Department of Mathematics and Systems Analysis

Duality in stochastic and dynamic optimization

Ari-Pekka Perkkiö

Aalto University

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Duality in stochastic and dynamic optimization

Ari-Pekka Perkkiö

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Aalto University
School of Science
Department of Mathematics and Systems Analysis

Supervising professor
Juha Kinnunen

Thesis advisor
Teemu Pennanen, King's College London, UK

Preliminary examiners
Rafael Goebel, Loyola University Chicago, USA
Miklos Rasonyi, University of Edinburgh, UK

Opponent
Boris Mordukhovich, Wayne State University, USA

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Abstract

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Ari-Pekka Perkkiö

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Abstract

The main theme in this dissertation is convex duality in stochastic and dynamic optimization. The analysis is based on the conjugate duality framework of Rockafellar and on the theory of convex integral functionals. The dissertation consists of an overview and of three articles.

In the first article we study dynamic stochastic optimization problems parameterized by a random variable. Such problems arise in many applications in operations research and mathematical finance. We give sufficient conditions for the existence of solutions and the absence of a duality gap. Our proof uses extended dynamic programming equations, whose validity is established under new relaxed conditions that generalize certain no-arbitrage conditions from mathematical finance.

The second article contributes to the theory of integral functionals that is closely connected with set-valued analysis. Given a strictly positive measure, we characterize inner semicontinuous solid convex-valued mappings for which continuous functions which are selections almost everywhere are selections. This class contains continuous mappings as well as fully lower semicontinuous closed-valued mappings that arise in variational analysis and optimization of integral functionals. The characterization allows for extending existing results on convex conjugates of integral functionals on continuous functions. We also give an application to integral functionals on left continuous functions of bounded variation.

In the third article we study duality in problems of Bolza over functions of bounded variation. We parameterize the problem by a general Borel measure which has direct economic interpretation in problems of financial economics. Using our results on conjugates of integral functionals, we derive a dual representation for the optimal value function in terms of continuous dual arcs and we give conditions for the existence of solutions. Combined with well-known results on problems of Bolza over absolutely continuous arcs, we obtain optimality conditions in terms of extended Hamiltonian conditions.

Keywords Convex analysis, optimization, stochastics

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Tiivistelmä

Tekijä
Ari-Pekka Perkkiö

Väitöskirjan nimi
Duaalisuus stokastisessa ja dynaamisessa optimoinnissa

Julkaisija Perustieteiden korkeakoulu

Yksikkö Matematiikan ja systeemianalyysin laitos

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Tiivistelmä

Väitöskirjan pääteema on konveksi duaalisuus stokastisessa ja dynaamisessa optimoinnissa. Analyysi pohjautuu Rockafellarin konjugaattiduaalisuuteen ja konveksien integraalifunktionaalien teoriaan. Väitöskirja sisältää johdannon ja kolme artikkelijulkaisua.

Ensimmäisessä artikkelissa tutkimme satunnaismuuttujilla parametrisoituja dynaamisia stokastisia optimointiongelmia. Tällaisia ongelmia esiintyy monissa sovelluksissa operaatiotutkimuksessa ja rahoitusteoriassa. Annamme riittävät ehdot ratkaisujen olemassaololle ja duaaliesitykselle. Todistuksemme hyödyntävät laajenettuja dynaamisen ohjelmoinnin yhtälöitä, joiden pätevyyden todistamme uusilla oletuksilla. Nämä yleistävät tunnettuja arbitraasiehtoja rahoitusteoriassa.

Toinen artikkeli käsittelee integraalifunktionaalien teoriaa, joka kytkeytyy joukkoarvoiseen analyysiin. Karakterisoimme sisältä puolijatkuvat kiinteät konveksiarvoiset kuvaukset, joille jatkuvat oleelliset selektiot ovat selektioita. Tämä luokka sisältää jatkuvat kuvaukset sekä alhaalta täysin puolijatkuvat suljettuarvoiset kuvaukset, joita esiintyy variaatioanalyysissa ja optimoinnissa integraalifunktionaalien yhteydessä. Karakterisointi mahdollistaa yleistyksen olemassa oleville tuloksille, jotka käsittelevät integraalifunktionaalien konvekseja konjugaatteja jatkuvilla funktioilla. Annamme myös sovelluksen integraalifunktionaaleille vasemmalta jatkuvilla, rajoitetusti heilahtevilla funktioilla.

Kolmannessa artikkelissa tutkimme duaalisuutta Bolzan tehtävissä rajoitetusti heilahtevilla funktioilla. Parametrisoimme tehtävän Borel-mitalla, jolla on taloudellinen tulkinta rahoitusteorian sovelluksissa. Johdamme duaaliesityksen arvofunktiolle ja annamme ehdot ratkaisujen olemassaololle käyttämällä uusia tuloksiamme integraalifunktionaaleille. Lisäksi esitämme optimaalisuusehdot laajennettujen Hamiltonin ehtojen avulla.

Avainsanat Konveksi analyysi, optimointi, stokastiikka

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Preface

I worked on this dissertation as a member of the Stochastics Group of the Department of Mathematics and Systems Analysis at Aalto University. The research was mainly funded by the Finnish Doctoral Programme in Stochastics and Statistics (FDPSS), for whom I also worked as a coordinator in 2009 – 2012.

I am very grateful to the late prof. Esko Valkeila who supervised me and lead our group until November 2012. I will always remember him as an enthusiastic scientist and a kind and sympathetic leader. I also want to express my deepest gratitude to my advisor, prof. Teemu Pennanen. He has influenced my scientific orientation and academic way of thinking the most.

