly due to thermal expansion of the structure, we will see a decrease in the frequencies as temperature increases. If this correlation is not included in the noise covariance matrix, then each affected measurement component will have a large variance on its own. If the correlations are included, however, the large variance is only associated with a single eigenvector of the covariance matrix.

Taking this correlation caused by environmental effects into account is the main reason for the clear difference in damage identification sensitivity between Publication I and Publication III. Both articles consider data from the same experiment. In Publication I the measurement was taken to include 48 components (frequencies and shapes for 6 modes), for which independence had to be assumed due to insufficient data. However, in Publication III the full covariance could be estimated as the measurement was taken to contain only 24 components (frequencies and shapes for 3 modes).

4. Some other methods of damage identification

4.1 The Kalman filter

The Kalman filter is a state estimation method, which under certain circumstances produces optimal estimates [3]. The damage identification problem can also be considered as a problem of state estimation. The state is then a combination of the standard state components of a structure and the damage parameters. As a simple example, the state vector $\boldsymbol{x}$ may be taken as

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{u} \\ \dot{\boldsymbol{u}} \\ \boldsymbol{d} \end{pmatrix}. \tag{4.1}$$

Assuming no external loading and a known damage state $\boldsymbol{d}$, we can define a discrete time propagation operator $\boldsymbol{F}(\boldsymbol{u}, \dot{\boldsymbol{u}}; \boldsymbol{d})$ for the dynamics of the structure, so that

$$\begin{pmatrix} \boldsymbol{u}_{k+1} \\ \dot{\boldsymbol{u}}_{k+1} \end{pmatrix} = \boldsymbol{F}(\boldsymbol{u}_k, \dot{\boldsymbol{u}}_k; \boldsymbol{d}). \tag{4.2}$$

The damage state is assumed to remain constant in time, so the state vector $\boldsymbol{x}$ is evolved as

$$\boldsymbol{x}_{k+1} = \begin{pmatrix} \boldsymbol{u}_{k+1} \\ \dot{\boldsymbol{u}}_{k+1} \\ \boldsymbol{d}_{k+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{F}(\boldsymbol{u}_k, \dot{\boldsymbol{u}}_k; \boldsymbol{d}_k) \\ \boldsymbol{d}_k \end{pmatrix}. \tag{4.3}$$

To simplify the notation, we denote the state evolution operator with $\boldsymbol{g}(\boldsymbol{x})$, so that the above equation becomes

$$\boldsymbol{x}_{k+1} = \boldsymbol{g}(\boldsymbol{x}_k). \tag{4.4}$$

The measurement model of Section 2.5 is then discretized in time as

$$\boldsymbol{m}_k = \tilde{\boldsymbol{H}} \boldsymbol{x}_k = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} \boldsymbol{x}_k. \tag{4.5}$$

As the dynamics of the structure have a nonlinear relationship with the damage parameters, i.e. the mapping $\boldsymbol{g}(\boldsymbol{x})$ in equation (4.4) is not linear, a nonlinear version of the Kalman filter is needed. In the following, the extended Kalman filter [3] is used.

In the extended Kalman filter, as with Kalman filters in general, the state is not thought as exactly known, but is assumed to have uncertainty associated with it. In the extended Kalman filter, we assume that the state is a random variable. At each time step k we keep track of the mean, denoted by $\boldsymbol{x}_k$, and of the covariance matrix, denoted by $\boldsymbol{P}_k$, of the distribution.

The extended Kalman filter consists of two phases. The first one is the prediction phase, in which the probability distribution of the state is evolved in time through the structure model, i.e. it predicts what the state of the structure should be after one time step, given that the state estimate is accurate for the current time step. The prediction of the mean value is taken as

$$\hat{\boldsymbol{x}}_k = \boldsymbol{g}(\boldsymbol{x}_{k-1}), \tag{4.6}$$

while the prediction of the covariance matrix is taken as

$$\hat{\boldsymbol{P}}_k = \boldsymbol{G}(\boldsymbol{x}_{k-1})\boldsymbol{P}_{k-1}\boldsymbol{G}(\boldsymbol{x}_{k-1})^T + \boldsymbol{Q}_k. \tag{4.7}$$

The matrix $\boldsymbol{G}(\boldsymbol{x}_{k-1})$ denotes the Jacobian of the mapping $\boldsymbol{g}$ evaluated at $\boldsymbol{x}_{k-1}$ and the matrix $\boldsymbol{Q}_k$ is the covariance matrix of the model noise, i.e. a representation of the inaccuracy of the structure model.