I thank the pre-examiners, assoc. prof. Rafael Goebel and Dr. Miklos Rasonyi, for their encouraging comments on the manuscript. Thanks to prof. Boris Mordukhovich, my opponent, for accepting the invitation to be at my defense. I thank prof. Juha Kinnunen who supervised me during the last year of the project.

Many thanks to new and former members of our stochastics group and to the whole community around the FDPSS. The scientific programs and lively social events of the FDPSS have brought me in touch with many researchers and students with similar interests both in Finland and abroad in a warm atmosphere.

Finally, thanks to my family, friends and especially to Lumi.

Iitti, September 2, 2013,

Ari-Pekka Perkkiö

Contents

List of Publications

This thesis consists of an overview and of the following publications which are referred to in the text by their Roman numerals.

I Pennanen, T. and Perkkiö, A.-P.. Stochastic programs without duality gaps. *Mathematical Programming*, Volume 136, Issue 1, pages 91–110, December 2012.

II Perkkiö A.-P.. Continuous essential selections and integral functionals. *Set-Valued and Variational Analysis*, DOI: 10.1007/s11228-013-0249-0, 14 pages, August 2013.

III Pennanen T. and Perkkiö A.-P.. Duality in convex problems of Bolza over functions of bounded variation. *http://math.aalto.fi/~aperkkio*, 18 pages, April 2013.

Author's Contribution

Publication I: "Stochastic programs without duality gaps"

The article is a joint work with Professor Teemu Pennanen.

Publication II: "Continuous essential selections and integral functionals"

The article is an individual work of the author.

Publication III: "Duality in convex problems of Bolza over functions of bounded variation"

The article is a joint work with Professor Teemu Pennanen.

1. Introduction

Convex duality is deep-rooted in stochastic and dynamic optimization. Dualization of a given problem often leads to new insights, computational techniques and optimality conditions. For instance one obtains in mathematical finance pricing formulas for financial instruments and characterizations of different types of no-arbitrage conditions. In dynamic optimization and mechanics, duality leads to optimality conditions in terms of Euler-Lagrange equations and Hamiltonian systems.

This thesis builds on the conjugate duality framework of Rockafellar and on the theory of convex integral functionals. Publication I studies stochastic optimization problems in discrete-time, Publication II contributes to the theory of integral functionals and Publication III addresses dynamic optimization problems in continuous time.

The analysis of Publication I is motivated by applications from mathematical finance, where convexity appears naturally in market models as well as risk preferences. We obtain stochastic dynamic programming equations under new relaxed conditions. These are used to give sufficient conditions for the existence of solutions and the closedness of the value function. The conditions subsume various no-arbitrage conditions used in mathematical finance.

The theory integral functionals is closely connected with set-valued analysis which is an integral part of modern variational analysis and optimization [2, 3, 12, 36]. In Publication II we give new results on continuous selections of set-valued mappings which we use to generalize existing results on conjugates of integral functionals.

The dynamic problems we study are problems of Bolza where an integral functional of state and velocity together with an end-point functional are optimized over a given class of trajectories. A problem of Bolza does not necessarily have a continuous optimal trajectory, while in some applications it is more natural to consider discontinuous trajectories to begin with. Both aspects lead one to consider problems of Bolza over functions of bounded variations which are studied in Publication III. We apply results of Publication II so that we get new conditions for the existence of solutions and the closedness of the value function. Moreover, the results of Publication II allow for new optimality conditions in terms of a generalized Hamiltonian equation.

We end this introduction by giving a short summary of conjugate duality and convex integral functionals.

1.1 Conjugate duality

The conjugate duality framework set forth by R.T. Rockafellar around 1970 unifies many other duality frameworks of optimization such as the linear programming duality, Lagrangian duality and Fenchel duality [31]; see the commentary of [36, Chapter 11] for the history of duality in optimization.

A basic idea in conjugate duality is to parameterize a given optimization problem. In many cases parameters have natural interpretations, while in other situations they serve only as perturbations whose effect on the problem is of interest. To this end, we consider the *parameterized primal problem*

$$\text{minimize } f(x,u) \quad \text{over } x \in X, \tag{P$_u$}$$

where f is a jointly convex extended real-valued ($\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$) function on $X \times U$ known as the *parameterized objective function* and X and U are the vector spaces of variables and parameters, respectively. The *value function* associated with (P$_u$) is the extended real-valued convex function on U defined by

$$\varphi(u) = \inf_{x\in X} f(x,u).$$

In order to dualize (P$_u$), we assume that U is in separating duality with a vector space Y with respect to a bilinear form $(u,y) \mapsto \langle u,y\rangle$. The *convex conjugate* of φ is the extended real-valued convex function on Y defined by

$$\varphi^*(y) = \sup_{u\in U}\{\langle u,y\rangle - \varphi(u)\}.$$

The mapping $\varphi \mapsto \varphi^*$ is known as the *Legendre-Fenchel transform*. When

φ is closed[1], the biconjugate theorem gives the *dual representation*

$$\varphi(u) = \sup_{y \in Y}\{\langle u, y\rangle - \varphi^*(y)\};$$

see [31, Theorem 5]. This formula is behind many important results, e.g., in economics and mathematical finance.

A central object in the conjugate duality framework is the *Lagrangian* which is the convex-concave function on $X \times Y$ defined by

$$L(x, y) = \inf_{u \in M}\{f(x, u) - \langle u, y\rangle\}.$$

We always have

$$\varphi^*(y) = -\inf_{x \in X} L(x, y).$$

In some situations this facilitates the derivation of an expression for φ^*. The Lagrangian is also involved in various optimality conditions. As usual, we denote the subdifferential of a convex function φ at a point $\bar{u}$ by

$$\partial\varphi(\bar{u}) = \{y \in Y \mid \varphi(\bar{u}) + \langle y, u - \bar{u}\rangle \le \varphi(u)\,\forall u \in U\}.$$

If X is in separating duality with another linear space V, then subdifferentials of a convex function on X can be defined analogously. If f is closed in u and $\partial\varphi(0) \neq \emptyset$, then $\bar{x}$ solves (P_u) for $u = 0$ if and only if there exists $\bar{y}$ such that

$$0 \in \partial_x L(\bar{x}, \bar{y}), \quad 0 \in \partial_y[-L](\bar{x}, \bar{y}).$$

Here $\partial_x L(\bar{x}, \bar{y})$ and $\partial_y[-L](\bar{x}, \bar{y})$ and denote subdifferentials of $x \mapsto L(x, \bar{y})$ at $\bar{x}$ and $y \mapsto -L(\bar{x}, y)$ at $\bar{y}$, respectively. This optimality relation is known as the (abstract) *Kuhn-Tucker condition*. For example this leads to Hamiltonian conditions in mechanics and to the Pontryagin maximum principle in optimal control.

As an illustration consider the nonlinear programming problem

$$\begin{aligned} &\text{minimize}\, f_0(x) \quad \text{over } x \in \mathbb{R}^d \\ &\text{subject to}\, f_i(x) \le 0 \quad i = 1, \ldots, m, \end{aligned}$$

where $f_0, \ldots, f_m$ are real-valued convex functions on $\mathbb{R}^d$. We embed the problem into the conjugate duality framework so that

$$f(x, u) = \begin{cases} f_0(x) & \text{if } f_i(x) \le u_i, \quad i = 1, \ldots, m, \\ +\infty & \text{otherwise,} \end{cases}$$

[1] A function is *closed* if it is lower semicontinuous (lsc) and either proper or a constant. A function is *proper* if it never takes the value $-\infty$ and it is finite at some point.

where $u = (u_1, \dots, u_m) \in \mathbb{R}^m$. The Lagrangian has the expression

$$L(x,y) = \begin{cases} f_0(x) + y_1 f_1(x) + \dots + y_m f_m(x) & \text{if } y_i \geq 0 \quad i = 1, \dots, m, \\ -\infty & \text{otherwise,} \end{cases}$$

so the Kuhn-Tucker condition can be written as

$$0 \in \partial f_0(\bar{x}) + \bar{y}_1 \partial f_1(\bar{x}) + \dots \bar{y}_m \partial f_m(\bar{x}), \quad \bar{y}_i \geq 0, f_i(\bar{x}) \leq 0, \bar{y}_i f_i(\bar{x}) = 0.$$

If $f_0, \dots f_m$ are differentiable, then the above conditions take the more familiar form

$$\nabla f_0(\bar{x}) + \bar{y}_1 \nabla f_1(\bar{x}) + \dots \bar{y}_m \nabla f_m(\bar{x}) = 0, \quad \bar{y}_i \geq 0, f_i(\bar{x}) \leq 0, \bar{y}_i f_i(\bar{x}) = 0.$$

1.2 Integral functionals

Many optimization problems in practice involve with integral functionals on spaces of measurable functions. Let $(\mathbb{T}, \mathcal{F}, \mu)$ be a measure space, S be a topological vector space equipped with the Borel σ-algebra and $L^0 = L^0(\Omega, \mathcal{F}, \mu; S)$ be the space of equivalence classes of $\mathcal{F}$-measurable S-valued functions that coincide μ-almost everywhere. Given a jointly measurable extended real-valued function h on $\mathbb{T} \times S$, the corresponding *integral functional* $I_h : L^0 \to \overline{\mathbb{R}}$ is defined by

$$I_h(u) = \int_{\mathbb{T}} h_t(u_t) d\mu_t,$$

where I_h is defined as $+\infty$ unless the positive part is integrable. Many kinds of constraints can be expressed as infinite penalties by allowing h to be extended-real valued; see e.g. [25] for examples from calculus of variations and optimal control.

The joint measurability is not sufficient to preserve measurability in some operations, especially under projections. Therefore, a slightly stronger property is more appropriate. Let $\Gamma : \mathbb{T} \rightrightarrows S$ be a set-valued mapping, that is, $\Gamma_t \subset S$ for every $t \in \mathbb{T}$. The mapping Γ is *measurable* if the preimage $\Gamma^{-1}(A) := \{t \in \mathbb{T} \mid \Gamma_t \cap A \neq \emptyset\}$ of every open $A \subset S$ is measurable. An extended real-valued function h on $S \times \mathbb{T}$ is a *convex normal integrand* if the set-valued mapping $t \mapsto \{(s, \alpha) \in S \times \mathbb{R} \mid h_t(s) \leq \alpha\}$ is closed convex-valued and measurable. The review article [33] contains the fundamental results on normal integrands, integral functionals and their convex conjugates and subdifferentials.