The second phase is the correction phase, in which measurement data is used to improve the predicted state estimate. The difference between the actual measurement and the predicted measurement is called the innovation, defined as

$$\boldsymbol{y}_k = \boldsymbol{m}_k - \tilde{\boldsymbol{H}}\hat{\boldsymbol{x}}_k. \tag{4.8}$$

The uncertainty involved in the innovation is determined by the uncertainty of the state estimate and the uncertainty related to the measurement process. The innovation covariance is given as

$$\boldsymbol{S}_k = \tilde{\boldsymbol{H}}\hat{\boldsymbol{P}}_k\tilde{\boldsymbol{H}}^T + \boldsymbol{R}_k, \tag{4.9}$$

where $\boldsymbol{R}_k$ is a covariance matrix representing the uncertainty in the measurement $\boldsymbol{m}_k$. Then, the estimated mean value and covariance matrix for time step k are obtained as

$$\begin{aligned} \boldsymbol{x}_k &= \hat{\boldsymbol{x}}_k + \hat{\boldsymbol{P}}_k\tilde{\boldsymbol{H}}^T\boldsymbol{S}_k^{-1}\boldsymbol{y}_k, \\ \boldsymbol{P}_k &= \hat{\boldsymbol{P}}_k - \hat{\boldsymbol{P}}_k\tilde{\boldsymbol{H}}^T\boldsymbol{S}_k^{-1}\tilde{\boldsymbol{H}}\hat{\boldsymbol{P}}_k. \end{aligned} \tag{4.10}$$

Starting from a suitable initial guess for the state mean and covariance, this process is repeated. Under favorable conditions, this results in a useful state estimate from which the damage state can then be read.

The measurement data can be used as is, that is no pre-processing is required. Also, to process the state estimate at time step k, only the measurement $\boldsymbol{m}_k$ is needed, i.e. it is not necessary to store past measurements nor does one need to know future measurements. These properties make the Kalman filter approach very suitable for a real-time damage identification application.

This simple example relies heavily on the accuracy of the structure model, which has problems especially with changing environmental conditions. In Publication III we used a more elaborate model, which in addition to just the damage parameters, includes other model parameters in the state as well. Also, instead of the extended Kalman filter presented here, the Ensemble Kalman filter [11, 12] was used to obtain an estimate of the damage state in a beam.

4.2 Method of sigma algebras on contour maps

The forward operator $\boldsymbol{f}(\boldsymbol{d})$, as defined in Section 2.7, is a function of damage such that we obtain our simulated measurement $\boldsymbol{m}$ as

$$\boldsymbol{m} = \boldsymbol{f}(\boldsymbol{d}) \tag{4.11}$$

for a given damage state $\boldsymbol{d}$.

The damage identification problem is to solve for the inverse of this, i.e. given measurement $\boldsymbol{m}$, find the damage state $\boldsymbol{d}$. As previously stated, this problem is ill-posed, and thus no unique solution exists. In Section 3 a solution based on Bayesian inference was introduced, but other methods exist as well. One other method in particular is the sigma algebras on contour maps (SACOM) method [7, 8, 9].

The measurement is assumed to have uncertainty associated with it, which defines a probability distribution on the measurement space M. Similar to the Bayesian inference approach, SACOM also defines a probability distribution in the damage parameter space. However, instead of using Bayes' theorem to define the distribution, it is defined through the set-valued inverses of the forward operator $\boldsymbol{f}(\boldsymbol{d})$. More precisely, the probability distribution on the measurement space is mapped to the damage parameter space through the pre-image map of $\boldsymbol{f}$. This is done in such a way, that given any set A in the measurement space, the probability of the pre-image $\boldsymbol{f}^{-1}(A)$ in the damage space is the same as the probability of A in the measurement space.

Computationally this approximated by generating a partition $\mathcal{V}$ of the damage parameter space D, as well as a partition $\mathcal{I}$ of the measurement outcome space M. A large number of damage samples $\boldsymbol{d}_i$ are generated in the damage parameter space D. The generated samples are mapped to the measurement space using the forward operator $\boldsymbol{m}_i = \boldsymbol{f}(\boldsymbol{d}_i)$. Each sample $\boldsymbol{d}_i$ belongs to some cell $V \in \mathcal{V}$, while the image of the sample $\boldsymbol{m}_i$ belongs to some cell $I \in \mathcal{I}$.