The formula for the conjugate of an integral functional on a decomposable space is used extensively in the analysis of problems of Bolza; see e.g. [25]. Let $S = \mathbb{R}^d$ and let μ be *σ-finite*, that is, $\mathbb{T}$ can be expressed as a countable union of sets with finite μ-measure. Assume that $\mathcal{U} \subset L^0$ is a linear space and that $\mathcal{U}$ is *decomposable* in the sense that $\mathbb{T}$ can be expressed as an increasing sequence of measurable sets $\mathbb{T}^\nu$ such that for every $\mathbb{T}^\nu$, for every bounded measurable $u' : \mathbb{T}^\nu \to \mathbb{R}^d$ and for every $u \in \mathcal{U}$, we have

$$1_{\mathbb{T}^\nu} u' + 1_{\mathbb{T}\setminus\mathbb{T}^\nu} u \in \mathcal{U}.$$

Spaces like L^p-spaces, Orlicz spaces and the space of measurable functions are decomposable.

Assume that $\mathcal{U}$ is in separating duality with another vector space $\mathcal{Y}$ under the bilinear form

$$\langle u, y\rangle = \int_{\mathbb{T}} u_t \cdot y_t d\mu_t,$$

and that there exists $u \in \mathcal{U}$ with $I_h(u) < \infty$. Then the conjugate of $I_h : \mathcal{U} \to \overline{\mathbb{R}}$ is given by $I_{h^*}(y)$, where

$$h^*_t(y) = \sup_{u\in\mathbb{R}^m} \{u \cdot y - h_t(u)\}$$

is a normal integrand on $\mathbb{R}^d \times \mathbb{T}$ known as the *conjugate* of h. Moreover, we have the subdifferential formula

$$\partial I_h(u) = \{y \in \mathcal{Y} \mid y_t \in \partial h_t(u_t)\ \mu\text{-a.e.}\},$$

where $\partial h_t(u)$ denotes the subdifferential of $u \mapsto h_t(u)$. Likewise, if $\mathcal{Y}$ is decomposable and there exists $y \in \mathcal{Y}$ with $I_{h^*}(y) < \infty$, then $I^*_{h^*}(u) = I_h(u)$. In particular, I_h is lower semicontinuous with respect to any topology compatible with the pairing between $\mathcal{U}$ and $\mathcal{Y}$.

However, many important spaces, such as the spaces of continuous functions or functions of bounded variation, are not decomposable. Let $\mathbb{T}$ be σ-compact, locally compact Hausdorff space and $C_0 = C_0(\mathbb{T};\mathbb{R}^d)$ be the space of continuous functions vanishing at infinity. The bilinear form

$$\langle y, \theta\rangle = \int_{\mathbb{T}} y_t d\theta_t$$

puts the space C_0 in separating duality with the space of finite regular $\mathbb{R}^d$-valued Borel measures on $\mathbb{T}$. In [29, Theorem 5] Rockafellar gave conditions under which the conjugate of $I_h : C_0 \to \overline{\mathbb{R}}$ can be expressed in terms of the conjugate of h as

$$J_{h^*}(\theta) = \int_{\mathbb{T}} h^*_t((d\theta/d\mu)_t)d\mu_t + \int_{\mathbb{T}} (h^*)^\infty_t((d\theta^s/d|\theta^s|)_t)d|\theta^s|_t,$$

where θ^s is the singular part of θ with respect to μ and $|\theta^s|$ denotes the total variation of θ^s. Here and in what follows, $(h_t^*)^\infty$ denotes the *recession function* of h_t^*. That is, $(h_t^*)^\infty$ is defined by

$$(h_t^*)^\infty(y) = \sup_{\alpha>0} \frac{h_t^*(\alpha y + \bar{y}) - h_t^*(\bar{y})}{\alpha},$$

which is independent of $\bar{y} \in \operatorname{dom} h_t^* := \{y \in \mathbb{R}^d \mid h_t^*(y) < \infty\}$; see [26, Chapter 8]. Rockafellar has applied [29, Theorem 5] to problems of Bolza over functions of bounded variation. More results for integral functionals on non-decomposable spaces can be found, e.g., from [11].

2. Summary of results

Publication I: Stochastic programs without duality gaps

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t=0}^T$ (an increasing sequence of sub-sigma-algebras of $\mathcal{F}$). We consider the stochastic optimization problem

$$\text{minimize} \quad Ef(x(\omega), u(\omega), \omega) \quad \text{over } x \in \mathcal{N}, \tag{P$_u$}$$

where, for given integers n_t and m,

$$\mathcal{N} = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\},$$

$u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ and f is a convex normal integrand on $\mathbb{R}^n \times \mathbb{R}^m \times \Omega$, where $n = n_0 + \ldots + n_T$. We refer to [22] for a series of examples how various problems in operations research and mathematical finance fit into the framework of normal integrands and integral functionals, and to [23] and references therein for a more comprehensive introduction to various aspects of convexity in financial applications.

The first main result is about dynamic programming recursion which is one of the main principles in dynamic optimization; see e.g. [5] or [6]. We show that in the convex case the inf-compactness assumption made in both [16] and [35] can be replaced by a weaker condition on the directions of recession of the normal integrand. This condition extends the no-arbitrage condition from mathematical finance.

Since the parameter u is irrelevant in the dynamic programming recursion, we simplify the notation by defining

$$h(x, \omega) = f(x, u(\omega), \omega).$$

We use the notation $x^t = (x_0, \ldots, x_t)$ and define extended real-valued functions $h_t, \tilde{h}_t : \mathbb{R}^{n_0+\cdots+n_t} \times \Omega \to \overline{\mathbb{R}}$ recursively for $t = T, \ldots, 0$ by

$$\begin{aligned}
\tilde{h}_T &= h, \\
h_t &= E_t \tilde{h}_t, \\
\tilde{h}_{t-1}(x^{t-1}, \omega) &= \inf_{x_t \in \mathbb{R}^{n_t}} h_t(x^{t-1}, x_t, \omega),
\end{aligned} \tag{2.1}$$

where $E_t\tilde{h}_t$ denotes the $\mathcal{F}_t$-conditional expectation of $\tilde{h}_t$; see [7, 12, 14, 37, 38] for the existence and uniqueness results for conditional normal integrands.

Theorem 1 *Assume that h has an integrable lower bound and that*

$$N_t(\omega) = \{x_t \in \mathbb{R}^{n_t} \mid h_t^\infty(x^t, \omega) \le 0,\ x^{t-1} = 0\}$$

is linear-valued for $t = T, \ldots, 0$. The functions h_t are then well-defined normal integrands and we have for every $x \in \mathcal{N}$ that

$$Eh_t(x_t(\omega), \omega) \ge \inf (\mathrm{P}_u) \quad t = 0, \ldots, T. \tag{2.2}$$

Optimal solutions for

$$\text{minimize } Eh(x(\omega), \omega) \quad \textit{over } x \in \mathcal{N}$$

exist and they are characterized by the condition

$$x_t(\omega) \in \operatorname*{argmin}_{x_t} h_t(x^{t-1}(\omega), x_t, \omega) \quad P\text{-a.s.} \quad t = 0, \ldots, T,$$

which is equivalent to having equalities in (2.2). Moreover, there is an optimal solution $x \in \mathcal{N}$ such that $x_t \perp N_t$ for every $t = 0, \ldots, T$.

The main result of Publication I is the following theorem. We assume that $\mathcal{U} \subset L^0(\Omega, F, P; \mathbb{R}^m)$ and $\mathcal{Y} \subset L^0(\Omega, F, P; \mathbb{R}^m)$ are decomposable spaces in separating duality with respect to

$$\langle u, y \rangle = Eu(\omega) \cdot y(\omega).$$

Theorem 2 *Assume that there is a $y \in \mathcal{Y}$ and an $m \in L^1(\Omega, \mathcal{F}, P)$ such that for P-almost every ω*

$$f(x, u, \omega) \ge u \cdot y(\omega) + m(\omega) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$$

and that $\{x \in \mathcal{N} \mid f^\infty(x(\omega), 0, \omega) \le 0 \text{ a.s.}\}$ is a linear space. Then

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u)$$

is lower semicontinuous on $\mathcal{U}$ and the infimum is attained for every $u \in \mathcal{U}$.

For the superhedging problem in a linear market model, the linearity condition in the above theorem reduces to the no-arbitrage condition. In Publication I we give an application of the above theorem to illiquid markets.

Publication II: Continuous essential selections and integral functionals

Given a set-valued mapping Γ from a topological space $\mathbb{T}$ to another X and a measure μ on $\mathbb{T}$, we say that a function $y : \mathbb{T} \to X$ is an *essential selection* of Γ if $y_t \in \Gamma_t$ μ-almost everywhere. The novel idea is to characterize a class of set-valued mappings for which continuous essential selections are selections (y is a *selection* if $y_t \in \Gamma_t$ for all t). Figure 2.1 illustrates a situation where y is a continuous essential selection of Γ, but y is not a selection since $y_t \notin \Gamma_t$.

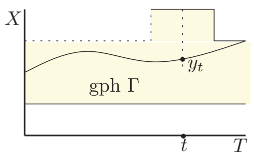

Figure 2.1. A continuous essential selection

To rule out situations such as in Figure 2.1, a set-valued mapping should not have irrelevant "vertical gaps" in its graph. On the other hand the set-valued mapping should not have "vertical boundaries" since such points can not belong to the graph of a continuous selection. A precise formulation of the characterization requires topological properties of set-valued mappings. The key concepts are the well-known inner semicontinuity which does not allow vertical boundaries as in Figure 2.1, and the new notion of outer regularity in measure which rules out vertical gaps supported on sets of measure zero.

Let $\mathcal{H}_t$ be the neighborhood system of $t \in \mathbb{T}$. Given a strictly positive[1] countably additive measure μ on the Borel σ-algebra $\mathcal{B}(\mathbb{T})$, we define $\mathcal{H}_t^\mu = \{B \in \mathcal{B}(\mathbb{T}) \mid \exists O \in \mathcal{H}_t : \mu(B \cap O) = \mu(O)\}$ and

$$(\mu\text{-liminf}\,\Gamma)_t = \bigcap_{B \in \mathcal{H}_t^{\mu\#}} \operatorname{cl}\Big(\bigcup_{t' \in B} \Gamma_{t'}\Big),$$

where $\mathcal{H}_t^{\mu\#} = \{B \in \mathcal{B}(\mathbb{T}) \mid B \cap O \neq \emptyset \;\forall O \in \mathcal{H}_t^\mu\}$. We say that Γ is *outer regular in measure* or *outer μ-regular* if

$$(\mu\text{-liminf}\,\Gamma)_t \subseteq \operatorname{cl}\Gamma_t \quad \forall t.$$

A topological space is T_1 if for every distinct points t and t' there is an $O_t \subset \mathcal{H}_t$ with $t' \notin O_t$. The space is *normal* if for every disjoint closed sets B and B' there are disjoint open sets O and O' such that $B \subset O$ and

[1] μ is strictly positive if $\mu(O) > 0$ for every nonempty open O.