The probability distribution of the measurement space is used to assign a probability to each cell $I \in \mathcal{I}$. Each sample $\boldsymbol{m}_i$ in a particular cell I gets assigned an equal fraction of that cell's probability. This associates a probability with each sample. The probability of a cell $V \in \mathcal{V}$ is then taken as the total probability of all samples for which $\boldsymbol{d}_i \in V$.

SACOM is used to estimate the damage state of a beam in Publication III.

4.3 Damage identification as an inverse source problem

Consider an undamaged structure under free vibration, characterized by the differential equation

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{0}. \tag{4.12}$$

Now, assume that the structure is damaged, which causes the stiffness to change by $\boldsymbol{\Delta K}$. The free vibrations are then described by

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + (\boldsymbol{K} + \boldsymbol{\Delta K})\boldsymbol{u} = \boldsymbol{0} \tag{4.13}$$

or equivalently

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = -\boldsymbol{\Delta K}\boldsymbol{u}. \tag{4.14}$$

That is, the damage can be thought of as an additional loading on the intact structure, and being able to reconstruct the loading condition is equivalent to reconstructing the damage.

Consider then the case of a vibrating structure which is driven by an external load $\boldsymbol{f}(t)$. It is described by the equation

$$\boldsymbol{M}\ddot{\boldsymbol{u}} + \boldsymbol{K}\boldsymbol{u} = \boldsymbol{f}(t). \tag{4.15}$$

Taking the Fourier transform in time gives

$$-\omega^2\boldsymbol{M}\hat{\boldsymbol{u}}(\omega) + \boldsymbol{K}\hat{\boldsymbol{u}}(\omega) = \hat{\boldsymbol{f}}(\omega), \tag{4.16}$$

where $\hat{\boldsymbol{u}}(\omega)$ and $\hat{\boldsymbol{f}}(\omega)$ are the Fourier transforms of $\boldsymbol{u}(t)$ and $\boldsymbol{f}(t)$ respectively. We also take the Fourier transform of the measurement model of Section 2.5, to obtain

$$\hat{\boldsymbol{m}}(\omega) = \boldsymbol{H}\hat{\boldsymbol{u}}(\omega). \tag{4.17}$$

For any given frequency ω, the problem then becomes an inverse source problem of finding the source term $\hat{\boldsymbol{f}}(\omega)$ when the measurement $\hat{\boldsymbol{m}}(\omega)$ is given. An interesting observation is that for the most part this problem is linear. This is important, as this may give greater opportunity for a rigorous mathematical analysis.

Inverse source problems of a related type are considered in Publication IV, where the main interest lies with the discretization error when solving the problem.

5. Concluding remarks

Publication I Vibration based damage identification is investigated as a statistical inverse problem. Using Bayesian inference the prior information of the damage state is updated with the information obtained in the measurement. The measurement and model noise is taken into account as a random variable assumed to be multivariate normal distributed. The maximum a posteriori estimate is used to give a point estimate of the damage state. The method is applied to experimental data obtained from acceleration measurements of a steel cantilever beam, with multiple levels of damage.

Publication II The use of hierarchically defined priors for the damage identification problem are investigated. Hierarchical priors can be very useful in specifying complex prior information, such as the sparsity of the damage parameter field, which was the motivation of this publication. One particular type of hierarchical prior is introduced and investigated using numerical experiments.

Publication III Three different parameter identification methods are applied to damage identification. The idea is to use the methods in a manner such as to complement the different strengths and weaknesses of the methods. The Kalman filter can be used to give a quick real time assess of the damage status without any data pre-processing. After processing the measurement data to extract modal data, the regularization method is used to give a more reliable estimate, as well as to define a low dimensional parametrization of the damage space for the SACOM method. SACOM is then used to get probability densities of the parameters. Both simulated and experimental measurements of a cantilever beam with multiple levels of damage are used.

Publication IV An a priori estimate is derived for the discretization error of an inverse source problem. Application to damage identification is not explicitly stated in the publication, as with the simplified setting presented in the article, the connection would have remained somewhat loose. Considering the damage identification problem as an inverse source problem, however, results in a set of problems, each of which is of a related type to what is considered in the article.

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