$B' \subset O'$. The space $\mathbb{T}$ is *perfectly normal* if it is normal and every closed set is a countable intersection of open sets. Recall that $\mathbb{T}$ is *Lindelöf* if every open cover of $\mathbb{T}$ has a countable subcover, and that $\mathbb{T}$ is *strongly Lindelöf* if every subspace of $\mathbb{T}$ is Lindelöf.

The following theorem is our main result on set-valued analysis. The sets of continuous selections and continuous essential selections of Γ will be denoted by $C(\mathbb{T};\Gamma)$ and $C(\mathbb{T},\mu;\Gamma)$, respectively. The mapping Γ is *inner semicontinuous* (isc) if $\Gamma^{-1}(A)$ is an open set for every open $A \subset X$. A convex-valued Γ is *solid* if it is closed-valued and $\operatorname{int}\Gamma_t \neq \emptyset$ for all t.

Theorem 3 *Assume that $\mathbb{T}$ is a Lindelöf perfectly normal T_1-space. An inner semicontinuous solid convex $\mathbb{R}^d$-valued mapping Γ is outer μ-regular if and only if $C(\mathbb{T},\mu;\Gamma) = C(\mathbb{T};\Gamma)$.*

For example, the standard finite-dimensional space, the Wiener space in stochastic analysis and the real-line equipped with the left half-open topology which allows for merely left continuous functions in the ordinary sense, are Lindelöf perfectly normal T_1-spaces. On the other hand the class of set-valued mappings which share the above property contains continuous set-valued mappings as well as, provided that we have a standard finite dimensional space $\mathbb{T}$, "fully lower semicontinuous" set-valued mappings introduced in [29].

We apply the above theorem to integral functionals on continuous functions. From now on we assume that $\mathbb{T}$ is a Lindelöf perfectly normal T_1-space and that h is a normal integrand on $\mathbb{T} \times \mathbb{R}^d$. The bilinear form

$$\langle y, \theta \rangle = \int_{\mathbb{T}} y_t d\theta_t$$

puts the spaces C_b and M_b in separating duality. Indeed, it follows from [9, p.71] that C_b separates the points in M_b, whereas it is evident that M_b separates the points of C_b.

We denote the relative interior of a set $A \subset \mathbb{R}^d$ by $\operatorname{rint} A$. The following result is our main theorem on integral functionals. Recall that

$$J_{h^*}(\theta) = \int_{\mathbb{T}} h_t^*((d\theta/d\mu)_t)d\mu_t + \int_{\mathbb{T}} (h^*)_t^\infty((d\theta^s/d|\theta^s|)_t)d|\theta^s|_t,$$

Theorem 4 *Assume that μ is a σ-finite Radon measure, $\operatorname{dom} h$ is inner semicontinuous, $\operatorname{dom} J_{h^*} \neq \emptyset$, $C_b(\mathbb{T};\operatorname{rint}\operatorname{dom} h) \cap \operatorname{dom} I_h \neq \emptyset$ and that for every $y \in C(\mathbb{T};\operatorname{rint}\mu\text{-liminf}\operatorname{dom} h)$ and for every t there exists $O_t \in \mathcal{H}_t$ such that*

$$\int_{O_t} |h_t(y_t)| d\mu_t < \infty.$$

If $\operatorname{dom} h$ *is outer* μ*-regular, then* I_h *and* J_{h^*} *are conjugates of each other. If* $\operatorname{int}\operatorname{dom} h_t \neq \emptyset$ *for all* t *and if* I_h *and* J_{h^*} *are conjugates of each other, then* $\operatorname{dom} h$ *is outer* μ*-regular.*

Given that $\mathbb{T} \subseteq \mathbb{R}^n$, the above theorem relaxes the "full lower semicontinuity" condition in [29, Theorem 5] while retaining the same explicit expression for the conjugate of I_h on continuous functions; see Corollary 1 in Publication II.

Publication III: Duality in convex problems of Bolza over functions of bounded variation

Problems of Bolza is a general class of problems in the calculus of variations and optimal control; see [21, Section 6.5] for an account on the history of the subject. Much as in Rockafellar's works [25, 28, 30, 32, 34], we will study convex problems of Bolza in the conjugate duality framework. We extend the results of [30, 32], where trajectories of bounded variation are also considered, in two directions. First, we relax the continuity assumptions on the domain of the Hamiltonian using the results of Publication II on conjugates of convex integral functionals. Second, we parameterize the problem with a general Borel measure that shifts the derivative rather than the state. This choice of parameterization is of interest in financial economics where, e.g., the parameter may represent endowments and/or liabilities of an economic agent. The relaxed continuity requirements allow discontinuous state constraints both in the primal and the dual.

Given $T > 0$, let X be the space of left-continuous functions $x : \mathbb{R}_+ \to \mathbb{R}^d$ of bounded variation such that x is constant after T. The space X may be identified with $\mathbb{R}^d \times M$ where M is the space of $\mathbb{R}^d$-valued Radon measures on $[0, T]$. Indeed, given $x \in X$ there is a unique $\mathbb{R}^d$-valued Radon measure Dx on $[0, T]$ such that $x_t = x_0 + Dx([0, t))$ for all $t \in [0, T]$ and $x_t = x_0 + Dx([0, T])$ for $t > T$; see e.g. [17, Theorem 3.29]. The value of $x \in X$ on (T, ∞) will be denoted by x_{T+}.

Given an atomless strictly positive Radon measure μ on $[0, T]$, a proper[2] convex normal integrand $K : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \overline{\mathbb{R}}$ and a proper convex lower semicontinuous function $k : \mathbb{R}^d \times \mathbb{R}^d \to \overline{\mathbb{R}}$, we study the parametric

[2]An integrand K is *proper* if K_t is proper for all t.

optimization problem

$$\text{minimize} \quad J_K(x, Dx+u) + k(x_0, x_{T+}) \quad \text{over } x \in X, \tag{P$_u$}$$

where $u \in M$ and $J_K : X \times M \to \overline{\mathbb{R}}$ is given by

$$J_K(x,\theta) = \int K_t(x_t, (d\theta^a/d\mu)_t)d\mu_t + \int K_t^\infty(0, (d\theta^s/d|\theta^s|)_t)d|\theta^s|_t.$$

Here the sum of finite collection of extended real numbers is defined as $+\infty$ if any of the terms equals $+\infty$. It follows that J_K as well as the objective in (P$_u$) are well-defined extended real-valued functions on $X \times M$. Problems of the form (P$_u$) with $u = 0$ extend the more familiar problems

$$\text{minimize} \quad \int K_t(x_t, \dot{x}_t)d\mu_t + k(x_0, x_T) \quad \text{over } x \in AC \tag{P$_{AC}$}$$

by allowing for discontinuous trajectories. Here AC is the space of absolutely continuous functions and $\dot{x}$ denotes the Radon–Nikodym derivative of Dx with respect to μ. In the context of optimal control, discontinuous trajectories correspond to impulsive control.

We relax the continuity assumptions made in [30, 32] on the domain of the associated Hamiltonian

$$H_t(x,y) = \inf_{u \in \mathbb{R}^d} \{K_t(x,u) - u \cdot y\}.$$

The Hamiltonian is convex in x and concave in y. By [36, Proposition 14.45 and Theorem 14.50], $(y,t) \mapsto -H_t(x_t, y)$ is a normal integrand for every $x \in X$, so the integral functional

$$I_H(x,y) = \int H_t(x_t, y_t)d\mu$$

is thus well defined on $X \times C$. We set $I_H(x,y) = +\infty$ unless the positive part of the integrand is integrable. The function I_H is convex in x and concave in y. The set $\operatorname{dom} H_t := \operatorname{dom}_1 H_t \times \operatorname{dom}_2 H_t$ where

$$\operatorname{dom}_1 H_t = \{x \in \mathbb{R}^d \mid H_t(x,y) < +\infty \ \forall y \in \mathbb{R}^d\},$$
$$\operatorname{dom}_2 H_t = \{y \in \mathbb{R}^d \mid H_t(x,y) > -\infty \ \forall x \in \mathbb{R}^d\}$$

is known as the *domain* of H_t. The domain of I_H is defined similarly.

We will say that a set-valued mapping S is *left-inner semicontinuous* (or left-isc) if it is isc with respect to τ_l, where τ_l is generated by sets of the form $\{(s,t] \mid s < t\}$. Similarly, S is said to be *left-outer μ-regular* if it is outer μ-regular with respect to τ_l.

The following theorem is one the main results. We define the functions $\tilde{K}$ and $\tilde{k}$ in terms of K and k as

$$\tilde{K}_t(y,v) = K_t^*(v,y)$$
$$\tilde{k}(\tilde{a},\tilde{b}) = k^*(\tilde{a},-\tilde{b})$$

and we denote by $\mathbb{B}(x,r)$ the open ball with center $x \in \mathbb{R}^d$ and radius $r > 0$.

Theorem 5 *Assume that*

1. $t \mapsto \operatorname{dom}_1 H_t$ *is left-isc and left-outer μ-regular*
2. $\emptyset \neq \{x \in X \mid \exists r > 0 : \mathbb{B}(x_t, r) \subset \operatorname{dom}_1 H_t\ \forall t\} \subset \operatorname{dom}_1 I_H$,
3. $t \mapsto \operatorname{dom}_2 H_t$ *is isc and outer μ-regular*
4. $\{y \in C \mid y_t \in \operatorname{int} \operatorname{dom}_2 H_t\ \forall t\} \subset \operatorname{dom}_2 I_H$,
5. *there exists a* $\bar{y} \in \operatorname{dom} g \cap AC$ *with* $\bar{y}_t \in \operatorname{int} \operatorname{dom}_2 H_t$ *for all* t,
6. $\{x \mid J_{K^\infty}(x, Dx+u) + k^\infty(x_0, x_{T+}) \le 0\}$ *is a linear space.*

Then the infimum in (P_u) *is attained for every* u *and*

$$\varphi(u) = \sup_{y \in C \cap X} \{\langle u, y\rangle - J_{\tilde{K}}(y, Dy) - \tilde{k}(y_0, y_T)\}.$$

Another main result is about optimality conditions for the problem (P_u) when $u = 0$. That is, we will be looking at the problem

$$\text{minimize} \int K_t(x_t, \dot{x}_t^a) d\mu_t + \int K_t^\infty(0, \dot{x}_t^s) d|Dx^s|_t + k(x_0, x_{T+}) \text{ over } x \in X, \quad \text{(P)}$$

where $\dot{x}^a = d(Dx^a)/d\mu$ and $\dot{x}^s = d(Dx^s)/d|Dx^s|$. We associate with (P) the problem

$$\text{minimize} \int \tilde{K}_t(y_t, \dot{y}_t^a) d\mu_t + \int \tilde{K}_t^\infty(0, \dot{y}_t^s) d|Dy^s|_t + \tilde{k}(y_0, y_T) \text{ over } y \in C \cap X. \quad \text{(D)}$$

For a mapping $S_t : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with $t \mapsto \operatorname{gph} S_t$ closed-valued and measurable, and for $z \in X$, we write $Dz \in S(z)$ if

$$\dot{z}_t^a \in S_t(z_t) \quad \mu\text{-a.e.},$$
$$\dot{z}_t^s \in S_t^s(z_t) \quad |Dz^s|\text{-a.e.},$$

where the mapping $S_t^s : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is defined for each t as the *graphical inner limit* (see [36, Chapter 5]) of the mappings $(\alpha S_t)(z) := \alpha S_t(z)$ as $\alpha \searrow 0$. This definition is suggested by [34, Section 14] as an extension of differential inclusions from absolutely continuous trajectories to trajectories of bounded variation.

We say that $x \in X$ and $y \in C \cap X$ satisfy the *generalized Hamiltonian equation* if

$$D(x,y) \in \Pi \tilde{\partial} H(x,y),$$

where $\Pi(v,u) = (u,v)$ and

$$\tilde{\partial} H_t(x,y) = \partial_x H_t(x,y) \times \partial_y[-H_t](x,y).$$

Since $\partial \tilde{H}_t$ is maximal monotone, $(\tilde{\partial} H)^s_t$ equals the normal cone mapping $N_{\operatorname{cl}\operatorname{dom} H_t}$ of $\operatorname{cl}\operatorname{dom} H_t$; see [36, Example 12.27, Theorem 12.37] and [26, Theorem 37.4].

When $\operatorname{dom} H_t = \mathbb{R}^d \times \mathbb{R}^d$, we have $N_{\operatorname{cl}\operatorname{dom} H_t}(x,y) = \{0\}$, so feasible trajectories are necessarily absolutely continuous and the generalized Hamiltonian equation reduces to an ordinary differential inclusion.

As usual $x \in X$ and $y \in C \cap X$ are said to satisfy the *transversality condition* if

$$(y_0, -y_T) \in \partial k(x_0, x_{T+}).$$

Theorem 6 *Assume that* $t \mapsto \operatorname{dom}_1 H_t$ *is left-outer* μ*-regular and that* $t \mapsto \operatorname{dom}_2 H_t$ *is outer* μ*-regular. Then* $\inf(P) \geq -\inf(D)$*. For* $\inf(P) = -\inf(D)$ *to hold with attainment at feasible* x *and* y *respectively, it is necessary and sufficient that* x *and* y *satisfy the generalized Hamiltonian equation and the transversality condition.*

The conditions of the theorem above generalize those in [32, Theorem 2]. On the other hand, in [32, Theorem 2] both trajectories are allowed to be discontinuous. As an application of the theorem above, we give conditions in Theorem 4.2 of Publication III under which $x \in X$ attains the infimum in (P) if and only if it satisfies the generalized Hamiltonian equation and the transversality condition with some $y \in AC$.

3. Further investigations

The results in Publication I have many applications in mathematical finance; see, e.g., [24]. However, it is assumed in the main theorems that the normal integrand has an integrable lower bound with respect to the strategies. This rules out expected utility maximization problems with unbounded utilities that are allowed, e.g., in [20]. Therefore, there certainly is a motivation to study to what extent the integrable lower bound condition can be generalized while allowing for a similar analysis as in Publication I.

In Publication II we introduced outer regularity in measure of set-valued mappings and analyzed it in sufficient detail in order to apply the concept to integral functionals. Just like outer and inner semicontinuity are dual notions in a sense, it is possible to introduce a dual notion to outer regularity in measure by defining that Γ is inner μ-regular if

$$\Gamma_t \subseteq \mu\text{-limsup}\,\Gamma_t \quad \text{for all } t;$$

see Remark 2 of Publication II. It seems reasonable to use inner μ-regularity much like local continuity was used in [4] to study arbitrage in non-semimartingale models.

In Publication III we obtained optimality conditions for problems of Bolza in terms of a generalized Hamiltonian equation. This is in turn based on a generalization of a differential inclusion from absolutely continuous functions to functions of bounded variation. Further analysis of such inclusions for functions of bounded variation seems a prominent line of research. On the other hand, another plausible area for applications of outer regularity in measure could be multi-dimensional generalizations of problems of Bolza over functions bounded variation which are studied e.g. in [1]. Indeed, Theorem 3 in Publication II is applicable in such situations, and Example 2 in Publication II indicates which kind of discontinuities can be handled using outer regularity in measure. Yet another possibility is to apply outer regularity in measure to non-convex problems of Bolza that are studied, e.g., in [13, 21].

Considering the thesis on the whole, a natural continuation is to study duality in stochastic optimization in continuous time. For instance many optimization problems in markets with transaction costs (see [19]) can be seen as stochastic problems of Bolza over predictable processes of bounded variation. The combination of the methods in the present publications gives tools which are well-suited to attack these problems. On the other hand, already during the 70's Bismut adapted ideas from conjugate duality in deterministic problems of Bolza to study optimal stochastic control in continuous time; see [8]. It would be interesting to combine Bismut's ideas with the techniques used in the thesis to study stochastic impulsive control.

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